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Variational principles and symmetries on fibered multisymplectic manifolds

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Abstract. The standard techniques of variational calculus are geometrically stated in the ambient of fiber bundles endowed with a (pre)multisymplectic structure. Then, for the corresponding variational equations, conserved quantities (or, what is equivalent, conservation laws), symmetries, Cartan (Noether) symmetries, gauge symmetries and different versions of Noether's theorem are studied in this ambient. In this way, this constitutes a general geometric framework for all these topics that includes, as special cases, first and higher order field theories and (non-autonomous) mechanics.

1 Introduction

As it is well known, the most of field equations of first and higher-order classical field theories and mechanics are locally variational; that is, they can be obtained starting from a variational principle. The phase spaces for all these theories have a similar geometric structure: they are fiber bundles $\kappa\colon \mathcal{M} \to M$ over an orientable manifold M (of dimension equal to 1 for mechanical systems, and greater than 1 for field theories), which are endowed with a multisymplectic or a pre-multisymplectic form (depending on the *regularity* of the theory).

The aim of this review work is to state a generic geometric framework which allows us to include these variational principles for all these kinds of theories in a single formulation (this is done in Section 2, after establishing some previous geometric and mathematical background). The variational equations, which are stated using multivector fields, include the Euler-Lagrange as well as the Hamilton

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In memory of Prof. G. Sardanashvily. He was a good colleage and a great contributor to this field.

equations. Then, we use this unified framework to study different kinds of symmetries for these equations, their conserved quantities (i.e., conservation laws), and giving general versions of Noether's theorem (in Section 3).

From this framework we can recover, as particular cases, the variational principles and several topics on the theory of symmetries and conserved quantities for classical field theories and (non-autonomous) mechanics of first and higher order, both in the Lagrangian and Hamiltonian formulations. In particular, let $\pi\colon E \to M$ be a fiber bundle. Then, If $\mathcal{M} \equiv J^k\pi$ (the kth-order jet bundle of π), and $\Omega \equiv \Omega_{\mathcal{L}}$ (the *Poincaré-Cartan form* associated to a Lagrangian density $\mathcal{L}$), we recover the classical *Hamilton variational Principle* and results on symmetries, conservation laws and Noether's theorem for classical Lagrangian field theories of first order (if $k = 1$) and higher-order (if $k > 1$) [1], [4], [7], [9], [12], [13], [14], [15], [16], [18], [19], [20], [23], [25]. Taking the suitable *multimomentum bundles* as $\mathcal{M}$, and the associated Hamiltonian counterparts of $\Omega_{\mathcal{L}}$, we recover the corresponding *Hamilton-Jacobi variational Principle* and symmetries for the Hamiltonian formalism of first and higher-order field theories [2], [7], [11], [8], [17]. Finally, if in the above situations we take $M = \mathbb{R}$, we obtain the analogous results for the Lagrangian and Hamiltonian formalisms of first and higher-order non-autonomous mechanics [3], [6], [21], [22], [24].

All the manifolds are real, second countable and C^∞. The maps and the structures are C^∞. Sum over repeated indices is understood.

2 Variational principle for multisymplectic systems

2.1 Multivector fields

(See [10] for details). Let $\mathcal{M}$ be a n-dimensional differentiable manifold.

Definition 1. Sections of $\Lambda^m \mathrm{T}\mathcal{M}$ are called *m-multivector fields* in $\mathcal{M}$; that is, they are the contravariant skew-symmetric tensors of order m in $\mathcal{M}$. The set of m-multivector fields in $\mathcal{M}$ is denoted as $\mathfrak{X}^m(\mathcal{M})$.

For every $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ and $p \in \mathcal{M}$, there exists a neighbourhood $U_p \subset \mathcal{M}$ and $X_1, \ldots, X_r \in \mathfrak{X}(U_p)$ such that

$$\mathbf{X}|_{U_p} = \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} X_{i_1} \wedge \ldots \wedge X_{i_m} ,$$

with $f^{i_1 \ldots i_m} \in C^\infty(U_p)$, $m \leqslant r \leqslant \dim \mathcal{M}$.

The classical operations with vector fields in differentiable manifolds can be extended to multivector fields.

Definition 2. Let $\Omega \in \Omega^k(\mathcal{M})$ be a differentiable k-form in $\mathcal{M}$ and let $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$; the *contraction* between $\mathbf{X}$ and Ω is defined as

$$\begin{aligned} i(\mathbf{X})\Omega \mid_{U_p} &:= \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} i(X_1 \wedge \ldots \wedge X_m)\Omega \\ &= \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} i(X_1) \ldots i(X_m)\Omega \end{aligned}$$

if $k \geq m$, and equal to zero if $k < m$. For $1 \leq j \leq k-1$, the k-form Ω is *j-nondegenerate* if, for every $p \in E$ and $\mathbf{X} \in \mathfrak{X}^j(\mathcal{M})$, we have that

$$i(\mathbf{X}_p)\Omega_p = 0 \Leftrightarrow \mathbf{X}_p = 0\,.$$

The *Lie derivative* with respect to $\mathbf{X}$ is defined as the graded bracket

$$[\mathrm{d}, i(\mathbf{X})] = \mathrm{d}i(\mathbf{X}) - (-1)^m i(\mathbf{X})\mathrm{d} := \mathrm{L}(\mathbf{X})$$

and it is an operation of degree $m-1$.

If $\mathbf{Y} \in \mathfrak{X}^i(\mathcal{M})$ and $\mathbf{X} \in \mathfrak{X}^j(\mathcal{M})$, another operation of degree $i+j-2$ is the *Schouten-Nijenhuis bracket* of $\mathbf{X}, \mathbf{Y}$, which is the bilinear assignment $\mathbf{Y}, \mathbf{X} \mapsto [\mathbf{Y}, \mathbf{X}]$, where $[\mathbf{Y}, \mathbf{X}]$ is a $(i+j-1)$-multivector field obtained as the graded commutator of $\mathrm{L}(\mathbf{Y})$ and $\mathrm{L}(\mathbf{X})$;that is,

$$\mathrm{L}([\mathbf{Y}, \mathbf{X}]) := [\mathrm{L}(\mathbf{Y}), \mathrm{L}(\mathbf{X})] = \mathrm{L}(\mathbf{Y})\mathrm{L}(\mathbf{X}) - (-1)^{i+j}\mathrm{L}(\mathbf{X})\mathrm{L}(\mathbf{Y})\,.$$

The following properties hold: for $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, multivector fields of degrees i, j, k, respectively, we have that:

1. $[\mathbf{X}, \mathbf{Y}] = -(-1)^{(i+1)(j+1)}[\mathbf{Y}, \mathbf{X}]$.
2. $[\mathbf{X}, \mathbf{Y} \wedge \mathbf{Z}] = [\mathbf{X}, \mathbf{Y}] \wedge \mathbf{Z} + (-1)^{(i+1)j}\mathbf{Y} \wedge [\mathbf{X}, \mathbf{Z}]$.
3. $(-1)^{(i+1)(k+1)}[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + (-1)^{(j+1)(i+1)}[\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]]$ $+(-1)^{(k+1)(j+1)}[\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0$.
4. For every $X \in \mathfrak{X}(\mathcal{M})$, $i([X, \mathbf{Y}])\Omega = \mathrm{L}(X)i(\mathbf{Y})\Omega - i(\mathbf{Y})\mathrm{L}(X)\Omega$.

