Correlation among runners and some results on the Lonely Runner Conjecture*

Guillem Perarnau
School of Mathematics
University of Birmingham
Birmingham, B15 2TT, UK.
g.perarnau@bham.ac.uk

Oriol Serra
Departament de Matemàtiques,
Universitat Politècnica de Catalunya, and
Barcelona Graduate School of Mathematics
Barcelona, Spain
oriol.serra@upc.edu

Abstract

The Lonely Runner Conjecture, posed independently by Wills and by Cusick, states that for any set of runners running along the unit circle with constant different speeds and starting at the same point, there is a time where all of them are far enough from the origin. We study the correlation among the time that runners spend close to the origin. By means of these correlations, we improve a result of Chen on the gap of loneliness. In the last part, we introduce dynamic interval graphs to deal with a weak version of the conjecture thus providing a new result related to the invisible runner theorem of Czerwiński and Grytczuk.

Keywords: Lonely Runner Conjecture

1 Introduction

The Lonely Runner Conjecture was posed independently by Wills [?] in 1967 and Cusick [?] in 1982. Its picturesque name comes from the following interpretation due to Goddyn [?]. Consider a set of k runners on the unit circle running with different constant speeds and starting at the origin. The conjecture states that, for each runner, there is a time where she is at distance at least $1/k$ on the circle from all the other runners.

For any real number x, denote by $\|x\|$, the distance from x to the closest integer

$$\|x\| = \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\} ,$$

and by

$$\{x\} = x - \lfloor x \rfloor,$$

its fractional part.

By assuming that one of the runners has zero speed, the conjecture can be easily seen to be equivalent to the following one.

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Conjecture 1 (Lonely Runner Conjecture)**.** For every $n \geq 1$ and every set of nonzero speeds $v_1, \dots, v_n$, there exists a time $t \in \mathbb{R}$ such that

$$\|tv_i\| \geq \frac{1}{n+1} ,$$

for every $i \in [n]$.

If true, the Lonely Runner Conjecture is best possible: for the set of speeds,

$$v_i = i \quad \text{for every } i \in [n] , \tag{1}$$

there is no time for which all the runners are further away from the origin than $\frac{1}{n+1}$. An infinite family of additional extremal sets for the conjecture can be found in [**?**].

The conjecture is obviously true for $n = 1$, since at some point $\|tv_1\| = 1/2$, and it is also easy to show that it holds for $n = 2$. Many proofs for $n = 3$ are given in the context of diophantine approximation (see [**?**, **?**]). A computer–assisted proof for $n = 4$ was given by Cusick and Pomerance motivated by a view-obstruction problem in geometry [**?**], and later Biena et al. [**?**] provided a simpler proof by connecting it to nowhere zero flows in regular matroids. The conjecture was proved for $n = 5$ by Bohmann, Holzmann and Kleitman [**?**]. Barajas and Serra [**?**] showed that the conjecture holds for $n = 6$ by studying the regular chromatic number of distance graphs.

In [**?**], the authors also showed that the conjecture can be reduced to the case where all speeds are positive integers and in the sequel we will assume this to be the case. In particular, we also may assume that t takes values on the $(0, 1)$ unit interval, since if $t \in \mathbb{Z}$, then $\|tv_i\| = 0$ for all $i \in [n]$.

Czerwiński [**?**] proved a strengthening of the conjecture if all the speeds are chosen uniformly at random among all the n-subsets of $[N]$. In particular, Czerwiński's result implies that, for almost all sets of runners, as $N \to \infty$ there is a time where all the runners are arbitrarily close to $1/2$. The dependence of N with respect to n for which this result is valid was improved by Alon [**?**] in the context of colorings of Cayley graphs.

Dubickas [**?**] used a result of Peres and Schlag [**?**] in lacunary integer sequences to prove that the conjecture holds if the sequence of increasing speeds grows fast enough; in particular, for n sufficiently large, if

$$\frac{v_{i+1}}{v_i} \geq 1 + \frac{22 \log n}{n} , \tag{2}$$

for every $1 \leq i < n$. These results introduce the use of the Lovász Local Lemma to deal with the dependencies among the runners. Recently, Tao showed in his blog [**?**], how to reduce the problem to the case where no speed is larger than n^{cn^2}, for some $c > 0$.

