# DIFFERENTIABILITY WITH RESPECT TO PARAMETERS IN GLOBAL SMOOTH LINEARIZATION

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Abstract. Let $\mathbb{X}$ be a Banach space and $T = L + f$, where $L$ is linear and $f$ is nonlinear with $f(0) = 0$, a family of invertible contractions. We give conditions for the existence of a family of linearization maps $H_{(L,f)}$, such that $H_{(L,f)}TH^{-1}_{(L,f)} = L$, with a smooth dependence on $(L, f)$. The results depend strongly on the choice of some appropriate spaces of maps, adapted norms and the use of a specific fixed point Theorem with smooth dependence on parameters.

Keywords: Linearization, Conjugacy, Smooth dependence on parameters

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## 1. Introduction and statement of the results

Suppose $\mathbb{X}$ is a Banach space and $T : \mathbb{X} \to \mathbb{X}$ is a map such that $T(0) = 0$. The global linearization of $T$ consists of finding a linear operator $L$ (usually, the Fréchet derivative $DT(0)$) and a change of variables $H : \mathbb{X} \to \mathbb{X}$ such that $HTH^{-1}x = Lx$ for all $x \in \mathbb{X}$. Linearization is an important tool in Dynamical Systems Theory with applications to both Ordinary and Partial Differential Equations, with a long history and also important current

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work. When $H$ is required to be merely of class $\mathcal{C}^0$, linearization is associated with the classical works of P. Hartman and D.M. Grobman, extended to infinite dimensions by Ch. Pugh [5]

We focuss on results applicable when $\dim(\mathbb{X}) = \infty$, and in the present paper we study the case where $T$ depends on parameters $T = T_\theta$ in a smooth way, and we study the smoothness of the conjugation map $H = H_\theta$ with respect to these parameters. When looking at the conjugation equation $H_\theta T_\theta H_\theta^{-1} = L_\theta$, if one wants to take derivatives with respect to $\theta$ it appears as almost unavoidable that $H_\theta(x)$ has to be smooth with respect to $x$. So, smooth linearization, and not only $\mathcal{C}^0$ linearization, appears as the natural frame to study smooth dependence on parameters.

Smooth linearization in infinite dimensions is still an active area of research. We mention the recent important results by M.S. ElBialy [2] and by W. Zhang, W. Zhang and W. Jarczyk [10] on resonant fixed points and on sharp regularity estimates, respectively. We also address to our previous paper [9] for more references and some open problems. In our previous work [8] we studied the continuous dependence on parameters for $\mathcal{C}^0$ linearization, and our present paper is strongly influenced by these previous results.

In the present paper we restrict ourselves to what we believe is the simplest case: when $T_\theta = L_\theta + f_\theta$ is a family of invertible contractions and their linear parts $L_\theta$ satisfy the simplest non-resonance condition $\rho(L_\theta^{-1})\rho(L_\theta)^2 < 1$, where $\rho$ is the spectral radius.

Even in this simplest case, our results have not been very easy to prove. The most difficult part has been to identify two spaces $E$ and $F$ of maps, $E \subset F$ such that the conjugacies will exist on $E$, but will depend smoothly on parameters only as functions of $F$. The right choice of the norms of $E$ and $F$ has been crucial in our proofs, together with the concept of weighted uniform continuity for the functions of $E$ (see Definition 1 below), a delicate condition that resulrs into one of the keypoints of our approach.

To be as general as possible, if $T = L + f$ we have taken as parameters $\theta = (L, f)$.

We proceed now to state our main result, Theorem 1, and two auxiliary results, Theorems 2 and 3, that we believe that have some interest by themselves.

Given a Banach space $(\mathbb{X}, |\cdot|)$ we consider the following spaces $E_f \subset E \subset F$ of maps from $\mathbb{X}$ to $\mathbb{X}$:

$$F = \left\{ h : \mathbb{X} \to \mathbb{X} \text{ of class } \mathcal{C}^0 : |h|_F := \sup_{x \in \mathbb{X} \setminus \{0\}} \frac{|h(x)|}{|x|^2} < \infty \right\}.$$

Before defining the next spaces we present the following

**Definition 1.** Given two Banach spaces $\mathbb{X}$ and $\mathbb{X}_1$ we will say that $g : \mathbb{X} \to \mathbb{X}_1$ is *uniformly continuous with the weight* $|x|_{\mathbb{X}}^{-1}$ if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, g) > 0$, independent of $x, y \in \mathbb{X}$, such that $|g(x+y) - g(x)|_{\mathbb{X}_1} \leq \varepsilon |x|_{\mathbb{X}}$ whenever $|y| \leq \delta |x|_{\mathbb{X}}$.

This definition will play an essential role in the proof of our main result, Theorem 1. The reader can convince itself that this property is not so unusual by checking that a globally Lipschitz map $g : \mathbb{X} \to \mathbb{X}_1$ is uniformly continuous with the weight $|x|_{\mathbb{X}}^{-1}$ by taking $\delta = \varepsilon / Lip(g)$. Also, it can be seen that a function $g$ that is uniformly continuous on bounded sets of $\mathbb{X}$ is uniformly continuous with the weight $|x|_{\mathbb{X}}^{-1}$ if it is Fréchet differentiable at $x = 0$ and $x = \infty$.

$$E = \Big\{ h \in F \text{ of class } \mathcal{C}^1 : Dh \text{ is uniformly continuous with the weight } |x|_{\mathbb{X}}^{-1}$$

$$\text{as a map from } \mathbb{X} \text{ to } \mathcal{L}(\mathbb{X}) \text{ and } |h|_E := \sup_{x \in \mathbb{X} \setminus \{0\}} \frac{||Dh(x)||}{|x|} < \infty \Big\}$$

$$E_f = \left\{ f \in E : |f|_{E_f} := |f|_E + \sup_{x \in \mathbb{X} \setminus \{0\}} |Df(x)| < \infty \right\}.$$

The proof that these three spaces with the given norms are complete will be postponed to Lemma 4 in the next section.