Definition 3. An m-multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *locally decomposable* if, for every $p \in \mathcal{M}$, there exists an open neighbourhood $U_p \subset \mathcal{M}$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$.

An m-dimensional distribution $D \subset T\mathcal{M}$ is *locally associated* with a non-vanishing m-multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ if there exists a connected open set $U \subseteq \mathcal{M}$ such that $\mathbf{X}|_U$ is a section of $\Lambda^m D|_U$. If $\mathbf{X}, \mathbf{X}' \in \mathfrak{X}^m(\mathcal{M})$ are non-vanishing m-multivector fields locally associated with the same distribution D, on the same set U, then there exists a non-vanishing function $f \in C^\infty(U)$ such that $\mathbf{X}'|_U = f\mathbf{X}|_U$. This defines an equivalence relation in the set of non-vanishing m-multivector fields in $\mathcal{M}$, whose equivalence classes are denoted by $\{\mathbf{X}\}_U$. Therefore, there is a one-to-one correspondence between the set of m-dimensional orientable distributions D in $T\mathcal{M}$ and the set of the equivalence classes $\{\mathbf{X}\}_\mathcal{M}$ of non-vanishing, locally decomposable m-multivector fields in $\mathcal{M}$. If $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is non-vanishing and locally decomposable, and $U \subseteq \mathcal{M}$, we denote by $\mathcal{D}_U(\mathbf{X})$ (or simply $\mathcal{D}(\mathbf{X})$, if $U = \mathcal{M}$) the distribution associated with the class $\{\mathbf{X}\}_U$.

Definition 4. A non-vanishing, locally decomposable multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is *integrable* or *involutive* if its associated distribution $\mathcal{D}_U(\mathbf{X})$ is integrable or involutive.

Obviously, if $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable or involutive, then so is every other in its equivalence class $\{\mathbf{X}\}$, and all of them have the same integral manifolds.

We are especially interested in the case where $\kappa\colon \mathcal{M} \to M$ is a fiber bundle and M is an m-dimensional orientable manifold with volume form $\eta \in \Omega^m(M)$.

Definition 5. A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is *κ-transverse* if, for every $\beta \in \Omega^m(M)$ with $\beta(\kappa(y)) \neq 0$, at every point $y \in \mathcal{M}$, we have that $(i(\mathbf{X})(\kappa^*\beta))_y \neq 0$. If $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, then it is κ-transverse if, and only if, its integral manifolds are local sections of $\kappa\colon \mathcal{M} \to M$. In this case, if $\psi\colon U \subset M \to \mathcal{M}$ is a local section with $\psi(x) = y$ and $\psi(U)$ is the integral manifold of $\mathbf{X}$ at y, then $T_y(\operatorname{Im}\psi) = \mathcal{D}_y(\mathbf{X})$ and ψ is said to be an *integral section* of $\mathbf{X}$.

Furthermore, there exists a unique m-multivector field $\mathbf{Y}_\eta\colon M \to \Lambda^m \mathrm{T}\, M$, such that $i(\mathbf{Y}_\eta)\eta = 1$; then the *canonical prolongation* of a section $\psi\colon U \subset M \to \mathcal{M}$ to $\Lambda^m\mathrm{T}\mathcal{M}$ is the section $\Lambda^m\psi\colon U \subset M \to \Lambda^m\mathrm{T}\mathcal{M}$ defined as $\Lambda^m\psi := \Lambda^m\mathrm{T}\psi \circ \mathbf{Y}_\eta$; where $\Lambda^m\mathrm{T}\psi\colon \Lambda^m\mathrm{T}M \to \Lambda^m\mathrm{T}\mathcal{M}$ is the natural extension of ψ to the corresponding multitangent bundles. Then, ψ is an integral section of $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ if, and only if,

$$\mathbf{X} \circ \psi = \Lambda^m\psi \,. \tag{1}$$

2.2 (Pre)multisymplectic systems

Let $\kappa\colon \mathcal{M} \to M$ be a fibred manifold which in what follows is assumed to be a fibre bundle, where $\dim M = m \geq 1$ and $\dim \mathcal{M} = n + m$, and M is an orientable manifold with volume form $\eta \in \Omega^m(M)$. We denote $\omega = \kappa^*\eta$. We write $(U; x^\mu, y^j)$, $\mu = 1, \ldots, m$, $j = 1, \ldots, n$, for local charts of coordinates in $\mathcal{M}$ adapted to the fibred structure, and such that $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$. We denote by $\mathfrak{X}^{V(\kappa)}(\mathcal{M})$ the set of κ-vertical vector fields in $\mathcal{M}$ (which is locally generated by $\left\{\dfrac{\partial}{\partial y^j}\right\}$).

Definition 6. A form $\Omega \in \Omega^{m+1}(\mathcal{M})$ $(m \geq 1)$ is a *(pre)multisymplectic form* if it is closed and 1-nondegenerate, that is, if the map $\flat_\Omega\colon \mathrm{T}\mathcal{M} \to \Lambda^m\mathrm{T}^*\mathcal{M}$, defined by $\flat_\Omega(x, v) = (x, i(v)\Omega_x)$, for every $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$, is injective. In this case, the system described by the triad (F, Ω, ω) is called a *multisymplectic system*. Otherwise, the form is said to be a *premultisymplectic form*, and the system is *premultisymplectic*. Finally, a multisymplectic form is *exact* if there exist $\Theta \in \Omega^m(\mathcal{M})$ such that $\Omega = -\mathrm{d}\Theta$.

From now on, we will assume this last condition (this does not represent a loss of generality since, by Poincaré Lemma, every closed form is locally exact).