Another approach to the conjecture is to reduce the *gap of loneliness*. That is, to show that, for some fixed $\delta \leq \frac{1}{n+1}$ and every set of nonzero speeds, there exists a time $t \in [0, 1)$ such that

$$\|tv_i\| \geq \delta \quad \text{for every } i \in [n] . \tag{3}$$

For this approach it is particularly useful to define the following sets,

$$A_i = \{t \in [0, 1) : \ \|tv_i\| < \delta\} .$$

For every $t \in A_i$, we will say that the i–th runner is δ–close to the origin at time t. Otherwise, we will say that the runner is δ–far from the origin at time t.

The set A_i can be thought of as an event in the probability space $[0,1)$ with the uniform distribution. Notice that we have $\Pr(A_i) = 2\delta$ independently from the value of v_i. In this setting, if

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0 , \tag{4}$$

then, there exists a time t for which (3) holds.

Here it is also convenient to consider the indicator random variables X_i for the events A_i. Let $X = \sum_{i=1}^n X_i$ count the number of runners which are δ–close to the origin at a time $t \in (0,1)$ chosen uniformly at random. Then, condition (4) is equivalent to $\Pr(X = 0) > 0$.

A first straightforward result in this direction is obtained by using the union bound in (4). For any $\delta < \frac{1}{2n}$, we have

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq 1 - \sum_{i=1}^{n} \Pr(A_i) = 1 - 2\delta n > 0 .$$

This result was improved by Chen [?] who showed that, for every set of n nonzero speeds, there exists a time $t \in \mathbb{R}$ such that

$$\|tv_i\| \geq \frac{1}{2n - 1 + \frac{1}{2n-3}} , \tag{5}$$

for every $i \in [n]$.

If $2n - 3$ is a prime number, then the previous result was extended by Chen and Cusick [?]. In this case, these authors proved that, for every set of n speeds, there exists a time $t \in \mathbb{R}$ such that

$$\|tv_i\| \geq \frac{1}{2n-3} ,$$

for every $i \in [n]$. We give a seemingly simpler proof of this result in Section 4. Unfortunately, both proofs strongly use the fact that $2n - 3$ is a prime.

In order to improve (5), we exactly compute the pairwise join probabilities $\Pr(A_i \cap A_j)$, the amount of time that two runners spend close to the origin at the same time. As a corollary, we give the following lower bound on $\mathbb{E}(X^2)$.

Proposition 2. *For every δ such that $\delta \to 0$ when $n \to +\infty$, we have*

$$\mathbb{E}(X^2) \geq 2\delta n \left(\delta\left(1 + \Omega\left(\frac{1}{\log \delta^{-1}}\right)\right) n + 1\right) .$$

Then, we are able to improve Chen's result on the gap of loneliness around the origin.

Theorem 3. *For every sufficiently large n and every set $v_1, \ldots, v_n$ of nonzero speeds there exists a time $t \in [0,1)$ such that*

$$\|tv_i\| \geq \frac{1}{2n - 2 + o(1)} ,$$

for each $i \in [n]$.

The proof of Theorem 3 uses a Bonferroni–type inequality due to Hunter [?] (see Lemma 13) that improves the union bound with the knowledge of pairwise intersections. While the improvement of Theorem 3 is modest, we point out that this is the best result up to date on the Lonely Runner Conjecture for a general sequence of speeds (provided that n is large).

The bound on δ in Theorem 3 can be substantially improved in the case of sets of speeds taken from a sequence with divergent sum of inverses. More precisely the following result is proven.

Theorem 4. *For every set $v_1, \dots, v_n$ of nonzero speeds there exists a time $t \in [0,1)$ such that*

$$\|tv_i\| \geq \frac{1}{2\left(n - \sum_{i=2}^{n} \frac{1}{v_i}\right)} ,$$

for each $i \in [n]$.