The following is the main result of the present paper:

**Theorem 1.** (*Smoothness with respect to parameters in $\mathcal{C}^1$ linearization*). *Let $(\mathbb{X}, |\cdot|)$ be a Banach space and $\Lambda \subset \mathcal{L}(\mathbb{X})$ an open set of linear invertible contractions with the following properties: there exist $M > 0$, $\varepsilon > 0$ and a positive integer $N$, all independent of $L \in \Lambda$, such that*

$$||L||, ||L^{-1}|| < M \text{ and } ||L^{-N}|| \, ||L^N||^2 < 1 - \varepsilon \text{ for all } L \in \Lambda. \tag{1.1}$$

*We claim that in this situation if we take* $\mu < \frac{1}{2}\min\{M^{-1}, \varepsilon''/M^{1/2}\}$, *where* $\varepsilon'' = 1-(1-\varepsilon)^{1/(2N)}$, *and* $\Theta \subset \Lambda \times E_f$ *is an open set of pairs* $(L, f)$ *with the property that* $\sup_{x\in\mathbb{X}} ||Df|| < \mu$, *then the following properties hold:*

*i) For each* $(L, f) \in \Theta$ *there exists a global diffeomorphism* $H_{(L,f)}$ *such that* $H_{(L,f)}(L+f)H^{-1}_{(L,f)}(x) = Lx$ *for all* $x \in \mathbb{X}$. $H_{(L,f)}$ *is of the form* $H_{(L,f)} = I + h_{(L,f)}$, *where* $h_{(L,f)}$ *belongs to* $E$, *it is unique with these properties, and the following estimate holds:*

$$|h|_E \leq \frac{4}{3}\frac{Q^2 M}{\varepsilon''}|f|_E, \tag{1.2}$$

*where* $Q = (1+(N-1)(1+M^{N-1})^2)^2$. *Also, the map* $(L, f) \to h_{(L,f)} \in E$ *is continuous.*

*ii) Moreover, the map* $(L, f) \to h_{(L,f)}$ *can also be seen as a map from* $\Theta \subset \Lambda \times E_f$ *into the larger space* $F$ *where it turns to be of class* $\mathcal{C}^1$ *with respect to the arguments* $(L, f)$. *The derivative* $\partial_{(L,f)}h_{L,f}$ *in the direction of* $(L_1, f_1)$ *is given by the formula*

$$(\partial_{(L,f)}h)[L_1, f_1](x) = (I - S_1)^{-1}\cdot \tag{1.3}$$

$$\cdot\Big( -L^{-1}L_1L^{-1}(h(Lx+f(x))+f(x)) + L^{-1}Dh(Lx+f(x))(L_1x+f_1(x)) + L^{-1}f_1(x)\Big),$$

*where we write* $S_1 = S_{1,(L,f)}$ *for the operator* $h \to L^{-1}h(L+f)$ *from* $F$ *to* $F$, *that will turn out to be a contraction in a suitable norm depending on* $L$.

The idea of Theorem 1 is that the map $h \in E$ that defines the smooth conjugacy will depend continuously on the parameters $(L, f)$ as a map of $\mathcal{L}(\mathbb{X}) \times E_f \to E$, and smoothly on the same arguments as a map of $\mathcal{L}(\mathbb{X}) \times E_f \to F$. The proof will be presented in Section 3 below, and will be based in the next two results: a theorem on differentiability with respect to parameters of fixed points of contractions and a theorem of global smooth linearization. We believe that these two auxiliary results have interest by themselves and we state them next:

**Theorem 2.** *Let* $(G, |\cdot|_G), (\Sigma, |\cdot|_\Sigma)$ *be Banach spaces,* $\widetilde{G} \subset G$ *and* $A \subset \Sigma$ *be open sets and* $\overline{G} \subset \widetilde{G}$ *a closed set. Let* $S : \widetilde{G} \times A \to G$ *be a family of maps depending on the parameter* $\sigma \in A$. *Suppose that for each* $\sigma \in A$ , $S(\overline{G}, \sigma) \subset \overline{G}$ *and suppose there exists* $\rho \in [0, 1)$, *such that*

$$|S(g_1, \sigma) - S(g_2, \sigma)|_\sigma \leq \rho|g_1 - g_2|_\sigma \tag{1.4}$$

*for every* $g_1, g_2 \in \overline{G}$ *and every* $\sigma \in A$ *for some noms* $|\cdot|_\sigma$ *of* $G$ *depending on* $\sigma$ *in such a way that*

$$q|g|_G \leq |g|_\sigma \leq Q|g|_G, \tag{1.5}$$

*for every* $g \in G$ *and every* $\sigma \in A$*, for some positive constants* $q,\ Q$*.*

*These assumptions imply that for each* $\sigma \in A$ *there exists a unique fixed point* $g = g(\sigma) \in \overline{G}$ *of* $S(\cdot, \sigma)$*, and* $g(\sigma)$ *will be a continuous function of* $\sigma$ *if also for all* $\sigma_0$ *the function* $\sigma \mapsto S(g(\sigma_0), \sigma)$ *is continuous at* $\sigma = \sigma_0$ *(see Rodrigues and Solà-Morales, 2010 (*[8]*), Theorem 2).*

*Moreover, let* $G_A$ *be the smallest convex set containing* $\{g(\sigma) : \sigma \in A\}$ *and suppose* $G_A \subset \overline{G}$*. We claim that if*