Furthermore, if $m \geq 2$, we assume that the following condition holds:

$$i(Z_1)i(Z_2)i(Z_3)\Omega = 0\,, \text{ for every } Z_1, Z_2, Z_3 \in \mathfrak{X}^{\mathrm{V}(\kappa)}(\mathcal{M})\,,$$

which is justified because this is the situation in the Lagrangian and Hamiltonian formalism of field theories. This condition means that, in a chart of adapted coordinates, we have that

$$\Omega|_U = \mathrm{d}F^\mu_j \wedge \mathrm{d}y^j \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}E \wedge \mathrm{d}^m x\,, \tag{2}$$

where $\mathrm{d}^{m-1}x_\mu = i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^m x$, and $F^\mu_j(x^\nu, y^i), E(x^\nu, y^i) \in \mathrm{C}^\infty(U)$.

2.3 Generalized variational principle and field equations

Let $\Gamma(\kappa)$ be the set of sections of κ. Consider the following functional (where the convergence of the integral is assumed)

$$\begin{aligned}\mathcal{F}\colon \Gamma(\kappa) &\to \mathbb{R}\\ \psi &\mapsto \int_M \psi^*\Theta\end{aligned}$$

Definition 7 (Generalized Variational Principle). The *generalized variational problem* for the (pre)multisymplectic system $(\mathcal{M},\Omega,\omega)$ is the search for the critical (local) sections of the functional $\mathcal{F}$ with respect to the variations of ψ given by $\psi_s=\sigma_s\circ\psi$, where $\{\sigma_s\}$ is a local one-parameter group of any compact-supported κ-vertical vector field Z in $\mathcal{M}$; that is,

$$\frac{d}{ds}\Big|_{s=0}\int_M \psi_s^*\Theta=0\,.$$

Theorem 1. *The following assertions on a section $\psi\in\Gamma(\kappa)$ are equivalent:*

1. *ψ is a solution to the generalized variational problem.*
2. *ψ is a section solution to the equation*
$$\psi^* i(Y)\Omega=0\,,\quad \textit{for every } Y\in\mathfrak{X}(\mathcal{M})\,. \tag{3}$$
3. *ψ is a section solution to the equation*
$$i(\Lambda^m\psi)(\Omega\circ\psi)=0\,. \tag{4}$$
4. *ψ is an integral section of a m-multivector field contained in a class of κ--transverse and integrable (and hence locally decomposable) m-multivector fields, $\{\mathbf{X}\}\subset\mathfrak{X}^m(\mathcal{M})$, satisfying the equation*
$$i(\mathbf{X})\Omega=0\,. \tag{5}$$

Proof. (The proof follows the patterns in [8] and [14]).

$(1\iff 2)$ Let $Z\in\mathfrak{X}^{V(\kappa)}(\mathcal{M})$ be a compact-supported vector field, and $U\subset M$ an open set such that ∂U is a $(m-1)$-dimensional manifold and $\kappa(\mathrm{supp}(Z))\subset U$. Then

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\int_M\psi_s^*\Theta &= \frac{d}{ds}\Big|_{s=0}\int_U\psi_s^*\Theta=\frac{d}{ds}\Big|_{s=0}\int_U\psi^*\sigma_s^*\Theta\\ &=\int_U\psi^*\left(\lim_{t\to 0}\frac{\sigma_s^*\Theta-\Theta}{t}\right)=\int_U\psi^*\mathrm{L}(Z)\Theta\\ &=\int_U\psi^*(i(Z)\mathrm{d}\Theta+\mathrm{d}i(Z)\Theta)=\int_U\psi^*(-i(Z)\Omega+\mathrm{d}i(Z)\Theta)\\ &=-\int_U\psi^*i(Z)\Omega+\int_U\mathrm{d}(\psi^*i(Z)\Theta)\\ &=-\int_U\psi^*i(Z)\Omega+\int_{\partial U}\psi^*i(Z)\Theta=-\int_U\psi^*i(Z)\Omega\,,\end{aligned}$$

as a consequence of Stoke's theorem and the assumptions made on the supports of the vertical vector fields. Thus, we conclude

$$\left.\frac{d}{ds}\right|_{s=0} \int_M \psi_s^*\Theta = 0 \quad \Longleftrightarrow \quad \psi^* i(Z)\Omega = 0\,,$$

for every compact-supported $Z \in \mathfrak{X}^{V(\kappa)}(\mathcal{M})$. However, since the compact-supported vector fields generate locally the $\mathrm{C}^\infty(\mathcal{M})$-module of vector fields in $\mathcal{M}$, it follows that the last equality holds for every κ-vertical vector field Z in $\mathcal{M}$. Now, recall that for every point $p \in \operatorname{Im}\psi$, we have a canonical splitting of the tangent space of $\mathcal{M}$ at p in a κ-vertical subspace and a κ-horizontal subspace,

$$\mathrm{T}_p\mathcal{M} = V_p(\kappa) \oplus \mathrm{T}_p(\operatorname{Im}\psi)\,.$$

Then, if $Y \in \mathfrak{X}(\mathcal{M})$ we have

$$Y_p = (Y_p - \mathrm{T}_p(\psi\circ\kappa)(Y_p)) + \mathrm{T}_p(\psi\circ\kappa)(Y_p) \equiv Y_p^V + Y_p^\psi\,,$$

with $Y_p^V \in V_p(\kappa)$ and $Y_p^\psi \in \mathrm{T}_p(\operatorname{Im}\psi)$. Therefore

$$\psi^* i(Y)\Omega = \psi^* i(Y^V)\Omega + \psi^* i(Y^\psi)\Omega = \psi^* i(Y^\psi)\Omega\,,$$

since $\psi^* i(Y^V)\Omega = 0$, by the conclusion in the above paragraph. Now, as $Y_p^\psi \in \mathrm{T}_p(\operatorname{Im}\psi)$ for every $p \in \operatorname{Im}\psi$, then the vector field Y^ψ is tangent to $\operatorname{Im}\psi$, and hence there exists a vector field $X \in \mathfrak{X}(M)$ such that X is ψ-related with Y^ψ; that is, $\psi_* X = \left.Y^\psi\right|_{\operatorname{Im}\psi}$. Then $\psi^* i(Y^\psi)\Omega = i(X)\psi^*\Omega$. However, as $\dim\operatorname{Im}\psi = \dim M = m$ and Ω is an $(m+1)$-form, we obtain that $\psi^*\Omega = 0$ and hence $\psi^* i(Y^\psi)\Omega = 0$. Therefore, we conclude that the equation (3) holds.

Taking into account the reasoning of the first paragraph, the converse is obvious since the equation (3) holds for every $Y \in \mathfrak{X}(\mathcal{M})$ and, in particular, for every $Z \in \mathfrak{X}^{V(\kappa)}(\mathcal{M})$.

