

# COMMENSURATIONS AND METRIC PROPERTIES OF HOUGHTON'S GROUPS

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ABSTRACT. We describe the automorphism groups and the abstract commensurators of Houghton's groups. Then we give sharp estimates for the word metric of these groups and deduce that the commensurators embed into the corresponding quasi-isometry groups. As a further consequence, we obtain that the Houghton group on two rays is at least quadratically distorted in those with three or more rays.

## INTRODUCTION

The family of Houghton groups  $\mathcal{H}_n$  was introduced by Houghton [7]. These groups serve as an interesting family of groups, studied by Brown [2], who described their homological finiteness properties, by Röver [10], who showed that these groups are all subgroups of Thompson's group  $V$ , and by Lehnert [9] who described the metric for  $\mathcal{H}_2$ . Lee [8] described isoperimetric bounds, and de Cornulier, Guyot, and Pitsch [4] showed that they are isolated points in the space of groups.

Here, we classify automorphisms and determine the abstract commensurator of  $\mathcal{H}_n$ . We also give sharp estimates for the word metric which are sufficient to show that the map from the abstract commensurator to the group of quasi-isometries of  $\mathcal{H}_n$  is an injection.

## 1. DEFINITIONS AND BACKGROUND

Let  $\mathbb{N}$  be the set of natural numbers (positive integers) and  $n \geq 1$  be an integer. We write  $\mathbb{Z}_n$  for the integers modulo  $n$  with addition and put  $R_n = \mathbb{Z}_n \times \mathbb{N}$ . We interpret  $R_n$  as the graph of  $n$  pairwise disjoint

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rays; each vertex  $(i, k)$  is connected to  $(i, k + 1)$ . We denote by  $\mathbf{Sym}_n$ ,  $\mathbf{FSym}_n$  and  $\mathbf{FAlt}_n$ , or simply  $\mathbf{Sym}$ ,  $\mathbf{FSym}$  and  $\mathbf{FAlt}$  if  $n$  is understood, the full symmetric group, the finitary symmetric group and the finitary alternating group on the set  $R_n$ , respectively.

The *Houghton group*  $\mathcal{H}_n$  is the subgroup of  $\mathbf{Sym}$  consisting of those permutations that are eventually translations (of each of the rays). In other words, the permutation  $\sigma$  of the set  $R_n$  is in  $\mathcal{H}_n$  if there exist integers  $N \geq 0$  and  $t_i = t_i(\sigma)$  for  $i \in \mathbb{Z}_n$  such that for all  $k \geq N$ ,  $(i, k)\sigma = (i, k + t_i)$ ; throughout we will use right actions.

Note that necessarily the sum of the translations  $t_i$  must be zero because the permutation needs of course to be a bijection. This implies that  $\mathcal{H}_1 \cong \mathbf{FSym}$ .

For  $i, j \in \mathbb{Z}_n$  with  $i \neq j$  let  $g_{ij} \in \mathcal{H}_n$  be the element which translates the line obtained by joining rays  $i$  and  $j$ , given by

$$\begin{aligned} (i, n)g_{ij} &= (i, n - 1) \text{ if } n > 1, \\ (i, 1)g_{ij} &= (j, 1), \\ (j, n)g_{ij} &= (j, n + 1) \text{ if } n \geq 1 \text{ and} \\ (k, n)g_{ij} &= (k, n) \text{ if } k \notin \{i, j\}. \end{aligned}$$

We also write  $g_i$  instead of  $g_{i, i+1}$ . It is easy to see that  $\{g_i \mid i \in \mathbb{Z}_n\}$ , as well as  $\{g_{ij} \mid i, j \in \mathbb{Z}_n, i \neq j\}$ , are generating sets for  $\mathcal{H}_n$  if  $n \geq 3$  as we can simply check that the commutator  $[g_0, g_1] = g_0^{-1}g_1^{-1}g_0g_1$  transposes  $(1, 1)$  and  $(2, 1)$ . In the special case of  $\mathcal{H}_2$ ,  $g_1$  is redundant as  $g_1 = g_0^{-1}$ . Further, an additional generator to  $g_0$  is required to generate the group; we choose  $\tau$  which fixes all points except for transposing  $(0, 1)$  and  $(1, 1)$ .

It is now clear that the commutator subgroup of  $\mathcal{H}_n$  is given by

$$\mathcal{H}'_n = \begin{cases} \mathbf{FAlt}, & \text{if } n \leq 2 \\ \mathbf{FSym}, & \text{if } n \geq 3 \end{cases}$$

For  $n \geq 3$ , we thus have a short exact sequence

$$(1) \quad 1 \longrightarrow \mathbf{FSym} \longrightarrow \mathcal{H}_n \xrightarrow{\pi} \mathbb{Z}^{n-1} \longrightarrow 1$$

where  $\pi(\sigma) = (t_0(\sigma), \dots, t_{n-2}(\sigma))$  is the abelianization homomorphism. We note that as the sum of all the eventual translations must be zero, we have the last translation is determined by the preceding ones:

$$(2) \quad t_{n-1}(\sigma) = - \sum_{i=0}^{n-2} t_i(\sigma).$$

We will use the following facts freely throughout this paper, see Dixon and Mortimer [5] or Scott [11].