*i)* $S(g, \sigma)$ *is differentiable with respect to its first variable at the points of* $G_A \times A$ *and* $\partial_g S(g, \sigma)$ *is continuous in both variables in* $G_A \times A$*,*

*ii) for each* $\sigma_0 \in A$*, the function* $A \ni \sigma \mapsto S(g(\sigma_0), \sigma)$ *is differentiable at* $\sigma = \sigma_0$*,*

*then the function* $g(\sigma)$ *is differentiable and*

$$g'(\sigma) = (I - \partial_g S(g(\sigma), \sigma))^{-1} \partial_\sigma S(g(\sigma), \sigma). \tag{1.6}$$

*Also, if* $\partial_\sigma S(g(\sigma), \sigma)$ *is a continuous function of* $\sigma$*, then* $g(\sigma)$ *is of class* $\mathcal{C}^1$*.*
*(See also Chow and Hale 1982* [1]*, Theorem 2.4, for a similar result, the main difference being that we deal with variable norms).*

**Theorem 3.** *(A global* $\mathcal{C}^1$ *linearization theorem) Let* $(\mathbb{X}, |\cdot|)$ *be a Banach space and* $L \in \mathcal{L}(\mathbb{X})$ *an invertible contraction such that*

$$||L||^2 ||L^{-1}|| < 1. \tag{1.7}$$

*Suppose that* $f \in E_f$*, that*

$$\sup_{x \in \mathbb{X}} ||Df(x)|| < \mu \tag{1.8}$$

*and define* $T = L + f$*.*

*(Observe that* $||L^{-1}||^{1/2} ||L|| < 1$ *because of* (1.7)*.) We claim that if*

$$\mu < \min\left\{ ||L^{-1}||^{-1}, \frac{1 - ||L^{-1}||^{1/2}||L||}{||L^{-1}||^{1/2}} \right\} \tag{1.9}$$

*then $T$ is a global invertible contraction of class $\mathcal{C}^1$ and there exists a global diffeomorphism $H$ of $\mathbb{X}$ of class $\mathcal{C}^1$ such that*

$$HTH^{-1}(x) = Lx \tag{1.10}$$

*for all $x \in \mathbb{X}$. The diffeomorphism $H$ will be of the form $H = I + h$, where $h$ belongs to the Banach space $E$ and $h$ is unique in this space. Moreover*

$$|h|_E \leq \frac{||L^{-1}||}{1 - ||L^{-1}||(||L|| + \mu)^2} |f|_E. \tag{1.11}$$

*(See also Mora and Solà-Morales 1987* [4] *for the local version of this result, and* [3] *for the classical local result in finite dimensions without non-resonance conditions ).*

## 2. Proofs of the auxiliary results

**Proof of Theorem 2:** We observe first that $\|\partial_g S(g,\sigma)\|_\sigma \leq \rho$ for every $(g,\sigma) \in G_A \times A$:

$$\frac{S(g + tg_1, \sigma_0) - S(g, \sigma_0)}{t} \to \partial_g S(g, \sigma_0) g_1$$

as $t \to 0$, so

$$\left| \frac{S(g + tg_1, \sigma_0) - S(g, \sigma_0)}{t} - \partial_g S(g, \sigma_0) g_1 \right|_\sigma = o(1),$$

$$|\partial_g S(g, \sigma_0) g_1|_\sigma \leq \left| \frac{S(g + tg_1, \sigma_0) - S(g, \sigma_0)}{t} \right|_\sigma + o(1) \leq \rho |g_1|_\sigma + o(1).$$

Then $\|\partial_g S(g,\sigma)\|_\sigma \leq \rho$.

Let us now try to find a candidate for the derivative of $g$. Suppose for a moment that $g$ is differentiable. Since

$S(g(\sigma), \sigma) = g(\sigma)$, we have that

$$\partial_g S(g(\sigma), \sigma) \cdot g'(\sigma) + \partial_\sigma S(g(\sigma), \sigma) = g'(\sigma).$$

Since $\|\partial_g S(g,\sigma)\|_\sigma \leq \rho < 1$ we obtain that

$$g'(\sigma) = (I - \partial_g S(g(\sigma), \sigma))^{-1} \partial_\sigma S(g(\sigma), \sigma).$$

Let $\sigma_0 \in A$ and $R(\sigma) := g(\sigma_0 + \sigma) - g(\sigma_0) - V\sigma$, where $V := [I - \partial_g S(g(\sigma_0), \sigma_0)]^{-1} \partial_\sigma S(g(\sigma_0), \sigma_0)$ and so $V = \partial_g S(g(\sigma_0), \sigma_0) V + \partial_\sigma S(g(\sigma_0), \sigma_0)$.

We have to prove that $|R(\sigma)| = o(|\sigma|)$, as $\sigma \to 0$. But we will prove that $|R(\sigma)|_{\sigma_0} = o(|\sigma|)$, as $\sigma \to 0$, that is an equivalent statement because of (1.5).