($2 \Longleftrightarrow 3$) In a chart of adapted coordinates $(U; x^\mu, y^j)$, for every $Y \in \mathfrak{X}(\mathcal{M})$ and for every $\psi \in \Gamma(\kappa)$ and $x \in M$, we have that

$$Y = f^\mu \frac{\partial}{\partial x^\mu} + g^j \frac{\partial}{\partial y^j}\,, \qquad \psi(x) = (x^\mu, \psi^j(x))$$

and

$$\Lambda^m \psi = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + \frac{\partial \psi^j}{\partial x^\mu}\frac{\partial}{\partial y^j} \right)\,.$$

Therefore, taking (2) into account, a simple calculation shows that equations (3) and (4) lead to the same expressions:

$$0 = \frac{\partial F_j^\mu}{\partial x^\mu}\frac{\partial \psi^j}{\partial x^\nu} + \frac{\partial F_j^\mu}{\partial y^i}\left(\frac{\partial \psi^i}{\partial x^\nu}\frac{\partial \psi^j}{\partial x^\mu} + \frac{\partial \psi^i}{\partial x^\mu}\frac{\partial \psi^j}{\partial x^\nu} \right)\,,$$

$$0 = \frac{\partial F_j^\mu}{\partial x^\mu} + \frac{\partial F_i^\mu}{\partial y^j}\frac{\partial \psi^i}{\partial x^\mu} - \frac{\partial F_j^\mu}{\partial y^i}\frac{\partial \psi^i}{\partial x^\mu} + \frac{\partial E}{\partial y^j}\,.$$

$(3 \iff 4)$ If $\psi\colon U \subset M \to \mathcal{M}$ is a solution to (4) then, for every $x \in U$ there exists a neigbourhood $U_x \subset U$ of x such that $\psi(U_x) \subset \psi(U)$. As $\psi|_{U_x}$ is an injective immersion (since ψ is a section and hence its image is an embedded submanifold), the map $\Lambda^m(\psi|_{U_x})$ defines a locally decomposable m-multivector field $\mathbf{X}^x$ in $\psi(U_x) \subset M$, which is tangent to $\psi(U_x)$ and has $\psi|_{U_x}$ as an integral section in U_x. Thus, as a consequence of (1), if equation (4) holds for $\psi|_{U_x}$, then (5) holds for $\mathbf{X}^x$, in U_x.

Conversely, if ψ is an integral section of an m-multivector field $\mathbf{X}$ in $U \subset M$, then (1) holds, and if (5) holds for $\mathbf{X}$, then (4) holds for ψ, in U. □

Remark 1. The equation (5), with the κ-transverse condition, can be written

$$i(\mathbf{X})\Omega = 0; \quad i(\mathbf{X})\omega \neq 0 \,. \tag{6}$$

Then, it is usual to fix the κ-transverse condition by taking a representative in the class $\{\mathbf{X}\}$ such that

$$i(\mathbf{X})\omega = 1 \,.$$

As it is usual, $\ker^m \Omega := \{\mathbf{X} \in \mathfrak{X}^m(\mathcal{M}) \,|\, i(\mathbf{X})\Omega = 0\}$. We denote by $\ker^m_\omega \Omega \subset \mathfrak{X}^m(\mathcal{M})$ the set of m-multivector fields satisfying equations (6), but not being locally decomposable necessarily. Then $\ker^m_{\omega(ld)} \Omega \subset \mathfrak{X}^m(\mathcal{M})$ and $\ker^m_{\omega(I)} \Omega \subset \mathfrak{X}^m(\mathcal{M})$ denote the sets of m-multivector fields satisfying equations (6) which are locally decomposable and integrable, respectively. Obviously we have that

$$\ker^m_{\omega(I)} \Omega \subset \ker^m_{\omega(ld)} \Omega \subset \ker^m_\omega \Omega \subset \ker^m \Omega \,. \tag{7}$$

Note: In general, if $(\mathcal{M}, \Omega, \omega)$ is a premultisymplectic system, then κ-transverse and integrable m-multivector fields $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ which are solutions to (6) could not exist. In the best of cases they exist only in some submanifold $j_{\mathcal{S}}\colon \mathcal{S} \hookrightarrow \mathcal{M}$ [5]. In this case, in the sets of (7) and in the following sections, we have to consider only multivector fields and vector fields which are tangent to $\mathcal{S}$.

3 Symmetries and conservation laws for multisymplectic systems

3.1 Conserved quantities and conservation laws

Next we recover the idea of *conservation law* or *conserved quantity*, and state Noether's theorem for (pre)multisymplectic systems. In this sense, a part of our discussion is a generalization of the results obtained for non-autonomous mechanical systems and field theories (see [6], [7], [11], [24], and references therein).

Definition 8. A *conserved quantity* of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$ is a form $\xi \in \Omega^{m-1}(\mathcal{M})$ such that $\mathrm{L}(\mathbf{X})\xi = 0$, for every $\mathbf{X} \in \ker^m_\omega \Omega$.

Observe that, in this case, $\mathrm{L}(\mathbf{X})\xi = (-1)^{m+1} i(\mathbf{X})\, \mathrm{d}\xi$.

Proposition 1. *If $\xi \in \Omega^{m-1}(\mathcal{M})$ is a first integral of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$, and $\mathbf{X} \in \ker^m_{\omega(I)} \Omega$, then ξ is closed on the integral submanifolds of $\mathbf{X}$; that is, if $j_S\colon S \hookrightarrow \mathcal{M}$ is an integral submanifold of $\mathbf{X}$, then $\mathrm{d}j_S^*\xi = 0$.*

Proof. Let $X_1, \ldots, X_m \in \mathfrak{X}(\mathcal{M})$ be independent vector fields tangent to the (m-dimensional) integral submanifold S. Then $\mathbf{X} = f X_1 \wedge \ldots \wedge X_m$, for some $f \in \mathrm{C}^\infty(\mathcal{M})$. Therefore, as $i(\mathbf{X})\,\mathrm{d}\xi = 0$, we have

$$j_S^*[\mathrm{d}\xi(X_1, \ldots, X_m)] = j_S^* i(X_1 \wedge \ldots \wedge X_m)\mathrm{d}\xi = 0\,.$$

□

Theorem 2. *A form $\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$ if, and only if, $\mathrm{L}(\mathbf{Z})\xi = 0$, for every $\mathbf{Z} \in \ker^m \Omega$.*

Proof. Let ξ be a conserved quantity. If $\mathbf{X}_0 \in \ker_\omega^m \Omega$ is a particular solution to the equations (6) then

$$\ker_\omega^m \Omega = \{ f\mathbf{X}_0 + \ker^m \Omega \cap \ker^m \omega ; f \in \mathrm{C}^\infty(\mathcal{M})\}\,.$$