**Lemma 1.1.** *The group  $\mathbf{FAlt}$  is simple and equal to the commutator subgroup of  $\mathbf{FSym}$ , and  $\mathbf{Aut}(\mathbf{FAlt}) = \mathbf{Aut}(\mathbf{FSym}) = \mathbf{Sym}$ .*

## 2. AUTOMORPHISMS OF $\mathcal{H}_n$

Here we determine the automorphism group of  $\mathcal{H}_n$ . First we establish that we have to look no further than  $\mathbf{Sym}$ . We let  $N_G(H)$  denote the normalizer, in  $G$ , of the subgroup  $H$  of  $G$ .

**Proposition 2.1.** *Every automorphism of  $\mathcal{H}_n$ ,  $n \geq 1$ , is given by conjugation by an element of  $\mathbf{Sym}$ , that is to say  $\mathbf{Aut}(\mathcal{H}_n) = N_{\mathbf{Sym}}(\mathcal{H}_n)$ .*

*Proof.* From the above, the finitary alternating group  $\mathbf{FAlt}$  is the second derived subgroup of  $\mathcal{H}_n$ , and hence characteristic in  $\mathcal{H}_n$ . So every automorphism of  $\mathcal{H}_n$  restricts to an automorphism of  $\mathbf{FAlt}$ . Since  $\mathbf{Aut}(\mathbf{FAlt}) = \mathbf{Sym}$ , this restriction yields a homomorphism  $\mathbf{Aut}(\mathcal{H}_n) \rightarrow \mathbf{Sym}$  and we need to show that it is injective with image equal to  $N_{\mathbf{Sym}}(\mathcal{H}_n)$ .

In order to see this let  $\psi \in \mathbf{Aut}(\mathcal{H}_n)$  be an automorphism. Compose this with an inner automorphism (of  $\mathbf{Sym}$ ) so that the result is an (injective) homomorphism  $\alpha : \mathcal{H}_n \rightarrow \mathbf{Sym}$  whose restriction to  $\mathbf{FAlt}$  is trivial. We let  $k \in \mathbb{N}$  and consider the following six consecutive points  $a_\ell = (i, k + \ell)$  of  $R_n$  for  $\ell \in \{0, 1, \dots, 5\}$ .

We denote by  $g_i^\alpha$  the image of  $g_i$  under  $\alpha$ , and by  $(xyz)$  the 3-cycle of the points  $x$ ,  $y$  and  $z$ . Using the identities

$$g_i^{-1}(a_1 a_2 a_3)g_i = (a_0 a_1 a_2) \text{ and } g_i^{-1}(a_3 a_4 a_5)g_i = (a_2 a_3 a_4)$$

and applying  $\alpha$ , which is trivial on  $\mathbf{FAlt}$ , we get

$$(g_i^\alpha)^{-1}(a_1 a_2 a_3)g_i^\alpha = (a_0 a_1 a_2) \text{ and } (g_i^\alpha)^{-1}(a_3 a_4 a_5)g_i^\alpha = (a_2 a_3 a_4).$$

Hence, we must have that  $g_i^\alpha$  maps  $\{a_1, a_2, a_3\}$  to  $\{a_0, a_1, a_2\}$ , and then also  $\{a_3, a_4, a_5\}$  to  $\{a_2, a_3, a_4\}$ . The conclusion is that it maps  $a_3$  to  $a_2$ . Applying a similar argument to all points in the branches  $i$  and  $i + 1$ , it follows that  $g_i^\alpha = g_i$ , and since  $i$  was arbitrary, this means that  $\alpha$  is the identity map.  $\square$

With Lemma 1.1 in mind we now present the complete description of  $\mathbf{Aut}(\mathcal{H}_n)$ .

**Theorem 2.2.** *For  $n \geq 2$ , the automorphism group  $\mathbf{Aut}(\mathcal{H}_n)$  of the Houghton group  $\mathcal{H}_n$  is isomorphic to the semidirect product  $\mathcal{H}_n \rtimes \mathcal{S}_n$ , where  $\mathcal{S}_n$  is the symmetric group that permutes the  $n$  rays.*

*Proof.* By the proposition, it suffices to prove that every  $\alpha \in \mathbf{Sym}$  which normalizes  $\mathcal{H}_n$  must map  $(i, k + m)$  to  $(j, l + m)$  for some  $k, l \geq 1$  and all  $m \geq 0$ .

