DIGRAPHS WITH WALKS OF EQUAL LENGTH BETWEEN VERTICES

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ABSTRACT

This paper studies digraphs that have walks of equal length \( \ell \) between vertices. When such a digraph models a communication network, this means that any message can be sent from its origin to its destination with precisely \( \ell \) delay time units. It is shown that a digraph \( D \) has this property unless it is a generalized cycle.

When \( D \) has the maximum possible order there is just one such walk between vertices. Among such digraphs are the Good-de Bruijn's. Other families are constructed by adequately modifying these digraphs and using some well-known constructions: line digraphs and conjunction of digraphs.
1. Equi-reachable digraphs

Let $D = (V, A)$ be a strongly connected digraph, and for any vertex $x$ call $\Gamma(x)$ the set of vertices adjacent from $x$. Analogously call $\Gamma(W)$ the set of vertices adjacent from the vertices of $W \subseteq V$, and $\Gamma^{n}(W) = \Gamma(\Gamma^{n-1}(W))$. The digraph $D$ has walks of equal length $m$ between vertices iff for all $x \in V$

$$\Gamma^{m}(x) = V$$

(1)

If $l$ is the smallest such $m$ we say that $D$ is $l$-reachable. For convenience, we shall use the term equi-reachable for digraphs that are $l$-reachable for some $l$, that is for digraphs with walks of equal length between vertices.

In fact, for $D$ to be equi-reachable it suffices that (1) holds for some $x \in V$, since then

$$\Gamma^{m+1}(x) = \Gamma(\Gamma^{m}(x)) = \Gamma(V) = V$$

because $D$ is strongly connected. Thus $\Gamma^{n}(x) = V$ for all $n \geq m$ and then, if $D$ has diameter $k$, there are walks through $x$ of length $m + k$ between any two vertices of $D$. Therefore $D$ is $l$-reachable for some $l \leq m + k$.

Trivially, (1) can not hold when $D$ is a cycle or a bipartite digraph. More generally, if $D$ is a generalized cycle in the sense that $V$ is the disjoint union of $r > 1$ subsets,

$$V = \bigcup_{i=0}^{r-1} V_i, \quad V_i \cap V_j = \emptyset \text{ for } i \neq j$$

(2)

and, for $i$ modulo $r$

$$\Gamma(V_i) = V_{i+1}^{i}, \quad 0 \leq i \leq r-1,$$  

(3)
it can not be equi-reachable. The following result shows that this
is necessarily the structure of such digraphs.

**Theorem 1.** A strongly connected digraph $D$ is equi-reachable un-
less it is a generalized cycle.

**Proof.** Suppose that $D$ is not equi-reachable and consider for $x \in V$
the sequence

$$x, \Gamma(x), \Gamma^2(x), \ldots, \Gamma^n(x), \ldots$$

(4)

of nontrivial subsets of $2^V$. Since necessarily repetitions will
occur, let $\Gamma^m(x) = \Gamma^m(x)$ be the first one. Then

$$\Gamma^{m+r+t}(x) = \Gamma^t(\Gamma^m(x)) = \Gamma^t(\Gamma^m(x)) = \Gamma^{m+t}(x)$$

so that for $n \geq m$ the sets

$$V_i = \Gamma^{m+i}(x) \quad 0 \leq i \leq r-1$$

(5)

recur periodically in the above sequence. As $D$ is strongly con-
connected any $y \in V$ must appear in the periodic part of the sequence.

Therefore

$$V = \{ U V_i, \quad 0 \leq i \leq r-1 \}$$

and in particular $r > 1$, since $V_i = \Gamma^{m+i}(x) \neq V$.

From its construction and the periodicity the sets $V_i$ satisfy
(3). Therefore to prove that $D$ is a generalized cycle it suffices
to show that they are disjoint. Suppose on the contrary that there
exists $y \in V_i \cap V_j$, $i \neq j$, and let $h = d(y, x)$, so that $x \in \Gamma^h(y)$.

Then

$$x \in \Gamma^h(y) \Rightarrow \Gamma^m(x) \subseteq \Gamma^{m+h}(y)$$
\[ y \in V_i \Rightarrow \Gamma^{m+h}(y) \subseteq \Gamma^{m+h+m+i}(x) = \Gamma^{m+t_1}(x) \]
\[ y \in V_j \Rightarrow \Gamma^{m+h}(y) \subseteq \Gamma^{m+h+m+j}(x) = \Gamma^{m+t_2}(x) \]

where \( 0 \leq t_1, t_2 \leq r-1 \) and \( t_1 \neq t_2 \) because \( i \neq j \). So \( \Gamma^m(x) \subseteq \Gamma^{m+t}(x) \) for some \( 1 \leq t \leq r-1 \). But this implies \( \Gamma^{m+t}(x) \subseteq \Gamma^{m+2t}(x) \) and then
\[ \Gamma^m(x) \subseteq \Gamma^{m+t}(x) \subseteq \ldots \subseteq \Gamma^{m+rt}(x) = \Gamma^m(x) \]

so that \( \Gamma^m(x) = \Gamma^{m+t}(x) \) with \( 1 \leq t \leq r-1 \) against the choice for \( r \).