Therefore, for every $\mathbf{Z} \in \ker^m \Omega \cap \ker^m \omega$, we have that $\mathbf{Z} = \mathbf{X}_1 - \mathbf{X}_2$, with $\mathbf{X}_1, \mathbf{X}_2 \in \ker_\omega^m \Omega$ such that $i(\mathbf{X}_1)\omega = i(\mathbf{X}_2)\omega$. Hence, if ξ is a conserved quantity, we have that $\mathrm{L}(\mathbf{Z})\xi = 0$. Furthermore, taking $\mathbf{X}_0 \in \ker_\omega^m \Omega$ with $i(\mathbf{X}_0)\omega = 1$, for every $\mathbf{Z} \in \ker^m \Omega$ we can write $\mathbf{Z} = (\mathbf{Z} - i(\mathbf{Z})\omega\mathbf{X}_0) + i(\mathbf{Z})\omega\mathbf{X}_0$ and it follows that $\mathbf{Z} - i(\mathbf{Z})\omega\mathbf{X}_0 \in \ker^m \Omega \cap \ker^m \omega$; therefore $\mathrm{L}(\mathbf{Z} - i(\mathbf{Z})\omega\mathbf{X}_0)\xi = 0$ and thus

$$\mathrm{L}(\mathbf{Z})\xi = \mathrm{L}(\mathbf{Z} - i(\mathbf{Z})\omega\mathbf{X}_0)\xi + \mathrm{L}(i(\mathbf{Z})\omega\mathbf{X}_0)\xi = (-1)^{m+1} i(\mathbf{Z})\omega i(\mathbf{X}_0)\mathrm{d}\xi = 0\,,$$

since $\mathrm{d}i(\mathbf{X}_0)\xi = 0$, because $\xi \in \Omega^{m-1}(\mathcal{M})$.

The converse is immediate. □

Now, given $\xi \in \Omega^{m-1}(\mathcal{M})$ and $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$, for every integral submanifold $\psi \colon M \to \mathcal{M}$ of $\mathbf{X}$, we can construct the form $\psi^*\xi \in \Omega^{m-1}(M)$. Then, using the volume form $\eta \in \Omega^m(M)$, we can obtain a unique $X_{\psi^*\xi} \in \mathfrak{X}(M)$ such that

$$i(X_{\psi^*\xi})\eta = \psi^*\xi\,,$$

(in the standard terminology, $\psi^*\xi$ is the so-called *form of flux* associated with the vector field $X_{\psi^*\xi}$). Then:

Proposition 2. *If $\mathrm{div} X_{\psi^*\xi}$ denotes the divergence of $X_{\psi^*\xi}$, we have that*

$$(\mathrm{div} X_{\psi^*\xi})\,\eta = \mathrm{d}\psi^*\xi\,.$$

Proof. In fact, $\mathrm{d}\psi^*\xi = i(X_{\psi^*\xi})\eta = \mathrm{L}(X_{\psi^*\xi})\eta = (\mathrm{div} X_{\psi^*\xi})\,\eta$. □

As a consequence of Proposition 1, this result allows to associate a *conservation law* in M to every conserved quantity in $\mathcal{M}$. In fact:

Proposition 3. *$\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity if, and only if, $\mathrm{div} X_{\psi^*\xi} = 0$, for every integral submanifold $\psi \colon M \to \mathcal{M}$ of $\mathbf{X}$. Therefore, by Stokes theorem, in every bounded domain $U \subset M$, we have*

$$\int_{\partial U} \psi^*\xi = \int_U (\mathrm{div} X_{\psi^*\xi})\,\eta = \int_U \mathrm{d}\psi^*\xi = 0\,.$$

The form $\psi^*\xi$ is called the current *associated with the conserved quantity ξ.*

3.2 Symmetries

Definition 9. 1. A *symmetry* of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$ is a diffeomorphism $\Phi\colon \mathcal{M} \to \mathcal{M}$ such that $\Phi_*(\ker^m \Omega) \subset \ker^m \Omega$.

2. An *infinitesimal symmetry* of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$ is a vector field $Y \in \mathfrak{X}(\mathcal{M})$ whose local flows are local symmetries; that is, if F_t is a local flow of Y, then $F_{t*}(\ker^m \Omega) \subset \ker^m \Omega$, in the corresponding open sets.

Another characterization of infinitesimal symmetries is the following:

Theorem 3. *Let $(\mathcal{M}, \Omega, \omega)$ be a (pre)multisymplectic system, $Y \in \mathfrak{X}(\mathcal{M})$. Then Y is an infinitesimal symmetry if, and only if,*

$$[Y, \ker^m \Omega] \subset \ker^m \Omega\,.$$

Proof. As $\ker^m \Omega$ is locally finite-generated, we can take a local basis $\mathbf{Z}_1, \ldots, \mathbf{Z}_r$ of $\ker^m \Omega$. Then, if $[Y, \ker^m \Omega] \subset \ker^m \Omega$, the assertion is equivalent to proving that, if F_t is a local flow of Y, then $[Y, \mathbf{Z}_i] = f_i^j \mathbf{Z}_j$ if, and only if, $F_{t*}\mathbf{Z}_i = g_i^j \mathbf{Z}_j$ (for every $i = 1, \ldots, r$), where g_i^j are functions defined on the corresponding open set, also depending on t.

First, it is clear that, if $F_{t*}\mathbf{Z}_i = g_i^j \mathbf{Z}_j$, then $[Y, \mathbf{Z}_i] = f_i^j \mathbf{Z}_j$.

For the converse, suppose that $[Y, \mathbf{Z}_i] = f_i^j \mathbf{Z}_j$, and consider and extended local basis to the whole $\mathfrak{X}^m(\mathcal{M})$: $\{\mathbf{Z_1}, \ldots, \mathbf{Z_r}, \mathbf{Z'_{r+1}}, \ldots, \mathbf{Z'_c}\}$, where c is the dimension of $\mathfrak{X}^m(\mathcal{M})$. Remember that $\left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} F_{t*}\mathbf{Z}_i = F_{s*}[Y, \mathbf{Z}_i]$. Hence, on the one hand we obtain

$$\begin{aligned} F_{s*}[Y, \mathbf{Z}_i] &= F_{s*}(f_i^j \mathbf{Z}_j) = (F_s^{-1})^* f_i^j F_{s*}\mathbf{Z}_j \\ &= (F_s^{-1})^* f_i^j (g_j^k \mathbf{Z}_k) + \sum_{k=r+1}^{c} (F_s^{-1})^* f_i^j (g_j^k \mathbf{Z'}_k)\,, \end{aligned}$$

and on the other hand, we have that

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} F_{t*}\mathbf{Z}_i &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} g_i^k \mathbf{Z}_k + \sum_{k=r+1}^{c} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} g_i^k \mathbf{Z'}_k \\ &= \left.\frac{\mathrm{d}g_i^k}{\mathrm{d}t}\right|_{t=s} \mathbf{Z}_k + \sum_{k=r+1}^{c} \left.\frac{\mathrm{d}g_i^k}{\mathrm{d}t}\right|_{t=s} \mathbf{Z'}_k\,. \end{aligned}$$