To this end, we pick  $\alpha \in N_{\mathbf{Sym}}(\mathcal{H}_n)$  and  $\sigma \in \mathcal{H}_n$ . Since  $\sigma^\alpha (= \alpha^{-1}\sigma\alpha)$  is in  $\mathcal{H}_n$  and maps the point  $x\alpha$  to  $(x\sigma)\alpha$ , these two points must lie on the same ray for all but finitely many  $x \in R_n$ . Similarly,  $x$  and  $x\sigma$  lie on the same ray for all but finitely many  $x \in R_n$ , as  $\sigma \in \mathcal{H}_n$ . In fact, given a ray, we can choose  $\sigma$  so that whenever  $x$  lies on that ray, then  $x$  and  $x\sigma$  are successors on the same ray. Combining these facts, we see that  $\alpha$  maps all but finitely many points of ray  $i$  to one and the same ray, say ray  $j$ . This defines a homomorphism from  $\text{Aut}(\mathcal{H}_n)$  onto  $\mathcal{S}_n$ , which is obviously split, since given a permutation of the  $n$  rays, it clearly defines an automorphism of  $\mathcal{H}_n$ .

So assume that  $\alpha$  is now in the kernel of that map, so it does not permute the rays, and take  $\sigma$  a  $g_{ji}$  generator of  $\mathcal{H}_n$ , i.e., an infinite cycle inside  $\mathbf{Sym}$ . Then, since conjugating inside  $\mathbf{Sym}$  preserves cycle types, the element  $\sigma^\alpha \in \mathcal{H}_n$  is also a single infinite cycle. This means that  $\sigma^\alpha$  has nonzero translations in only two rays, and these translations are 1 and  $-1$ . For all points in the support of  $\sigma$ , we have that  $\sigma^k$  sends this point into the  $i$ -th ray for all sufficiently large  $k$ . Therefore, as  $\alpha$  sends almost all points in the  $i$ -th ray into the  $i$ -th ray, the same is true for  $\alpha^\sigma$ . Hence  $t_i(\sigma^\alpha)$  is positive, so it must be  $t_i(\sigma^\alpha) = 1$ . It is quite clear now that  $\alpha$  translates by an integer in the ray  $i$ , sufficiently far out. This finishes the proof since this could be done for any  $i$ , and hence  $\alpha \in \mathcal{H}_n$ .  $\square$

### 3. COMMENSURATIONS OF $\mathcal{H}_n$

First, we recall that a commensuration of a group  $G$  is an isomorphism  $A \xrightarrow{\phi} B$ , where  $A$  and  $B$  are subgroups of finite index in  $G$ . Two commensurations  $\phi$  and  $\psi$  of  $G$  are equivalent if there exists a subgroup  $A$  of finite index in  $G$ , such that the restrictions of  $\phi$  and  $\psi$  to  $A$  are equal. The set of all commensurations of  $G$  modulo this equivalence relation forms a group, known as the (abstract) *commensurator of  $G$* , and is denoted  $\text{Com}(G)$ . In this section we will determine the commensurator of  $\mathcal{H}_n$ .

For a moment, we let  $H$  be a subgroup of a group  $G$ . The *relative commensurator of  $H$  in  $G$*  is denoted  $\text{Com}_G(H)$  and consists of those  $g \in G$  such that  $H \cap H^g$  has finite index in both  $H$  and  $H^g$ . Similar to the homomorphism from  $N_G(H)$  to  $\text{Aut}(H)$ , there is a homomorphism from  $\text{Com}_G(H)$  to  $\text{Com}(H)$ ; Its kernel consists of those elements of  $G$  that centralize a finite index subgroup of  $H$ .

In order to pin down  $\text{Com}(\mathcal{H}_n)$ , we first establish that every commensuration of  $\mathcal{H}_n$  can be seen as conjugation by an element of  $\text{Sym}$ , and then characterize  $\text{Com}_{\text{Sym}}(\mathcal{H}_n)$ .

Since a commensuration  $\phi$  and its restriction to a subgroup of finite index in its domain are equivalent, we can restrict our attention to the following family of subgroups of finite index in  $\mathcal{H}_n$ , in order to exhibit  $\text{Com}(\mathcal{H}_n)$ . For every integer  $p \geq 1$ , we define the subgroup  $U_p$  of  $\mathcal{H}_n$  by

$$U_p = \langle \text{FAlt}, g_i^p \mid i \in \mathbb{Z}_n \rangle.$$

We collect some useful properties of these subgroup first, where  $A \subset_f B$  means that  $A$  is a subgroup of finite index in  $B$ .