In fact,

$$R(\sigma) = S(g(\sigma_0+\sigma),\sigma_0+\sigma) - S(g(\sigma_0),\sigma_0) - \partial_g S(g(\sigma_0),\sigma_0)V\sigma - \partial_\sigma S(g(\sigma_0),\sigma_0)\sigma =$$

$$= S(g(\sigma_0+\sigma),\sigma_0+\sigma) - S(g(\sigma_0),\sigma_0+\sigma) - \partial_g S(g(\sigma_0),\sigma_0)V\sigma +$$

$$+ S(g(\sigma_0),\sigma_0+\sigma) - S(g(\sigma_0),\sigma_0) - \partial_\sigma S(g(\sigma_0),\sigma_0)\sigma =$$

$$= \int_0^1 \frac{d}{dt} S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma)\ dt - \partial_g S(g(\sigma_0),\sigma_0)V\sigma + o(|\sigma|) =$$

$$= \int_0^1 \partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma)\ [g(\sigma_0+\sigma)-g(\sigma_0)]\ dt - \partial_g S(g(\sigma_0),\sigma_0)V\sigma + o(|\sigma|) =$$

$$= \int_0^1 \partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma)\ [g(\sigma_0+\sigma)-g(\sigma_0)-V\sigma]\ dt +$$

$$\int_0^1 \partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma)\ V\sigma\ dt - \int_0^1 \partial_g S(g(\sigma_0),\sigma_0)V\sigma\ dt + o(|\sigma|) =$$

$$= \int_0^1 \partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma)\ R(\sigma)\ \ dt +$$

$$+ \int_0^1 [\partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma) - \partial_g S(g(\sigma_0),\sigma_0)]\ V\sigma\ dt + o(|\sigma|) =$$

$$= \int_0^1 \partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0),\sigma_0+\sigma)\ R(\sigma)\ dt + o(|\sigma|).$$

Then it follows that

$$|R(\sigma)|_{\sigma_0} \leq \int_0^1 \|\partial_g S(tg(\sigma_0+\sigma)+(1-t)g(\sigma_0)),\sigma_0+\sigma)\|_{\sigma_0}\ dt\ |R(\sigma)|_{\sigma_0} + o(|\sigma|) \leq$$

$$\leq \rho |R(\sigma)|_{\sigma_0} + o(|\sigma|).$$

Since $\rho < 1$ we conclude that $|R(\sigma)|_{\sigma_0}$ is $o(|\sigma|)$, as $\sigma \to 0$.

Finally, to see that $g'(\sigma)$ is continuous when $\partial_\sigma S(g(\sigma),\sigma)$ is continuous, is an immediate consequence of formula (1.6) and hypothesis i). ■

Before going into the proof of Theorem 3 let us prove the next

**Lemma 4.** *The three spaces $F, E, E_f$ defined in Section 1 are complete, so are Banach spaces with their respective norms. They are continuously embedded as $E_f \subset E \subset F$.*

**Proof:** It is straightforward to see that $F$ is complete.

For $h \in E$, since
$$|h(x)| \leq \int_0^1 |Dh(tx)||x| \; dt \leq \int_0^1 |h|_E t|x|^2 dt$$
it is clear that $E$ is continuously embedded in $F$ and $|h|_F \leq |h|_E$.

Let now $(h_n)$ be a Cauchy sequence in $E$ with its norm. Then, $Dh_n(x)$ is a Cauchy sequence in the sense that
$$\sup_{x\neq 0} \frac{|Dh_n(x) - Dh_m(s)|}{|x|} < \varepsilon$$
if $n$ and $m$ are larger than some $n_0(\varepsilon)$. So,
$$|Dh_n(x) - Dh_m(x)| \leq \varepsilon|x| \tag{2.1}$$
for all $x \in \mathbb{X}$. If we take $R > 0$ and restrict to $|x| \leq R$ we deduce from (2.1) that there exists a limit function $g$ such that $Dh_n \to g$ uniformly on bounded sets, by making $R \to \infty$.

So $g(x)$ is also continuous, and using that
$$\frac{|g(x)|}{|x|} \leq \frac{|g(x) - Dh_{n_0}(x)|}{|x|} + \frac{|Dh_{n_0}(x)|}{|x|}$$
we see that $\sup_{x\neq 0} \frac{|g(x)|}{|x|} < \infty$.

Using now that $h_n(x) = \int_0^1 Dh_n(tx)x \; dt$ one can also prove that $h_n$ converges in the norm of $E$ to the function $h(x) = \int_0^1 g(tx)x \; dt$.

So, $|h|_E < \infty$ and $|h - h_n|_E \to 0$, but we have to prove that $h \in E$. We have now to check its uniform continuity with the weight $|x|^{-1}$. Given $\varepsilon > 0$ we choose $n_0 = n_0(\varepsilon)$ sufficiently large, and then take $\delta = \delta(\varepsilon, Dh_{n_0})$. So,

$$|Dh(x+y)-Dh(x)| \leq |Dh(x+y)-Dh_{n_0}(x+y)|+|Dh_{n_0}(x+y)-Dh_{n_0}(x)|+|Dh_{n_0}(x)-Dh(x)|$$
$$\leq |h - h_{n_0}|_E|x + y| + \varepsilon|x| + |h - h_{n_0}|_E|x| \leq |h - h_{n_0}|_E(2 + \delta)|x| + \varepsilon|x|$$

provided that $|y| \leq \delta|x|$. This proves that $h \in E$ and concludes the proof of the completeness of $E$.

To prove that $E_f$ with the norm $|f|_{E_f} = |f|_E + \sup_{x\in\mathbb{X}} |Df(x)|$ is complete is a straightforward consequence of the fact that $E$ is complete and $\sup_{x\in\mathbb{X}} |Df(x)|$ is a seminorm. It is also clear that $E_f \subset E$ continuously. ■

**Proof of Theorem 3:** To see that $T = L + f : \mathbb{X} \to \mathbb{X}$ is a global invertible contraction of class $\mathcal{C}^1$ if (1.9) is satisfied, we observe first that $\mu$ is the Lipschitz constant of $f$ and that $T$ is an invertible contraction because (1.9) implies both $\mu < ||L^{-1}||^{-1}$ and $\mu < 1 - ||L||$, since $||L^{-1}|| > 1$ and so $\mu < ||L^{-1}||^{1/2}\mu < 1 - ||L^{-1}||^{1/2}||L|| < 1 - ||L||$. These two conditions imply that $L+f$ is one-to-one and onto, respectively. The fact that $T^{-1}$ is of class $\mathcal{C}^1$ follows from the Inverse Function Theorem.