Therefore, comparing these expressions, we conclude that $\dfrac{\mathrm{d}g_i^k}{\mathrm{d}t} = (F_t^{-1})^* f_i^j g_j^k$, for $k = 1, \ldots, c$. This is a system of ordinary linear differential equations for the functions g_i^k. With the initial condition $g_i^k(0) = \delta_i^k$ for $k \leq r$ and $g_i^k(0) = 0$ for $k > r$, has a unique solution, defined for every t on the domain of F_t. Then, taking this solution, we have proved the existence of functions g_i^j such that $F_{t*}\mathbf{Z}_i = g_i^j \mathbf{Z}_j$, and the result holds. □

Bearing in mind the properties of multivector fields we obtain the basic properties:

- If $Y_1, Y_2 \in \mathfrak{X}(\mathcal{M})$ are infinitesimal symmetries, then so is $[Y_1, Y_2]$.
- If $Y \in \mathfrak{X}(\mathcal{M})$ is an infinitesimal symmetry and Ω is a premultisymplectic form, for every $Z \in \ker \Omega$, then $Y + Z$ is also an infinitesimal symmetry.

The classical interpretation that a symmetry of a system of differential equations transforms solutions into solutions is recovered from the following result:

Theorem 4. *Let $\Phi \in \mathrm{Diff}(\mathcal{M})$ be a symmetry of a (pre)multisymplectic system.*

1. *If $\mathbf{X} \in \ker^m \Omega$ is an integrable multivector field, then Φ transforms integral submanifolds of $\mathbf{X}$ into integral submanifolds of $\Phi_* \mathbf{X}$.*
2. *In particular, if $\Phi \in \mathrm{Diff}(\mathcal{M})$ restricts to a diffeormorphism $\varphi \colon M \to M$ (that is, $\varphi \circ \kappa = \kappa \circ \Phi$), then, for every $\mathbf{X} \in \ker^m_{\omega(I)} \Omega$, Φ transforms integral submanifolds of $\mathbf{X}$ into integral submanifolds of $\Phi_* \mathbf{X}$, and hence $\Phi_* \mathbf{X} \in \ker^m_{\omega(I)} \Omega$.*

Proof. 1. Let $X_1, \ldots, X_m \in \mathfrak{X}(\mathcal{M})$ be vector fields locally expanding the involutive distribution associated with $\mathbf{X}$. Then $\Phi_* X_1, \ldots, \Phi_* X_m$ generate another distribution which is also involutive, and, hence, is associated with a class of locally decomposable multivector fields whose representative is just $\Phi_* \mathbf{X}$, by construction. The assertion about the integral submanifolds is then immediate.

2. As $\Phi \in \mathrm{Diff}(\mathcal{M})$ restricts to a diffeomorphism φ in M such that $\varphi \circ \kappa = \kappa \circ \Phi$ then, for every $\psi \colon M \to \mathcal{M}$, integral section of $\mathbf{X}$, we can define $\psi_M \colon M \to \mathcal{M}$ as $\Phi \circ \psi = \psi_M \circ \varphi$, which is also a section of κ because

$$\kappa \circ \psi_M = \kappa \circ \Phi \circ \psi \circ (\varphi)^{-1} = \varphi \circ \kappa \circ \psi \circ (\varphi)^{-1} = \varphi \circ (\varphi)^{-1} = \mathrm{Id}_M \, ,$$

since $\kappa \circ \psi = \mathrm{Id}_M$. Then, by construction, $\mathrm{Im}\, \psi_M = \Phi(\mathrm{Im}\, \psi)$ is an integral submanifold of $\Phi_* \mathbf{X}$, and as is a section of κ, it is κ-transverse. Hence $\Phi_* \mathbf{X}$ (which belongs to $\ker^m \Omega$, by Theorem 3) is integrable (then locally decomposable), and as its integral submanifolds are sections of κ, then $\Phi_* \mathbf{X}$ is κ-transverse, and thus $\Phi_* \mathbf{X} \in \ker^m_{\omega(I)} \Omega$. □

From this result we obtain as an immediate corollary the following:

Theorem 5. *Let $Y \in \mathfrak{X}(\mathcal{M})$ be an infinitesimal symmetry of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$, and F_t a local flow of Y.*

1. *If $\mathbf{X} \in \ker^m \Omega$ is an integrable multivector field, then F_t transforms integral submanifolds of $\mathbf{X}$ into integral submanifolds of $F_{t*} \mathbf{X}$.*

2. *In particular, if $Y \in \mathfrak{X}(\mathcal{M})$ is κ-projectable (this means that there exists $Z \in \mathfrak{X}(M)$ such that the local flows of Z and Y are κ-related), then, for every $\mathbf{X} \in \ker^m_{\omega(I)} \Omega$, F_t transforms integral submanifolds of $\mathbf{X}$ into integral submanifolds of $F_{t*}\mathbf{X}$, and hence $F_{t*}\mathbf{X} \in \ker^m_{\omega(I)} \Omega$.*

Symmetries allows us to obtain new conserved quantities from another one:

Proposition 4. *1. If $\Phi \in \mathrm{Diff}(\mathcal{M})$ is a symmetry and $\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$, then $\Phi^*\xi$ is also a conserved quantity.*

2. *If $Y \in \mathfrak{X}(\mathcal{M})$, is an infinitesimal symmetry and $\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$, then $\mathrm{L}(Y)\xi$ is also a conserved quantity.*

Proof. For every $\mathbf{X} \in \ker^m \Omega$, we have that:

1. As $\Phi_*\mathbf{X} \in \ker^m \Omega$, we obtain:

$$\mathrm{L}(\mathbf{X})(\Phi^*\xi) = \Phi^*\mathrm{L}(\Phi_*\mathbf{X})\xi = 0\,.$$

2. As $[\mathbf{X}, Y] \in \ker^m \Omega$, as a consequence of Theorem 2 we get

$$\mathrm{L}(\mathbf{X})\mathrm{L}(Y)\xi = \mathrm{L}([\mathbf{X}, Y])\xi + \mathrm{L}(Y)\mathrm{L}(\mathbf{X})\xi = \mathrm{L}([\mathbf{X}, Y])\xi = 0\,.$$

□

3.3 Cartan symmetries. Noether's theorem

Now we introduce the concept that generalizes the notion of *Cartan (Noether) symmetry* for non-autonomous mechanical systems [6], [24].