**Lemma 3.1.** *Let  $n \geq 3$ .*

(i) *For every  $p$ , the group  $U_p$  coincides with  $\mathcal{H}_n^p$ , the subgroup generated by all  $p^{\text{th}}$  powers in  $\mathcal{H}_n$ , and hence is characteristic in  $\mathcal{H}_n$ .*

$$(ii) \ U'_p = \begin{cases} \text{FAlt}, & p \text{ even} \\ \text{FSym}, & p \text{ odd} \end{cases} \quad \text{and} \quad |\mathcal{H}_n : U_p| = \begin{cases} 2p^{n-1}, & p \text{ even} \\ p^{n-1}, & p \text{ odd} \end{cases}.$$

(iii) *For every finite index subgroup  $U$  of  $\mathcal{H}_n$ , there exists a  $p \geq 1$  with  $\text{FAlt} = U'_p \subset U_p \subset_f U \subset_f \mathcal{H}_n$ .*

The same is essentially true for the case  $n = 2$ , except that  $U'_p$  is always equal to  $\text{FAlt}$  in this case, with the appropriate change in the index.

*Proof.* First we establish (ii). We know that  $[g_i, g_j]$  is either trivial, when  $j \notin \{i-1, i+1\}$ , or an odd permutation. So the commutator identities  $[ab, c] = [a, c]^b [b, c]$  and  $[a, bc] = [a, c][a, b]^c$  imply the first part, and the second part follows immediately using the short exact sequence (1) from Section 1.

Part (i) is now an exercise, using that  $\text{FAlt}^p = \text{FAlt}$ .

In order to show (iii), let  $U$  be a subgroup of finite index in  $\mathcal{H}_n$ . The facts that  $\text{FAlt}$  is simple and  $U$  contains a normal finite index subgroup of  $\mathcal{H}_n$ , imply that  $\text{FAlt} \subset U$ . Let  $p$  be the smallest even integer such that  $(p\mathbb{Z})^{n-1}$  is contained in the image of  $U$  in the abelianisation of  $\mathcal{H}_n$ . It is now clear that  $U_p$  is contained in  $U$ .  $\square$

Noting that  $\text{Com}(\mathcal{H}_1) = \text{Aut}(\mathcal{H}_1) = \text{Sym}$ , we now characterize the commensurators of Houghton's groups.

**Theorem 3.2.** *Let  $n \geq 2$ . Every commensuration of  $\mathcal{H}_n$  normalizes  $U_p$  for some even integer  $p$ . The group  $N_p = N_{\text{Sym}}(U_p)$  is isomorphic*

to  $\mathcal{H}_{np} \rtimes (\mathcal{S}_p \wr \mathcal{S}_n)$ , and  $\text{Com}(\mathcal{H}_n)$  is the direct limit of  $N_p$  with even  $p$  under the natural embeddings  $N_p \longrightarrow N_{pq}$  for  $q \in \mathbb{N}$ .

*Proof.* Let  $\phi \in \text{Com}(\mathcal{H}_n)$ . By Lemma 3.1, we can assume that  $U_p$  is contained in the domain of both  $\phi$  and  $\phi^{-1}$  for some even  $p$ . Let  $V$  be the image of  $U_p$  under  $\phi$ . Then  $V$  has finite index in  $\mathcal{H}_n$  and so contains  $\text{FAlt}$ , by Lemma 3.1. However, the set of elements of finite order in  $V$  equals  $[V, V]$ , whence  $[V, V] = \text{FAlt}$ , as  $\text{FAlt}$  and  $\text{FSym}$  are not isomorphic. This means that the restriction of  $\phi$  to  $\text{FAlt}$  is an automorphism of  $\text{FAlt}$ , and hence yields a homomorphism

$$\iota: \text{Com}(\mathcal{H}_n) \longrightarrow \text{Sym}.$$

That  $\iota$  is an injective homomorphism to  $\text{Com}_{\text{Sym}}(\mathcal{H}_n)$  follows from a similar argument to the one in Proposition 2.1 applied to  $g_i^p$  and six points of the form  $a_\ell = (i, k + p\ell)$  with  $\ell \in \{0, 1, \dots, 5\}$ . Since the centralizer in  $\text{Sym}$  of  $\text{FAlt}$ , and hence of any finite index subgroup of  $\mathcal{H}_n$ , is trivial, the natural homomorphism from  $\text{Com}_{\text{Sym}}(\mathcal{H}_n)$  to  $\text{Com}(\mathcal{H}_n)$  mentioned above is also injective, and we conclude that  $\text{Com}(\mathcal{H}_n)$  is isomorphic to  $\text{Com}_{\text{Sym}}(\mathcal{H}_n)$ , and that  $\iota$  is the isomorphism:

$$\iota: \text{Com}(\mathcal{H}_n) \longrightarrow \text{Com}_{\text{Sym}}(\mathcal{H}_n).$$