We write now equation (1.10) as $(I + h)(L + f) = L(I + h)$ or $f + h(L + f) = Lh$ that allows us to seek for $h$ as a fixed point in

$$h = S(h) := L^{-1}h(L + f) + L^{-1}f. \tag{2.2}$$

Let us now show that $S$ is a contraction with respect to the norm $|\cdot|_E$. Let us write $S(h) =: S_1(h)+L^{-1}f$, and, since $S_1(h)$ is linear in $h$, it is enough to see that $|S_1(h)|_E \leq r|h|_E$ for some $r < 1$. We have $S_1(h) = L^{-1}h(L + f)$ and

$$DS_1(h)(x) = L^{-1}Dh(Lx + f(x))(L + Df(x)),$$

$$|S_1(h)|_E \leq \sup_{x\in\mathbb{X}\setminus\{0\}} \left\{ ||L^{-1}|| \frac{||Dh(Lx + f(x))||}{|Lx + f(x)|} \frac{|Lx + f(x)|}{|x|} ||L + Df(x)|| \right\} \leq$$

$$\leq ||L^{-1}||\ |h|_E\, (||L|| + \mathrm{Lip}(f)) \left( ||L|| + \sup_{x\in\mathbb{X}} ||Df(x)|| \right) \leq ||L^{-1}||(||L|| + \mu)^2|h|_E,$$

and it is clear, because of the hypothesis $||L||^2||L^{-1}|| < 1$, that $S$ is a contraction in $E$ if

$$\mu < \frac{1 - ||L^{-1}||^{1/2}||L||}{||L^{-1}||^{1/2}}.$$

To prove that $S(E) \subset E$ we recall first that $f \in E_f \subset E$, so $L^{-1}f \in E$. Let us now prove that $S_1(E) \subset E$.

We have to see that the map

$$\mathbb{X} \ni x \mapsto L^{-1}Dh(Lx+f(x))\cdot(L+Df(x)) \in \mathcal{L}(\mathbb{X})$$

is uniformly continuous with the weight $|x|^{-1}$ as in Definition 1:

$$||L^{-1}Dh(L(x+y)+f(x+y))\cdot(L+Df(x+y)) - L^{-1}Dh(Lx+f(x))\cdot(L+Df(x))||$$

$$\le ||L^{-1}||||Dh(L(x+y)+f(x+y)) - Dh(Lx+f(x))||||L+Df(x+y)||$$

$$+||L^{-1}||||Dh(Lx+f(x))||||L+Df(x+y)-L-Df(x)||$$

$$\le ||L^{-1}||\ \varepsilon\ ||Lx+f(x)||||L+Df(x+y)|| + ||L^{-1}||||Dh(Lx+f(x))||\ \varepsilon\ |x|$$

provided that

$$|Ly+f(x+y)-f(x)| \le \delta(\varepsilon,h)|Lx+f(x)| \tag{2.3}$$

for the first term, and $|y| \le \delta(\varepsilon, f)|x|$ for the second term.

To achieve (2.3) we note that

$$|Ly+f(x+y)-f(x)| \le (||L||+\mu)|y|$$

and also that

$$|Lx+f(x)| \ge \frac{||L^{-1}||}{||L^{-1}||}|Lx| - \mu|x| \ge \frac{|x|}{||L^{-1}||} - \mu|x| = \left(\frac{1}{||L^{-1}||} - \mu\right)|x| = \kappa|x|$$

for some $\kappa > 0$. So $|y| \le \frac{\kappa}{||L||+\mu}\delta(\varepsilon,h)|x|$ ensures that (2.3) holds.

Once we know that a unique fixed point $h$ of $S$ exists we want to prove that $I+h$ is a global diffeomorphism of $\mathbb{X}$ of class $\mathcal{C}^1$. Let us write again $T = L+f$ and $H = I+h$ and recall that $Dh(0) = 0$. Because of the Inverse Function Theorem there exist a small radius $r_0$ such that $H$ is a $\mathcal{C}^1$ diffeomorphism of $B_{r_0}(0)$ into its image $W_{r_0}$, that is also a neighbourhood of 0.

We have obtained that $HT(x) = LH(x)$ for all $x \in \mathbb{X}$. From this, one easily deduces that $HT^n(x) = L^nH(x)$, also for all $x \in \mathbb{X}$. We prove first that $H$ is one-to-one and onto. If $H(x_1) = H(x_2)$ then $L^nH(x_1) = L^nH(x_2)$, and so $HT^n(x_1) = HT^n(x_2)$, but for $n$ sufficiently large, since $T$ is a contraction and $T(0) = 0$, the points $T^n(x_1)$ and $T^n(x_2)$ reach the ball $B_{r_0}(0)$, and consequently $T^n(x_1) = T^n(x_2)$, and from the global invertibility of $T$ we deduce that $x_1 = x_2$.

To prove that $H$ is onto we want to see that an equation of the form $H(y) = x$ has a solution $y$ for each $x \in \mathbb{X}$. The global invertibility of $T$ implies that this equation is equivalent to $HT(z) = x$ or $LHz = x$, and, for the same reason for $L$, equivalent to $HT^{n+1}(z) = L^n x$. We can now pick $n$ sufficiently large such that $L^n x \in W_{r_0}$, and then we see that the equation is solvable. We have now seen that $H$ is one-to-one and onto.