The condition (2) of Dubickas [?] implies that the conjecture is true if the speeds grow sufficiently fast. Theorem 4 is interesting in the sense that it provides meaningful bounds for the opposite case, that is when the speeds grow slowly. In particular, if $v_i, 1 \leq i \leq n$ is a sequence of speeds satisfying $\sum_{i=1}^{n} \frac{1}{v_i} = \omega(1)$, then there exists a time $t \in [0,1)$ such that

$$\|tv_i\| \geq \frac{1}{2n - \omega(1)} ,$$

for every $i \in [n]$. The last inequality holds under a natural density condition on the set of speeds which covers the more difficult cases where the speeds grow slowly.

Another interesting result on the Lonely Runner Conjecture, was given by Czerwiński and Grytczuk [?]. Given a set of n runners, we say that a runner $k \in [n]$ is *almost alone at time t* if there exists a $j \neq k$ such that

$$\|t(v_i - v_k)\| \geq \frac{1}{n+1} ,$$

for every $i \neq j, k$. If this case we say that j leaves k almost alone. We do a slight abuse of notation by saying that a point $x \in [0,1)$ is almost alone at time t if there exists a $j \neq k$ such that $\|t(v_i - x)\| \geq \frac{1}{n+1}$, for every $i \neq j, k$.

In [?], the authors showed that every runner is almost alone at some time. This means that Conjecture 1 is true if we are allowed to make one runner invisible, that is, there exists a time when all runners but one are far enough from the origin.

Theorem 5 ([?])**.** *For every $n \geq 1$ and every set of nonzero speeds $v_1, \dots, v_n$, there exist a time $t \in [0,1)$ such that the origin is almost alone at time t.*

A similar result can be derived by using a model of dynamic circular interval graphs. By using this model we can show that either there is a runner alone at some time or at least four runners are almost alone at the same time.

Theorem 6. *For every set of different speeds $v_1, \dots, v_{n+1}$, there exist a time $t \in (0,1)$ such that, at time t, either there is a runner alone or four different runners are almost alone.*

The paper is organized as follows. In Section 2 we compute the pairwise join probabilities for the events A_i and give a proof for Proposition 2. As a corollary of these results, we also prove Theorem 3 and Theorem 4 (Subsection 2.1). In Section 3 we introduce an approach on the problem based on dynamic circular interval graphs and prove Theorem 6. Finally, in Section 4 we give some conclusions, discuss some open questions and give a proof of the improved bound $1/(2n-3)$ when $2n-3$ is a prime which uses a combination of the ideas presented in this paper and a technique from [?].

2 Correlation among runners

In this section we want to study the pairwise join probabilities $\Pr(A_i \cap A_j)$, for every $i, j \in [n]$. Notice first, that, if A_i and A_j were independent events, then we would have $\Pr(A_i \cap A_j) = 4\delta^2$, since $\Pr(A_i) = 2\delta$ for every $i \in [n]$. This is not true in the general case, but, as we will see later on, some of these pairwise probabilities can be shown to be large enough.

For each ordered pair (i, j) with $i, j \in [n]$, we define

$$\varepsilon_{ij} = \left\{ \frac{v_i}{\gcd(v_i, v_j)} \delta \right\} , \tag{6}$$

where $\gcd(v_i, v_j)$ denotes the greatest common divisor of v_i and v_j and $\{\cdot\}$ is the fractional part.

Let us also consider the function $f : [0,1)^2 \to \mathbb{R}$, defined by

$$f(x, y) = \min(x, y) + \max(x + y - 1, 0) - 2xy . \tag{7}$$

The proofs of Propositions 7 and 8 below are based on the proofs of lemmas 3.4 and 3.5 in Alon and Ruzsa [?]. Let us start by studying the case when the speeds v_i and v_j are coprime.

Proposition 7. *Let v_i and v_j be coprime positive integers and $0 < \delta < 1/4$. Then*

$$\Pr(A_i \cap A_j) = 4\delta^2 + \frac{2f(\varepsilon_{ij}, \varepsilon_{ji})}{v_i v_j} .$$

Proof. Observe that A_i and A_j can be expressed as the disjoint unions of intervals

$$A_i = \bigcup_{k=0}^{v_i - 1} \left(\frac{k}{v_i} - \alpha, \frac{k}{v_i} + \alpha \right) \qquad A_j = \bigcup_{l=0}^{v_j - 1} \left(\frac{l}{v_j} - \beta, \frac{l}{v_j} + \beta \right)$$

where $\alpha = \delta/v_i$ and $\beta = \delta/v_j$ and we consider the elements in $[0,1)$ modulo 1. We observe that $\alpha + \beta = \delta(v_i + v_j)/v_i v_j < 1/2$ since $\delta < 1/4$.