From now on, we assume that  $\phi \in \text{Com}_{\text{Sym}}(\mathcal{H}_n)$ . In particular, the action of  $\phi$  is given by conjugation, and our hypothesis is that  $U_p^\phi \subset \mathcal{H}_n$ . Now we can apply the argument from the proof of Theorem 2.2 to  $\sigma \in U$  and  $\sigma^\phi$  (instead of  $\sigma^\alpha$ ). Namely, consider the  $i^{\text{th}}$  ray, and choose a  $\sigma = g_{ji}^p \in U_p$ . So we have  $t_i(\sigma) = p$ ,  $t_j(\sigma) = -p$  and zero translation elsewhere. Now, except for finitely many points,  $\sigma^\phi$  preserves the rays and sends  $x\phi$  to  $x\sigma^\phi$ . Thus there is an infinite subset of the  $i^{\text{th}}$  ray which is sent to the same ray by  $\phi$ , say ray  $k$ . The infinite subset should be thought of as a union of congruence classes modulo  $p$ , except for finitely many points. We claim that no infinite subset of ray  $j$  can be mapped by  $\phi$  to ray  $k$ . This is because, if it were, the infinite subset would contain a congruence class modulo  $p$  (except for finitely many points) from which we would be able to produce two points,  $x, y$ , in the support of  $\sigma^\phi$  such that all sufficiently large positive powers of  $\sigma^\phi$  send  $x$  into ray  $k$  and all sufficiently large negative powers of  $\sigma^\phi$  send  $y$  into ray  $k$ , and this is not possible for an element of  $\mathcal{H}_n$ . This means that  $\phi$  maps an infinite subset of ray  $i$  onto almost all of ray  $k$  (observe that ray  $k$  is almost contained in the support of  $\sigma^\phi$  so must be almost contained in the image of the unions of rays  $i$  and  $j$ ). Now applying similar arguments to  $\phi^{-1}$  we get that  $\phi$  is a bijection between rays  $i$  and  $k$  except for finitely many points. In fact,  $\phi$  must induce

bijections between the congruence classes modulo  $p$  (except for finitely many points) inside the two ray systems.

Thus  $\phi$  induces a permutation of the ray system. Again, looking at large positive powers we deduce that  $t_k(\sigma^\phi) > 0$  and since the support of  $\sigma^\phi$  is almost equal to the union of two rays, we must have that  $t_k(\sigma^\phi) = p$ , as  $\sigma$  and hence  $\sigma^\phi$  have exactly  $p$  orbits. In particular, we may deduce that  $\phi$  normalises  $U_p$ . So  $U_p^\phi = U_p$ .

In order to proceed, it will be useful to change the ray system. Specifically, each ray can be split into  $p$  rays, preserving the order. This realises  $U_p$  as a (normal) subgroup of  $\mathcal{H}_{np}$ . We say two of these new rays are equivalent if they came from the same old ray. Thus there are  $n$  equivalence classes, each having  $p$  elements. The group  $U_p$  acts on this new ray system as the subgroup of  $\mathcal{H}_{np}$  consisting of all  $\sigma \in \mathcal{H}_{np}$  such that  $t_i(\sigma) = t_j(\sigma)$  whenever the  $i^{\text{th}}$  and  $j^{\text{th}}$  rays are equivalent. Note that because we have split the rays, these translation amounts can be arbitrary in  $U_p$  (as a subgroup of  $\mathcal{H}_{np}$ ) and not just multiples of  $p$ . In particular, we have that  $U_p = U_p^\phi \subset \mathcal{H}_{np}$  and the previous arguments imply that  $\phi$  induces a bijection on the new ray system and sends equivalent rays to equivalent rays (since it is actually permuting the old ray system). Since  $\phi$  permutes the rays, but must preserve equivalence classes, we get a homomorphism from  $N_{\text{Sym}}(U_p)$  to the subgroup of  $\mathcal{S}_{np}$  which preserves the equivalence classes - this is easily seen to be  $(\mathcal{S}_p \wr \mathcal{S}_n)$  and the above homomorphism is split.

As in Theorem 2.2 we now conclude that the kernel of this homomorphism is  $\mathcal{H}_{np}$  and hence we get that  $N_p = N_{\text{Sym}}(U_p)$  is  $\mathcal{H}_{np} \rtimes (\mathcal{S}_p \wr \mathcal{S}_n)$ .  $\square$

We note that  $\text{Com}(\mathcal{H}_n)$  is not finitely generated, for if it were, it would lie in some maximal  $N_p$ .

#### 4. METRIC ESTIMATES FOR $\mathcal{H}_n$

In this section we will give sharp estimates for the word length of elements of Houghton's groups. This makes no sense for  $\mathcal{H}_1$  which is not finitely generated. As mentioned in the introduction, the metric in  $\mathcal{H}_2$  was described by Lehnert [9]. In order to deal with  $\mathcal{H}_n$  for  $n \geq 3$ , we introduce the following measure of complexity of an element.

Given  $\sigma \in \mathcal{H}_n$ , we define  $p_i(\sigma)$ , for  $i \in \mathbb{Z}_n$ , to be the largest integer such that  $(i, p_i(\sigma))\sigma \neq (i, p_i(\sigma) + t_i(\sigma))$ . Note that if  $t_i(\sigma) < 0$ , then necessarily  $p_i(\sigma) \geq |t_i(\sigma)|$ , as the first element in each ray is numbered 1.