**Remarks**

1.- When the digraph \( D \) is the Cayley diagram of a (finite) group \( G \), the above decomposition of \( V \) corresponds to the partitioning of \( G \) into cosets given by the normal subgroup \( H \) of those elements that can be expressed in terms of the generators in such a way that the sum of exponents equals zero. Then \( r \) equals the greatest common divisor of the sums of exponents in the defining relators.

2.- We can characterize \( \ell \)-reachable digraphs as those whose adjacency matrix \( A \) is such that \( A^\ell \) is positive (that is, \( (A^\ell)_{ij} > 0 \) for all \( i, j \)), and therefore by the fact that its spectral radius \( \rho(A) \) is a simple eigenvalue—greater in magnitude than any other eigenvalue, see, for instance, \( |1| \).

2. Some constructions

We examine in this section the behaviour of equi-reachable and non-equireachable digraphs under some well-known graph constructions. It is evident that a digraph \( D \) is \( \ell \)-reachable if and
only if its converse digraph (i.e., the digraph obtained by reversing the orientation of the arcs in D) is \( l \)-reachable. More important are the results on the line digraph of D and the conjunction of two digraphs \( D_1 \) and \( D_2 \).

2.1. The line digraph

In the line digraph \( L(D) \) of a digraph \( D = (V,A) \) each vertex represents an arc of \( D \), that is

\[
V(L(D)) = \{uv \mid [u,v] \in A(D)\},
\]

and two vertices are adjacent when the corresponding arcs are adjacent in \( D \). Its order is the size (i.e., number of arcs) of \( D \). Then if \( D \) is strongly regular of degree \( d \), the order of \( L(D) \) is \( d \) times the order of \( D \).

The main result in our context is

Theorem 2.- A digraph is \( l \)-reachable if and only if its line digraph is \((l+1)\)-reachable.

Proof.- If \( D \) is \( l \)-reachable there is a walk of length \( l+1 \) between any two vertices \( uv, wz \) of \( L(D) \) that uses the walk of length \( l \) from \( v \) to \( w \) in \( D \). And the argument can be reversed.

Using Theorem 1 it follows that \( D \) is a generalized cycle if and only if \( L(D) \) is a generalized cycle. In fact, in this case both decompose into the same number \( r \) of subsets.

2.2. Conjunction of digraphs

The conjunction \( D_1 \ast D_2 \) of two digraphs \( D_1 = (V_1,A_1) \) and \( D_2 = (V_2,A_2) \) is the digraph with set of vertices \( V = V_1 \times V_2 \) and adjacency rule

\[
[(x,y),(z,t)] \in A \iff [x,z] \in A_1 \text{ and } [y,t] \in A_2
\]

(7)
It follows that its order is the product of the orders of $D_1$ and $D_2$ and its maximum out-degree the corresponding product of out-degrees, and that $D_1 \ast D_2$ is strongly regular if and only if both $D_1$ and $D_2$ are strongly regular.

When $D_1$ is $\ell_1$-reachable and $D_2$ is $\ell_2$-reachable their conjunction is $\ell$-reachable with $\ell = \max \{\ell_1, \ell_2\}$, (use $\ell$ length walks from $x$ to $z$ in $D_1$ and from $y$ to $t$ in $D_2$ to construct a length $\ell$ walk from $(x, y)$ to $(z, t)$ in $D_1 \ast D_2$). Analogously it is easily seen that if one or both digraphs are not equi-reachable nor is their conjunction.

3. Largest digraphs

If $D$ is $\ell$-reachable and has maximum out-degree $d$ its order is at most $N = d^\ell$, since this is the maximum number of different walks of length $\ell$. To attain this bound there should be just one walk of length $\ell$ between any two vertices. It follows that the adjacency matrix $A$ of $D$ must verify the matrix equation $A^\ell = J$, and therefore $D$ ought to be strongly regular of degree $d$ (i.e., with in- and out-degree of all vertices equal to $d$), see $|4|$. Note also that these digraphs must be geodetic (i.e., with just one shortest path between any two vertices).

The $\ell$-reachable digraphs with $d^\ell$ vertices have been studied by N.S. Mendelsohn in $|6|$ as "UPP digraphs" (digraphs with the unique path property of order $\ell$), and by Conway and Guy, unaware of the work of Mendelsohn, in $|2|$ as "tight precisely $\ell$-steps digraphs", using them to construct large transitive digraphs of given diameter. We use here Mendelsohn's terminology.

Among the UPP digraphs are the well-known Good-de Bruijn digraphs, whose set of vertices consist of all length $\ell$ words from an alphabet of $d$ letters, and with a vertex $x$ adjacent to $y$ if the last $\ell-1$ letters of $x$ coincide with the first $\ell-1$ letters of $y$. 
But they are not the only UPP digraphs. For instance, for \( d=3 \) and \( \ell=2 \) Mendelsohn presents in \(|6|\) five other non-isomorphic such digraphs that can be seen as models of groupoids. More generally, UPP digraphs can be seen as models of a universal algebra, see \(|7|\).