Definition 10. 1. A *Cartan symmetry* of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$ is a diffeomorphism $\Phi \colon \mathcal{M} \to \mathcal{M}$ such that, $\Phi^*\Omega = \Omega$.

2. An *infinitesimal Cartan symmetry* of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$ is a vector field $Y \in \mathfrak{X}(\mathcal{M})$ satisfying that $\mathrm{L}(Y)\Omega = 0$.

Remarks:

- It is immediate to prove that, if $Y_1, Y_2 \in \mathfrak{X}(\mathcal{M})$ are infinitesimal Cartan symmetries, then so is $[Y_1, Y_2]$.
- The condition $\mathrm{L}(Y)\Omega = 0$ is equivalent to demanding that $i(Y)\Omega$ is a closed m-form in $\mathcal{M}$. Therefore, for every $p \in \mathcal{M}$, there exists an open neighborhood $U_p \ni p$, and $\xi_Y \in \Omega^{m-1}(U_p)$, such that $i(Y)\Omega = \mathrm{d}\xi_Y$ (on U_p). Thus, an infinitesimal Cartan symmetry of a (pre)multisymplectic system is just a *locally Hamiltonian vector field* for the multisymplectic form Ω, and ξ_Y is the corresponding *local Hamiltonian form*, which is unique, up to a closed $(m-1)$-form.

Proposition 5. *1. Every Cartan symmetry of a (pre)multisymplectic system $(\mathcal{M},\Omega,\omega)$ is a symmetry.*

2. Every infinitesimal Cartan symmetry of a (pre)multisymplectic system $(\mathcal{M},\Omega,\omega)$ is an infinitesimal symmetry.

Proof. For every $\mathbf{X}\in\ker^m\Omega$, we have that:

1. If $\Phi\in\mathrm{Diff}(\mathcal{M})$ is a Cartan symmetry then

$$\Phi^* i(\Phi_*\mathbf{X})\Omega = i(\mathbf{X})(\Phi^*\Omega) = i(\mathbf{X})\Omega = 0 \iff i(\Phi_*\mathbf{X})\Omega = 0 \iff \Phi_*\mathbf{X}\in\ker^m\Omega\,.$$

2. If $Y\in\mathfrak{X}(\mathcal{M})$ is an infinitesimal Cartan symmetry, then

$$i([Y,\mathbf{X}])\Omega = \mathrm{L}(Y)i(\mathbf{X})\Omega - i(\mathbf{X})\mathrm{L}(Y)\Omega = 0 \iff [Y,\mathbf{X}]\subset\ker^m\Omega\,.$$

(Also, if $Y\in\mathfrak{X}(\mathcal{M})$ is an infinitesimal Cartan symmetry, by definition, its local flows are local Cartan symmetries, then the result is a consequence of the above item). □

Then, the classical *Noether's theorem* can be generalized as follows:

Theorem 6 (Noether). *If $Y\in\mathfrak{X}(\mathcal{M})$ is an infinitesimal Cartan symmetry of a (pre)multisymplectic system $(\mathcal{M},\Omega,\omega)$, with $i(Y)\Omega = \mathrm{d}\xi_Y$. Then, for every $\mathbf{X}\in\ker^m_\omega\Omega$ (and hence for every $\mathbf{X}\in\ker^m_{\omega(I)}\Omega$), we have that*

$$\mathrm{L}(\mathbf{X})\xi_Y = 0\,;$$

that is, any Hamiltonian $(m-1)$-form ξ_Y associated with Y is a conserved quantity of $(\mathcal{M},\Omega,\omega)$. (It is usually called a Noether current*, in this context).*

Proof. If $Y\in\mathfrak{X}(\mathcal{M})$ is a Cartan symmetry then

$$\mathrm{L}(\mathbf{X})\xi_Y = \mathrm{d}i(\mathbf{X})\xi_Y - (-1)^m i(\mathbf{X})\mathrm{d}\xi_Y = -(-1)^m i(\mathbf{X})i(Y)\Omega = -i(Y)i(\mathbf{X})\Omega = 0\,.$$

□

To our knowledge, given a conserved quantity of a (pre)multisymplectic system, there is no a straightforward way of associating to it an infinitesimal Cartan symmetry Y since, given a $(m-1)$-form ξ, the existence of a solution to the equation $i(Y)\Omega = \mathrm{d}\xi$ is not assured (even in the case Ω being 1-nondegenerate). Hence, in general, the *converse Noether theorem* cannot be stated for (pre)multisymplectic systems.

Finally, as a particular case, we have:

Proposition 6. *Let $Y\in\mathfrak{X}(\mathcal{M})$ be an infinitesimal Cartan symmetry of an exact (pre)multisymplectic system $(\mathcal{M},\Omega,\omega)$ (with $\Omega = -\mathrm{d}\Theta$). Therefore:*

1. *$\mathrm{L}(Y)\Theta$ is a closed form, hence, in an open set $U \subset \mathcal{M}$, there exist $\zeta_Y \in \Omega^{m-1}(U)$ such that $\mathrm{L}(Y)\Theta = \mathrm{d}\zeta_Y$.*
2. *If $i(Y)\Omega = \mathrm{d}\xi_Y$, in an open set $U \subset \mathcal{M}$, then*

$$\mathrm{L}(Y)\Theta = \mathrm{d}(i(Y)\Theta - \xi_Y) = \mathrm{d}\zeta_Y \quad (\text{in } U)\,,$$

and hence $\xi_Y = i(Y)\Theta - \zeta_Y$ (up to a closed $(m-1)$-form).

As a particular case, if $\mathrm{L}(Y)\Theta = 0$, we can take $\xi_Y = i(Y)\Theta$, and Y is said to be an exact infinitesimal Cartan symmetry.

Proof. 1. The first item is immediate since $\mathrm{dL}(Y)\Theta = \mathrm{L}(Y)\mathrm{d}\Theta = 0$.

2. For the second item we have

$$\mathrm{L}(Y)\Theta = \mathrm{d}i(Y)\Theta + i(Y)\mathrm{d}\Theta = \mathrm{d}i(Y)\Theta - i(Y)\Omega = \mathrm{d}(i(Y)\Theta - \xi_Y)\,.$$

Hence we can write $\xi_Y = i(Y)\Theta - \zeta_Y$ (up to a closed $(m-1)$-form). □

In the case that $\ker\Omega := \{Y \in \mathfrak{X}(\mathcal{M}) \,|\, i(Y)\Omega = 0\} \neq \{0\}$, these vector fields are Cartan symmetries. Then:

Definition 11. Let $(\mathcal{M}, \Omega, \omega)$ be a premultisymplectic system such that the equations (6) have solutions on $\mathcal{M}$. Then $Y \in \mathfrak{X}(\mathcal{M})$ is a *gauge symmetry* of $(\mathcal{M}, \Omega, \omega)$ if $Y \in \ker\Omega \cap \mathfrak{X}^{V(\kappa)}(\mathcal{M})$.