The *complexity* of  $\sigma \in \mathcal{H}_n$  is the natural number  $P(\sigma)$ , defined by

$$P(\sigma) = \sum_{i \in \mathbb{Z}_n} p_i(\sigma).$$

And the *translation amount* of  $\sigma$  is

$$T(\sigma) = \frac{1}{2} \sum_{i \in \mathbb{Z}_n} |t_i(\sigma)|.$$

The above remark combined with (2) immediately implies  $P(\sigma) \geq T(\sigma)$ . It is easy to see that an element with complexity zero is trivial, and only the generators  $g_{ij}$  have complexity one.

**Theorem 4.1.** *Let  $n \geq 3$  and  $\sigma \in \mathcal{H}_n$ , with complexity  $P = P(\sigma) \geq 2$ . Then the word length  $|\sigma|$  of  $\sigma$  with respect to any finite generating set satisfies*

$$P/C \leq |\sigma| \leq KP \log P,$$

where the constants  $C$  and  $K$  only depend on the choice of generating set.

*Proof.* Since the word length with respect to two different finite generating sets differs only by a multiplicative constant, we can and will choose  $\{g_{ij} \mid i, j \in \mathbb{Z}_n, i \neq j\}$  as generating set to work with, and show that the statement holds with  $C = 1$  and  $K = 7$ .

The lower bound is established by examining how multiplication by a generator can change the complexity. Suppose  $\sigma$  has complexity  $P$  and consider  $\sigma g_{ij}$ . It is not difficult to see that

$$(3) \quad p_k(\sigma g_{ij}) = \begin{cases} p_k(\sigma) + 1, & \text{if } k = i \text{ and } (i, p_i(\sigma) + 1)\sigma = (i, 1) \\ p_k(\sigma) - 1, & \text{if } k = j, (j, p_j(\sigma) + 1)\sigma = (j, 1) \text{ and} \\ & (j, p_j(\sigma))\sigma = (i, 1) \\ p_k(\sigma), & \text{otherwise} \end{cases}$$

where the first two cases are mutually exclusive, as  $i \neq j$ . Thus  $|P(\sigma g_{ij}) - P(\sigma)| \leq 1$ , which establishes the lower bound.

The upper bound is obtained as follows. Suppose  $\sigma \in \mathcal{H}_n$  has complexity  $P$ . First we show by induction on  $T = T(\sigma)$  that there is a word  $\rho$  of length at most  $T \leq P$  such that the complexity of  $\sigma\rho$  is  $\bar{P}$  with  $\bar{P} \leq P$  and  $T(\sigma\rho) = 0$ . The case  $T = 0$  is trivial. If  $T > 0$ , then there are  $i, j \in \mathbb{Z}_n$  with  $t_i(\sigma) > 0$  and  $t_j(\sigma) < 0$ . So  $T(\sigma g_{ij}) = T - 1$ . Moreover,  $P(\sigma g_{ij}) \leq P$ , because the first case of (3) is excluded, as it implies that  $t_i(\sigma) = -p_i(\sigma) \leq 0$ , contrary to our assumption. This completes the induction step.

We are now in the situation that  $\sigma\rho \in \text{FSym}$  and loosely speaking we proceed as follows.

- (1) We push all irregularities into ray 0, i.e. multiply by  $\prod g_{i0}^{p_i(\sigma\rho)}$ .
- (2) We push all points back into the ray to which they belong, except for those from ray 0 which we mix into ray 1, say.
- (3) We push out of ray 1 separating the points belonging to rays 0 and 1 into ray 0 and any other ray, say ray 2, respectively.
- (4) We push the points belonging to ray 1 back from ray 2 into it.

These four steps can be achieved by multiplying by an element  $\mu$  of length at most  $4\bar{P}$ , such that  $\sigma\rho\mu$  is an element which, for each  $i$ , permutes an initial segment  $I_i$  of ray  $i$ . Notice that  $\sigma\rho\mu$  is now an element of  $\mathcal{H}_n$  which maps each ray to itself, and hence  $t_i(\sigma\rho\mu) = 0$  for all  $i$ .

It is clear that  $\mu$  can be chosen so that the total length of the moved intervals,  $\sum |I_i| \leq \bar{P}$ . Finally, we sort each of these intervals using a recursive procedure, modeled on standard merge sort.

In order to sort the interval  $I = I_2$  say, we push each of its points out of ray 2 and into either ray 0 if it belongs to the lower half, or to ray 1 if it belongs to the upper half of  $I$ . If each of the two halves occurs in the correct order, then we only have to push them back into ray 2 and are done, having used  $2|I|$  generators. If the two halves are not yet sorted, then we use the same “separate the upper and lower halves” approach on each of them recursively in order to sort them. In total this takes at most  $2|I|\log_2 |I|$  steps.

