OFFPRINTS FROM GRAPH THEORY AND ITS APPLICATIONS TO ALGORITHMS AND COMPUTER SCIENCE
Edited by Y. Alavi et al.

DIGRAPHS WITH WALKS OF EQUAL LENGTH BETWEEN VERTICES

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ABSTRACT

This paper studies digraphs that have walks of equal length ℓ between vertices. When such a digraph models a communication network, this means that any message can be sent from its origin to its destination with precisely ℓ delay time units. It is shown that a digraph D has this property unless it is a generalized cycle.

When D has the maximum possible order there is just one such walk between vertices. Among such digraphs are the Good-de Bruijn's. Other families are constructed by adequately modifying these digraphs and using some well-known constructions: line digraphs and conjunction of digraphs.

1. Equi-reachable digraphs

Let $D = (V,A)$ be a strongly connected digraph, and for any vertex x call $\Gamma(x)$ the set of vertices adjacent from x. Analogously call $\Gamma(W)$ the set of vertices adjacent from the vertices of $W \subset V$, and $\Gamma^n(W) = \Gamma(\Gamma^{n-1}(W))$. The digraph D has walks of equal length m between vertices iff for all $x \in V$

$$\Gamma^m(x) = V \tag{1}$$

If ℓ is the smallest such m we say that D is ℓ-reachable. For convenience, we shall use the term equi-reachable for digraphs that are ℓ-reachable for some ℓ, that is for digraphs with walks of equal length between vertices.

In fact, for D to be equi-reachable it suffices that (1) holds for some $x \in V$, since then

$$\Gamma^{m+1}(x) = \Gamma(\Gamma^m(x)) = \Gamma(V) = V$$

because D is strongly connected. Thus $\Gamma^n(x) = V$ for all $n \geq m$ and then, if D has diameter k, there are walks through x of length $m + k$ between any two vertices of D. Therefore D is ℓ-reachable for some $\ell \leq m + k$.

Trivially, (1) can not hold when D is a cycle or a bipartite digraph. More generally, if D is a generalized cycle in the sense that V is the disjoint union of $r > 1$ subsets,

$$V = \{ \bigcup V_i,\ 0 \leq i \leq r-1 \},\quad V_i \cap V_j = \emptyset \ \text{ for } i \neq j \tag{2}$$

and, for i modulo r

$$\Gamma(V_i) = V_{i+1},\quad 0 \leq i \leq r-1, \tag{3}$$

it can not be equi-reachable. The following result shows that this is necessarily the structure of such digraphs.

Theorem 1.- A strongly connected digraph D is equi-reachable unless it is a generalized cycle.

Proof.- Suppose that D is not equi-reachable and consider for $x \in V$ the sequence

$$x,\ \Gamma(x),\ \Gamma^2(x),\ \ldots,\ \Gamma^n(x),\ \ldots \tag{4}$$

of nontrivial subsets of 2^V. Since necessarily repetitions will occur, let $\Gamma^{m+r}(x) = \Gamma^m(x)$ be the first one. Then

$$\Gamma^{m+r+t}(x) = \Gamma^t(\Gamma^{m+r}(x)) = \Gamma^t(\Gamma^m(x)) = \Gamma^{m+t}(x)$$

so that for $n \geqslant m$ the sets

$$V_i = \Gamma^{m+i}(x) \qquad 0 \leqslant i \leqslant r-1 \tag{5}$$

recur periodically in the above sequence. As D is strongly connected any $y \in V$ must appear in the periodic part of the sequence. Therefore

$$V = \{\bigcup V_i,\ \ 0 \leqslant i \leqslant r-1\}$$

and in particular $r > 1$, since $V_i = \Gamma^{m+i}(x) \neq V$.

From its construction and the periodicity the sets V_i satisfy (3). Therefore to prove that D is a generalized cycle it suffices to show that they are disjoint. Suppose on the contrary that there exists $y \in V_i \cap V_j$, $i \neq j$, and let $h = d(y,x)$, so that $x \in \Gamma^h(y)$. Then

$$x \in \Gamma^h(y) \Rightarrow \Gamma^m(x) \subset \Gamma^{m+h}(y)$$