We describe below two direct methods of constructing digraphs of this kind for any \( d \geq 3 \) and \( \ell \geq 2 \) by adequately modifying the Good-de Bruijn digraphs, that when \( d=3 \) and \( \ell=2 \) produce the five above-mentioned digraphs.

Since, from Theorem 2, the line digraph of an \( \ell \)-reachable digraph of order \( N = d^\ell \) is a \((\ell+1)\)-reachable digraph of order \( dN = d^{\ell+1} \), it suffices to construct \( 2 \)-reachable digraphs with order \( N = d^2 \) and then their iterated line-digraphs.

Consider then the Good-de Bruijn digraph of all length 2 sequences of \( d \) digits, \( d \geq 3 \), that is

\[ V = \{x_0x_1, \ x_1 \in X\}, \quad X = \{0,1,\ldots,d-1\} \]

and \( x_0x_1 \) is adjacent to \( x_1x_2 \) for all choices of \( x_2 \in X \).

For the first method, let \( a_0, a_1, \ldots, a_{d-1} \) be \( d \) (not necessarily different) permutations of \( X \) such that \( a_j(0) = 0 \), and modify the Good-de Bruijn digraph by replacing each arc \([0i,ij]\) with the arc \([0i,a_j(i)j]\) for \( i,j = 0,1,\ldots,d-1 \) (in fact for \( i=0 \) there is no alteration). To see that the resulting digraph is still \( 2 \)-reachable (hence strongly regular) it suffices to consider the following walks from \( x_0x_1 \) to \( y_0y_1 \):

1. If \( x_0 \neq 0, x_1 \neq 0 \)
   \[ x_0x_1 \to x_1y_0 \to y_0y_1. \]
2. If \( x_0 = 0, x_1 \neq 0 \), setting \( j = y_0 \)
   \[ 0x_1 \to a_j(x_1)j \to jy_1 = y_0y_1 \]
   since \( x_1 \neq 0 \) implies \( a_j(x_1) \neq 0 \).
(3) If \( x_1 = 0 \), setting \( j = y_1, i = a_{j}^{-1}(y_0) \)
\[
x_00 \to 0i \to a_{j}(i) j = y_0y_1.
\]

Alternatively, consider 2 permutations of \( X, \alpha \) and \( \beta \), such that \( \alpha(1)=1 \) and \( \beta(0)=0 \), and modify the Good-de Bruijn digraph by replacing each arc \([0i,11]\) with the arc \([0i,\alpha(i)1]\) for all \( i \), and also each arc \([1i,00]\) with the arc \([1i,\beta(i)0]\) for all \( i \). The new digraph is still 2-reachable as it can be seen by adequately adapting the previous reasoning. For example the length 2 walk from \( x_00 \) to \( y_01 \), \( x_0 \neq 0 \), go through \( 0y_0 \) if \( y_1 \neq 1 \) and through \( 0\alpha^{-1}(y_0) \) if \( y_1 = 1 \).

When \( d=3 \) the choices of \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) as

\[
\begin{align*}
\alpha_0 &= (1,2) & \alpha_1 &= 1 & \alpha_2 &= 1 & (D_1) \\
\alpha_0 &= 1 & \alpha_1 &= (1,2) & \alpha_2 &= 1 & (D_2)
\end{align*}
\]

for the first method, and the choices of \( \alpha \) and \( \beta \) as

\[
\begin{align*}
\alpha &= (0,2) & \beta &= (1,2) & (D_3)
\end{align*}
\]

for the second, yield three non-isomorphic 2-reachable digraphs.

The other two, besides Good-de Bruijn's, are the converse digraphs of \( D_2 \) and \( D_3 \).

The digraphs \( D_2 \) and \( D_3 \) are non-planar, so they are counterexamples to the conjecture of Mendelsohn in \(|5| \) and \(|6| \).

The above constructions require \( d \geq 3 \). For \( d=2 \) and \( k \geq 3 \) analogous techniques yield non-isomorphic UPP digraphs. For instance if \( \ell=3 \), besides the Good-de Bruijn digraph, there are two other 3-reachable digraphs. One is shown in Figure 1 and the other one is its converse digraph.
This digraph is useful to verify a question raised by Mendelsohn in [6]. For given values of $l$ and $d$, he calculates the number of elementary circuits of length $n \leq l+1$ in the Good-de Bruijn digraphs. He shows that all UPP digraphs have the same number of elementary circuits of length $n \leq l$ and wonders if this result still holds for $n > l(n \leq d^l)$. However, already for $d=2$, $l=3$ and $n=5$ the Good-de Bruijn digraph has 2 elementary circuits while the digraph of Figure 1 has 3. Moreover, their line digraphs may be used to verify that even for $n=l+1$ the result does not hold.

REFERENCES