Altogether we have used at most

$$P + 4\bar{P} + 2 \sum_{i \in \mathbb{Z}_n} |I_i| \log_2 |I_i| \leq 7P \log_2 P$$

generators to represent the inverse of  $\sigma$ ; we used the hypothesis  $P \geq 2$  in the last inequality.  $\square$

We note that because there are many permutations, the fraction of elements which are close to the lower bound goes to zero in much the same way as shown for Thompson's group  $V$  by Birget [1] and its generalization  $nV$  by Burillo and Cleary [3].

**Lemma 4.2.** *Let  $n \geq 3$ . For  $\mathcal{H}_n$  take the generating set  $g_1, \dots, g_{n-1}$  with  $n - 1$  elements. Consider the following sets:*

- $B_k$  is the ball of radius  $k$ ,
- $C_k$  is the set of elements in  $\mathcal{H}_n$  which have complexity  $P = k$ ,
- $D_k \subset C_k$  is the set of elements of  $C_k$  which have word length at most  $k \log_{2n-2} k$

Then, we have that

$$\lim_{k \rightarrow \infty} \frac{|D_k|}{|C_k|} = 0$$

An element of complexity  $P$ , according to the metric estimates proved above, has word length between  $P$  and  $P \log P$ . What this lemma means is that most elements with complexity  $P$  will have word length closer to  $P \log P$  than to  $P$ .

*Proof.* Observe that

$$\frac{|D_k|}{|B_{k \log_{2^{n-2}} k}|} \leq 1$$

because it is a subset. Now, introduce the  $C_k$  as

$$\frac{|D_k|}{|C_k|} \frac{|C_k|}{|B_{k \log_{2^{n-2}} k}|}$$

and the proof will be complete if we show that

$$\lim_{k \rightarrow \infty} \frac{|C_k|}{|B_{k \log_{2^{n-2}} k}|} = \infty.$$

In  $C_k$  there are at least  $(nk - 2)!$  elements. This is because we can take a transposition involving a point at distance  $k$  down one of the rays, with another point. Since this already ensures  $P = k$ , we are free to choose any permutation of the other  $nk - 2$  points which are at one of the first  $k$  positions in each ray. And inside  $|B_{k \log_{2^{n-2}} k}|$ , counting grossly according to the number of generators, there are at most  $(2n - 2)^{k \log_{2^{n-2}} k} = k^k$  elements. Now the limit becomes

$$\lim_{k \rightarrow \infty} \frac{(nk - 2)!}{k^k}$$

which is easily seen that it approaches infinity using Stirling's formula and the fact that  $n \geq 3$ .  $\square$

Consequentially, these estimates give an easy way to see that the group has exponential growth. We note that exponential growth also follows easily from the fact that  $g_{01}$  and  $g_{02}$  generate a free subsemi-group.

**Proposition 4.3.** *Let  $n \geq 3$ . Then  $\mathcal{H}_n$  has exponential growth.*

*Proof.* Consider a finitary permutation of complexity  $P$ , and observe that there are at least  $P!$  of those. By the metric estimate, its word length is at most  $KP \log P$ . Using again as in the previous lemma the notation  $B_k$  for a ball, we will have that the group has exponential growth if

$$\lim_{k \rightarrow \infty} \frac{\log |B_k|}{k} > 0.$$

In our case, this amounts to

$$\lim_{P \rightarrow \infty} \frac{\log |B_{KP \log P}|}{KP \log P} \geq \lim_{P \rightarrow \infty} \frac{\log P!}{KP \log P} = \frac{1}{K}.$$

□

## 5. SUBGROUP EMBEDDINGS

We note that each  $\mathcal{H}_n$  is a subgroup of  $\mathcal{H}_m$  for  $n < m$  and that our estimates together with work of Lehnert are enough to give at least quadratic distortion for some of these embeddings.

**Theorem 5.1.** *The group  $\mathcal{H}_2$  is at least quadratically distorted in  $\mathcal{H}_m$  for  $m \geq 3$ .*

*Proof.* We consider the element  $\sigma_n$  of  $\mathcal{H}_2$  which has  $T(\sigma_n) = 0$  and transposes  $(0, k)$  and  $(1, k)$  for all  $k \leq n$ . Then  $\sigma_n$  corresponds to the word  $g_n$  defined in Theorem 8 of [9], where it is shown to have length of the order of  $n^2$  with respect to the generators of  $\mathcal{H}_2$  in Lemma 10 there, which are exactly the generators for  $\mathcal{H}_2$  given in the introduction. One easily checks that  $\sigma_n = g_{02}^n g_{12}^n g_{02}^{-n} g_{12}^{-n}$  in  $\mathcal{H}_3$ . Thus a family of words of quadratically growing length in  $\mathcal{H}_2$  has linearly growing length in  $\mathcal{H}_3$ , which proves the theorem. □

A natural, but seemingly difficult, question is whether  $\mathcal{H}_n$  is distorted in  $\mathcal{H}_m$  for  $3 \leq n < m$ . Another question, which also seems difficult, is to ask whether  $\mathcal{H}_n$  is distorted in Thompson's group  $V$ , under the embeddings mentioned in the introduction, [10].