This last equivalence shows us that we can write $y = H^{-1}(x) = T^{-n}H^{-1}L^n x$, where the $H^{-1}$ written in the right hand side is the function defined in $W_{r_0}$. This proves that $H^{-1}$ is of class $\mathcal{C}^1$ in a neighbourhood of $x$, and this concludes the proof that $H$ is a $\mathcal{C}^1$ diffeomorphism of $\mathbb{X}$. ■

**Remark 5.** *Observe that in terms of the spectral radius $\rho$ of the operators involved, when $\rho(L)^2\rho(L^{-1})^2 < 1$ then there exists an equivalent norm $|\cdot|_1$ in $\mathbb{X}$ such that the hypothesis $||L||_1^2||L^{-1}||_1 < 1$ is satisfied (see* [6]*).*

## 3. Proof of Theorem 1

The next lemma will be used subsequently in the proof of Theorem 1. It proves the existence of suitable norms adapted to given operators. This has always been done adapting the norm to the spectral radius of the operator $L$ (see our previous papers [6] or [7], for example) but here we need uniformity conditions when dealing with families of operators, and the spectral radius does not seem to be suitable for this purpose, because it does not depend continuously on $L$. So, we use $||L^N||^{1/N}$ for some uniform $N$, that is something that approaches the spectral radius if $N$ is large, but that keeps a better continuity property. Apart from that, the construction of the new norm is not very different from the classical construction.

**Lemma 6.** *Let $\mathbb{X}$ be a Banach space, $L \in \mathcal{L}(\mathbb{X})$ invertible, $||L||, ||L^{-1}|| \leq M$ and $N \in \mathbb{N}$.*

*i) We claim that for all $n \in \mathbb{N}$*

$$||L^n|| \leq A \ \left(||L^N||^{1/N}\right)^n ,$$

*and $A$ with $A = (1+M^{N-1})^2$.*

*ii) Suppose that $||L^N|| \leq B$, for some $B > 0$. We claim that there exists a new equivalent norm $|\cdot|'$ in $\mathbb{X}$ such that $|x| \leq |x|' \leq Q'|x|$ for all $x \in \mathbb{X}$, such that $||L||' \leq B^{1/N}$ and such that also if $L_0$ commutes with $L$, then $||L_0||' \leq ||L_0||$. In particular, $||L||' \leq ||L||$. Also, $Q'$ can be taken as $Q' = 1 + (N-1)A$, where $A$ is given in i).*

**Proof**: i) Let us write $n = kN + j$ for some $k \in \{0, 1, 2 \dots\}$ and some $j \in \{0, 1, \dots N-1\}$. Then

$$||L^n|| = ||L^{kN+j}|| \leq ||L^j|| \left(||L^N||^{1/N}\right)^{kN},$$

but also

$$1 \leq \left(||L^{-N}||\ ||L^N||\right)^{j/N} = ||L^{-N}||^{j/N}||L^N||^{j/N} \leq ||L^{-1}||^{N(j/N)}||L^N||^{j/N} = ||L^{-1}||^j||L^N||^{j/N},$$

so

$$||L^n|| \leq ||L^j||\ ||L^{-1}||^j||L^N||^{j/N} \left(||L^N||^{1/N}\right)^{kN} \leq$$
$$\leq \max\{1, ||L||, ||L||^2, \dots ||L||^{N-1}\} \max\{1, ||L^{-1}||, ||L^{-1}||^2, \dots ||L^{-1}||^{N-1}\} \left(||L^N||^{1/N}\right)^n \leq$$
$$\leq (1+M^{N-1})^2 \left(||L^N||^{1/N}\right)^n.$$

ii) Define

$$|x|' = |x| + B^{-1/N}|Lx| + B^{-2/N}|L^2x| + \dots B^{-(N-1)/N}|L^{N-1}x|.$$

Note that $|x| \leq |x|'$ and also that, by using the inequality obtained in part i) above

$$|x|' \leq \left(1 + B^{-1/N}AB^{1/N} + B^{-2/N}AB^{2/N} + \dots B^{-(N-1)/N}AB^{(N-1)/N}\right)|x| = (1+(N-1)A)\,|x|.$$

Besides that,

$$|Lx|' = B^{1/N}\left(B^{-1/N}|Lx| + B^{-2/N}|L^2x| + B^{-3/N}|L^3x| + \dots B^{-1}|L^Nx|\right) \leq B^{1/N}|x|',$$

since $||B^{-1}L^N|| \leq 1$.

And the inequality $||L||' \leq B^{1/N}$ follows.

The proof that $||L_0||' \leq ||L_0||$ when $L_0$ commutes with $L$ is straightforward. ■

**Proof of Theorem 1:** Let us show first, with the use of Lemma 6, that condition (1.1) implies that for each $L \in \Lambda$ there exists a norm $|\cdot|_L$ in $\mathbb{X}$ such that $||L^{-1}||_L||L||_L^2 < 1 - \varepsilon'$

for some $\varepsilon' > 0$, independent of $L \in \Lambda$ and a number $Q$, also independent of $L \in \Lambda$, such that $|x| \leq |x|_L \leq Q|x|$ for all $x \in \mathbb{X}$.

We apply first Lemma 6 to $L$ with $B = ||L^N||$ and obtain the existence of a first new norm $|\cdot|'_L$ such that $||L||'_L \leq B^{1/N} = ||L^N||^{1/N}$. Then, by considering $L^{-1}$, that commutes with $L$, obtain, with $B = ||L^{-N}||$ a second new norm, that we will indicate by $|\cdot|_L$, such that $||L^{-1}||_L \leq ||L^{-N}||^{1/N}$, that preserves the previous inequality $||L||_L \leq ||L^N||^{1/N}$.

So, we get

$$||L^{-1}||_L||L||_L^2 \leq ||L^{-N}||^{1/N}||L^N||^{2/N} = \left(||L^{-N}||||L^N||^2\right)^{1/N} < (1-\varepsilon)^{1/N} = 1-\varepsilon',$$

where $\varepsilon' = 1-(1-\varepsilon)^{1/N} > 0$.