3.4 Higher-order Cartan symmetries. Generalized Noether's theorem

Noether's theorem associates conserved quantites to Cartan symmetries. But there are symmetries which are not of Cartan type. Different attempts have been made to extend Noether's theorem in order to obtain the corresponding conservation laws for these kinds of symmetries. Next we present a generalization of Theorem 6, which is based in the approach of [24] for mechanical systems.

Definition 12. An *infinitesimal Cartan symmetry of order n* of a (pre) multisymplectic system $(\mathcal{M}, \Omega, \omega)$ is a vector field $Y \in \mathfrak{X}(\mathcal{M})$ satisfying that:

1. Y is a symmetry of the (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$.
2. $\mathrm{L}^n(Y)\Omega = 0$, but $\mathrm{L}^k(Y)\Omega \neq 0$, for $k < n$.

Cartan symmetries of order $n > 1$ are not necessarily Hamiltonian vector fields for the (pre)multisymplectic form Ω. Nevertheless we have that:

Proposition 7. *If $Y \in \mathfrak{X}(\mathcal{M})$ is an infinitesimal Cartan symmetry of order n of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$, then the form $\mathrm{L}^{n-1}(Y)i(Y)\Omega \in \Omega^m(\mathcal{M})$ is closed.*

Proof. In fact, from the definition 12 we obtain

$$0 = \mathrm{L}^n(Y)\Omega = \mathrm{L}^{n-1}(Y)\mathrm{L}(Y)\Omega = \mathrm{L}^{n-1}(Y)\,\mathrm{d}i(Y)\Omega = \mathrm{d}\mathrm{L}^{n-1}(Y)i(Y)\Omega\,.$$

□

This condition is equivalent to demanding that, for every $p \in \mathcal{M}$, there exists an open neighborhood $U_p \ni p$, and $\xi_Y \in \Omega^{m-1}(U_p)$, such that $\mathrm{L}^{n-1}(Y)i(Y)\Omega = \mathrm{d}\xi_Y$ (on U_p). Then, theorem 6 can be generalized as follows:

Theorem 7. *If $Y \in \mathfrak{X}(\mathcal{M})$ is an infinitesimal Cartan symmetry of order n of a (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$, with $\mathrm{L}^{n-1}(Y)i(Y)\Omega = \mathrm{d}\xi_Y$. Then, for every $\mathbf{X} \in \ker^m_\omega \Omega$ (and hence for every $\mathbf{X} \in \ker^m_{\omega(I)} \Omega$), we have that*

$$\mathrm{L}(\mathbf{X})\xi_Y = 0$$

that is, the $(m-1)$-form ξ_Y associated with Y is a conserved quantity.

Proof. If $Y \in \mathfrak{X}(\mathcal{M})$ is an infinitesimal Cartan symmetry of order n then it is a symmetry, and then $[Y, \mathbf{X}] = \mathbf{Z} \in \ker\,\Omega$. Therefore

$$\begin{aligned}\mathrm{L}(\mathbf{X})\xi_Y &= (-1)^{m+1} i(\mathbf{X})\mathrm{d}\xi_Y = (-1)^{m+1} i(\mathbf{X})\mathrm{L}^{n-1}(Y)i(Y)\Omega \\ &= (-1)^{m+1} i(\mathbf{X})\mathrm{L}(Y)\mathrm{L}^{n-2}(Y)i(Y)\Omega \\ &= (-1)^{m+1}(\mathrm{L}(Y)i(\mathbf{X})\mathrm{L}^{n-2}(Y)i(Y)\Omega - i([Y,\mathbf{X}])\mathrm{L}^{n-2}(Y)i(Y)\Omega) \\ &= (-1)^{m+1}((\mathrm{L}(Y)i(\mathbf{X}) - i(\mathbf{Z}))\mathrm{L}^{n-2}(Y)i(Y)\Omega)\,,\end{aligned}$$

and repeating the reasoning $n-2$ times we arrive at the result

$$\mathrm{L}(\mathbf{X})\xi_Y = (-1)^{m+1}((\mathrm{L}(Y)i(\mathbf{X}) - i(\mathbf{Z}))^{n-1} i(Y)\Omega) = 0\,,$$

since $i(\mathbf{X})i(Y)\Omega = 0$ and $i(\mathbf{Z})i(Y)\Omega = 0$. □

Proposition 8. *Let $Y \in \mathfrak{X}(\mathcal{M})$ be an infinitesimal Cartan symmetry of order n of an exact (pre)multisymplectic system $(\mathcal{M}, \Omega, \omega)$. Therefore:*

1. *$\mathrm{L}^n(Y)\Theta$ is a closed form, hence, in an open set $U \subset \mathcal{M}$, there exist $\zeta_Y \in \Omega^{m-1}(U)$ such that $\mathrm{L}^n(Y)\Theta = \mathrm{d}\zeta_Y$.*
2. *If $\mathrm{L}^{n-1}(Y)i(Y)\Omega = \mathrm{d}\xi_Y$, in an open set $U \subset \mathcal{M}$, then*

$$\mathrm{L}^n(Y)\Theta = \mathrm{d}(\mathrm{L}^{n-1}(Y)i(Y)\Theta - \xi_Y) = \mathrm{d}\zeta_Y \quad (\text{in } U)\,.$$

Proof. 1. The first item is immediate since $\mathrm{d}\mathrm{L}^n(Y)\Theta = \mathrm{L}^n(Y)\,\mathrm{d}\Theta = 0$.

2. For the second item we have

$$\begin{aligned}\mathrm{L}^n(Y)\Theta &= \mathrm{L}^{n-1}(Y)\mathrm{L}(Y)\Theta = \mathrm{L}^{n-1}(Y)(\mathrm{d}i(Y)\Theta + i(Y)\,\mathrm{d}\Theta) \\ &= \mathrm{d}\mathrm{L}^{n-1}(Y)i(Y)\Theta + \mathrm{L}^{n-1}(Y)i(Y)\,\mathrm{d}\Theta \\ &= \mathrm{d}\mathrm{L}^{n-1}(Y)i(Y)\Theta - \mathrm{d}\xi_Y = \mathrm{d}(\mathrm{L}^{n-1}(Y)i(Y)\Theta - \xi_Y)\,.\end{aligned}$$

Hence we can write $\xi_Y = \mathrm{L}^{n-1}(Y)i(Y)\Theta - \zeta_Y$. □

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