## 6. SOME QUASI-ISOMETRIES OF $\mathcal{H}_n$

Commensurations give rise to quasi-isometries and are often a rich source of examples of quasi-isometries. Here we show that the natural map from the commensurator of  $\mathcal{H}_n$  to the quasi-isometry group of  $\mathcal{H}_n$ , which we denote by  $\text{QI}(\mathcal{H}_n)$ , is an injection. That is, we show that each commensuration is not within a bounded distance of the identity. That this is an injection also follows from the more general argument of Whyte which appears as Proposition 7.5 in Farb-Mosher [6].

**Theorem 6.1.** *The natural homomorphism from  $\text{Com}(\mathcal{H}_n)$  to  $\text{QI}(\mathcal{H}_n)$  is an embedding for  $n \geq 2$ .*

*Proof.* We will show that for each non-trivial  $\phi \in \text{Com}(\mathcal{H}_n)$  and every  $N \in \mathbb{N}$  we can find a  $\sigma \in \mathcal{H}_n$  such that  $d(\sigma, \sigma^\phi) \geq N$ , so none of the non-trivial images are within a bounded distance of the identity.

By Theorem 3.2, we can and will view  $\phi$  as a non-trivial element of  $N_p \subset \text{Sym}$  for some even  $p$ .

If  $\phi$  eventually translates a ray  $i$  non-trivially to a possibly different ray  $j$ , then we let  $\sigma = ((i, N) (i, N + 1))$ , a transposition in the translated ray. The image of  $\sigma$  under conjugation by  $\phi$  is the transposition  $((j, N + t), (j, N + t' + 1))$ , and the distance  $d(\sigma, \sigma^\phi)$  is the length of  $\sigma^{-1}\sigma^\phi$ , which is at least  $N$  since it moves at least one point at distance  $N$  down one of the rays.

If  $\phi$  does not eventually translate a ray but eventually non-trivially permutes ray  $i$  with another ray  $j$ , then we can show boundedness away from the identity by taking  $\sigma = ((j, N) (j, N + 1))$ . The point  $(i, N)$  is fixed by  $\sigma$  but is moved to  $(i, N + 1)$  under  $\sigma^\phi$  ensuring that the length of  $\sigma^{-1}\sigma^\phi$  is at least  $N$ .

Finally, if  $\phi$  does not have the preceding two properties, then  $\phi$  is a non-trivial finitary permutation. Since Houghton's group is  $k$ -transitive, for every  $k$ , we can find a  $\sigma \in \mathcal{H}_n$  such that  $\phi^\sigma$  has support disjoint from that of  $\phi$ , and at distance at least  $N$  down one of the rays. Hence  $\sigma^{-1}\phi^{-1}\sigma\phi = \sigma^{-1}\sigma^\phi$  has length at least  $N$ . □

## 7. CO-HOPFICITY

Houghton's groups are long known to be Hopfian although they are not residually finite, see [4]. In this section we will prove that  $\mathcal{H}_n$  is not co-Hopfian, by exhibiting a map which is injective but not surjective. The map is the following:

$$\begin{aligned} f: \mathcal{H}_n &\longrightarrow \mathcal{H}_n \\ s &\mapsto f(s) \end{aligned}$$

defined by: if  $(i, n)s = (j, m)$ , then:

$$(i, 2n - 1)f(s) = (j, 2m - 1) \quad \text{and} \quad (i, 2n)f(s) = (j, 2m).$$

It is straightforward to show that  $f$  is a homomorphism. It is injective, because if  $s$  is not the identity with  $(i, n)s \neq (i, n)$ , then  $(i, 2n)f(s) \neq (i, 2n)$ . And clearly the map is not surjective, because the permutation always sends adjacent points  $(i, 2n - 1)$ ,  $(i, 2n)$  to adjacent points, and a permutation which does not do this cannot be in the image.

In fact,  $\mathcal{H}_n$  has many proper subgroups isomorphic to the whole group. The following argument was pointed out to us by Peter Kropholler.

One can well-order the ray system by taking a lexicographic order. The group  $\mathcal{H}_n$  is then the group of all almost order preserving bijections of the well-ordered ray system. It is then clear that the ray system

minus a point is order isomorphic to the original ray system, which demonstrates that a point stabiliser is a subgroup isomorphic to  $\mathcal{H}_n$ .

**Theorem 7.1.** *Houghton's groups  $\mathcal{H}_n$  are not co-Hopfian.*

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