In the two applications of Lemma 6 we have taken $A = (1+M^{N-1})^2$, so $Q' = 1+(N-1)(1+M^{N-1})^2$ each time, and after the two changes we get $Q = Q'^2$ and

$$|x| \leq |x|_L \leq \left(1+(N-1)((1+M^{N-1})^2\right)^2 |x| = Q|x|. \tag{3.1}$$

Then for the condition (1.9) to be satisfied in each of the norms $|\cdot|_L$ we observe first that $||L^{-1}||_L^{-1} \geq M^{-1}$, since $||L^{-1}||_L \leq ||L^{-1}|| \leq M$. Also, $||L^{-1}||_L^{1/2}||L||_L < (1-\varepsilon')^{1/2} = 1-\varepsilon''$, where $\varepsilon'' = 1-(1-\varepsilon')^{1/2} = 1-(1-\varepsilon)^{1/(2N)} > 0$. Then,

$$1-||L^{-1}||_L^{1/2}||L||_L > \varepsilon'' \tag{3.2}$$

and

$$\frac{1-||L^{-1}||_L^{1/2}||L||_L}{||L^{-1}||_L^{1/2}} \geq \frac{\varepsilon''}{M^{1/2}}.$$

So, summarizing, the condition

$$\mu < \min\{M^{-1}, \frac{\varepsilon''}{M^{1/2}}\}$$

depends only on $M, \varepsilon$ and $N$ and insures that (1.9) is satisfied in each of the norms $|\cdot|_L$ for all $L \in \Lambda$. To obtain the following estimates, we take from now on

$$\mu \leq \frac{1}{2}\min\{M^{-1}, \frac{\varepsilon''}{M^{1/2}}\}. \tag{3.3}$$

Part *i)* follows now from Theorem 3 in the norms $|\cdot|_L$. Let us see that the estimates (1.11) imply now the estimate (1.2), that does not depend on the auxiliary norm $|\cdot|_L$. We have that

$$\mu \leq \frac{1}{2}\frac{\varepsilon''}{M^{1/2}} \leq \frac{1-||L^{-1}||_L^{1/2}||L||_L - \frac{1}{2}\varepsilon''}{||L^{-1}||_L^{1/2}}$$

because of (3.2), and then

$$||L||_L + \mu \leq \frac{1-\frac{1}{2}\varepsilon''}{||L^{-1}||_L^{1/2}}$$

and

$$||L^{-1}||_L(||L||_L+\mu)^2 \leq (1-\frac{1}{2}\varepsilon'')^2 \leq 1-\frac{3}{4}\varepsilon'',$$

$$1-||L^{-1}||_L(||L||_L+\mu)^2 \geq \frac{3}{4}\varepsilon''$$

and finally

$$\frac{||L^{-1}||_L}{1-||L^{-1}||_L(||L||_L+\mu)^2} \leq \frac{4M}{3\varepsilon''}.$$

So, the inequality 1.11 now implies

$$|h|_{E(L)} \leq \frac{4M}{3\varepsilon''}|f|_{E(L)},$$

where the subindex means that the norm of $E$ is defined after the norm $|\cdot|_L$ of $\mathbb{X}$. Written in the original norm onde gets (1.2).

The map $h_{L,f}$ has been obtained as a fixed point of a contraction with parameters in a norm that depends on the values of these parameters. Since the contraction depends continuously on these parameters, a straightforward application of Theorem 2 of [8] gives that the map $(L,f)\mapsto h_{L,f}$ is continuous.

Let us see that the map $S(h,(L,f)) = L^{-1}h(L+f)+L^{-1}f$ as a map of $E\times\mathcal{L}(\mathbb{X})\times E_f \to F$ is differentiable with respect to $(L,f)$ at the point $(h_0,(L_0,f_0))$ when $h_0$ is precisely the fixed point: $h_0 = L_0^{-1}h_0(L_0+f_0)+L_0^{-1}f_0$. We can say that this is the central point of our proof, the place where the differentiability of $h_0(x)$ plays a key role.

The differentiability of $(L,f)\to L^{-1}f$ is a consequence of the differentiability of $L\mapsto L^{-1}$ in $\mathcal{L}(\mathbb{X})$ and the fact that $(L',f)\mapsto L'f$ is bilinear continuous from $\mathcal{L}(\mathbb{X})\times E_f\to F$.

To prove the Fréchet differentiability of $(L,f)\mapsto L^{-1}h_0(L+f)$ from $\mathcal{L}(\mathbb{X})\times E_f\to F$ when $h_0\in E$ we can concentrate on the hard part $(L,f)\mapsto h_0(L+f)$. The natural candidate for the differential of this map at the point $(L_0, f_0)$ when applied to the direction $(L_1, f_1)$ is $Dh_0(L_0x + f_0(x))(L_1x + f_1(x))$.

Let us start by seeing that $(L_1, f_1) \mapsto Dh_0(L_0x + f_0(x))(L_1x + f_1(x))$ is a bounded linear operator from $\mathcal{L}(\mathbb{X}) \times E_f$ to $F$:

$$\frac{|Dh_0(L_0x + f_0(x))(L_1x + f_1(x))|}{|x|^2} \leq |h_0|_E(||L_0|| + Lip(f_0))(||L_1|| + Lip(f_1))$$
$$\leq |h_0|_E(||L_0|| + |f_0|_{E_f})(||L_1|| + |f_1|_{E_f}).$$

Let us show that this natural candidate is the true Fréchet differential, that is to show that if we use the auxiliary function $g(x)$

$$g(x) := h_0(L_0x + f_0(x) + L_1x + f_1(x)) - h_0(L_0x + f_0(x)) - Dh_0(L_0x + f_0(x))(L_1x + f_1(x)),$$

then we have to prove that $|g|_F = o(||L_1|| + |f_1|_{E_f})$. We have for each $x \in \mathbb{X}$,

$$|g(x)| = |\int_0^1 \Big(Dh_0(L_0x + f_0(x) + tL_1x + tf_1(x))(L_1x + f_1(x))$$
$$-Dh_0(L_0x + f_0(x))(L_1x + f_1(x))\Big)\, dt|$$
$$\leq \int_0^1 ||Dh_0(L_0x + f_0(x) + tL_1x + tf_1(x)) - Dh_0(L_0x + f_0(x))||\ |L_1x + f_1(x)|\ dt.$$

The first expression under the integral sign has the form $||Dh_0(X + tY) - Dh_0(X)||$. Because of the weighted uniform continuity of $Dh_0$ when $h_0 \in E$ one has that this last expression is bounded by $\varepsilon|X|$ provided that $|tY| \leq \delta(\varepsilon)|X|$, so

$$|g(x)| \leq \varepsilon\ (|L_0x| + |f_0(x)|)|L_1x + f_1(x)| \leq \varepsilon\ (||L_0|| + Lip(f_0))(||L_1|| + Lip(f_1))|x|^2$$
$$\leq \varepsilon\ (||L_0|| + |f_0|_{E_f})(||L_1|| + |f_1|_{E_f})|x|^2,$$

provided that

$$|L_1x + f_1(x)| \leq \delta(\varepsilon)|L_0x + f_0(x)|. \tag{3.4}$$

We have to check that (3.4) holds when $||L_1||+|f_1|_{E_f}$ is sufficiently small. Remember from (3.3) that $Lip(f_0) \leq \mu \leq \frac{1}{2}M^{-1}$ and $||L_0^{-1}|| \leq M$. So $x \mapsto L_0x + f_0(x)$ is globally invertible and $|L_0x + f_0(x)| \geq \frac{||L_0^{-1}||}{||L_0^{-1}||}|L_0x| - \mu|x| \geq \frac{|x|}{||L_0^{-1}||} - \mu|x| \geq \left(\frac{1}{M} - \frac{1}{2M}\right)|x| = \frac{1}{2M}|x|$.

And so, to get (3.4) it is enough that $|L_1x + f_1(x)| \leq \delta(\varepsilon)(2M)^{-1}|x|$, and this is achieved if $(||L_1|| + ||f_1||_{E_f})$ is less than $\delta(\varepsilon)(2M)^{-1}$.

This proves that $|g|_F = o(||L_1|| + |f_1|_{E_f})$, as we wanted to prove.

The rest of the statement *ii)* follows now from the application of Theorem 2 to this situation. To see the validity of (1.3) it remains to see that $S_1 = S_{1,(L,f)}$ is a contraction also from $F$ to $F$. This will happen in the norm $|\cdot|_L$ of $\mathbb{X}$ and will ensure the invertibility of $I - S_1$:

$$\frac{|S_1 h_1(x) - S_1 h_2(x)|_L}{|x|_L^2} \leq ||L^{-1}||_L \frac{|h_1(Lx + f(x)) - h_2(Lx + f(x))|_L}{|Lx + f(x)|_L^2} \frac{|Lx + f(x)|_L^2}{|x|_L^2}$$

$$\leq ||L^{-1}||_L |h_1 - h_2|_F (||L||_L + Lip(f))^2,$$

where $Lip(f)$ is here considered in the $L$-norm.

But we already know that $||L^{-1}||_L(||L||_L + Lip(f))^2 < 1$ because condition (3.3) implies that (1.9) is satisfied in each of the norms $|\cdot|_L$. ■

## References


[1] S.N Chow and J.K. Hale *Methods of bifurcation theory,* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 251. Springer-Verlag, New York-Berlin, 1982.

[2] M.S. ElBialy, *Smooth conjugacy and linearization near resonant fixed points in Hilbert spaces*, Houston J. Math., **40**, 467-509 (2014).

[3] P. Hartman, *On local homeomorphisms of euclidean spaces,* Boletin de la Sociedad Matematica Mexicana, Volumen **5** Numero **2**, 220-241 (1960).

[4] X. Mora and J. Solà-Morales, *Existence and non-existence of finite-dimensional globally attracting invariant manifolds in semilinear damped wave equations,* Dynamics of infinite-dimensional systems (Lisbon, 1986), 187-210, NATO Adv. Sci. Inst. Ser. F: Comput. Systems Sci., 37, Springer, Berlin-New York, (1987).

[5] C. C. Pugh, *On a theorem of P. Hartman*, American J. of Math., **91**, 363-367 (1969).

[6] Rodrigues, H. M. and Solà-Morales, J., *Linearization of Class $C^1$ for Contractions on Banach Spaces*, J. Differential Equations **201** (2004), no. 2, 351-382.

[7] Rodrigues, H. M. and Solà-Morales, J., *A note on the relationship between spectral radius and norms of bounded linear operators*, Mat. Contemp. **36** (2009), 131-137.

[8] Rodrigues, H. M. and Solà-Morales, J., *On the Hartman-Grobman Theorem with Parameters*, J. Dynam. Differential Equations **22** (2010), 473-489.

[9] Rodrigues, H. M. and Solà-Morales, J., *Known Results and Open Problems on C1 linearization in Banach Spaces*, São Paulo Journal of Mathematical Sciences **6** (2012), 375-384.

[10] W. Zhang, W. Zhang and W. Jarczyc, *Sharp regularity of linearization for C1,1 hyperbolic diffeomorphisms*, Math. Ann., **358**, 69-113 (2014).