Denote by $I = (-\alpha, \alpha)$ and $J = (-\beta, \beta)$. We have

$$\begin{aligned}
\Pr(A_i \cap A_j) &= \Pr\left(\bigcup_{k \le v_i, l \le v_j} (I + k/v_i) \cap (J + l/v_j) \right) \\
&= \sum_{k \le v_i, l \le v_j} \Pr((I + k/v_i) \cap (J + l/v_j)) \\
&= \sum_{k \le v_i, l \le v_j} \Pr(I \cap (J + l/v_j - k/v_i)) \\
&= \sum_{k=0}^{v_i v_j - 1} \Pr\left(I \cap (J + k/v_j v_i)\right) ,
\end{aligned}$$

where in the last equality we used the fact that $\gcd(v_i, v_j) = 1$.

For each $-1/2 < x < 1/2$, define $d(x) = \Pr(I \cap (J + x))$. Let us assume that $v_j < v_i$. We can write $d(x)$ as follows (see Figure 1):

$$d(x) = \begin{cases} \beta + \alpha + x, & x \in [-(\beta+\alpha), -(\beta-\alpha)] \\ 2\alpha, & x \in [-(\beta-\alpha), \beta-\alpha] \\ \beta + \alpha - x, & x \in [\beta-\alpha, \beta+\alpha] \\ 0 & \text{otherwise} \end{cases}$$

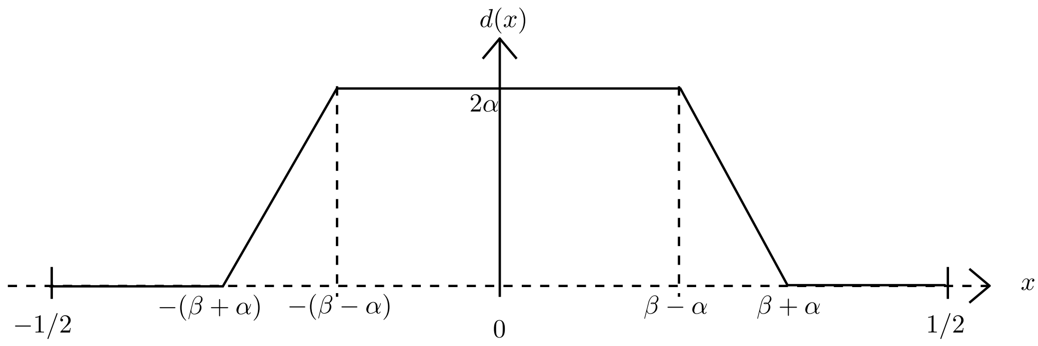

Figure 1: Plot of $d(x)$ in $(-1/2, 1/2)$.

By symmetry, we have

$$\frac{\Pr(A_i \cap A_j)}{2} = \frac{d(0)}{2} + \sum_{1 \le k \le (\beta+\alpha)v_iv_j} d\left(\frac{k}{v_iv_j}\right) = \alpha + \sum_{1 \le k \le (\beta+\alpha)v_iv_j} \min\left(2\alpha, \beta + \alpha - \frac{k}{v_iv_j}\right) .$$

Write $\alpha v_i v_j = p + \varepsilon_{ji}$ and $\beta v_i v_j = q + \varepsilon_{ij}$, where p and q are integers and $0 \le \varepsilon_{ji}, \varepsilon_{ij} < 1$.

Observe that

$$d\left(\frac{q-p}{v_iv_j}\right) v_iv_j = \begin{cases} 2(p + \varepsilon_{ji}), & \text{if } \varepsilon_{ji} \le \varepsilon_{ij} \\ 2p + \varepsilon_{ji} + \varepsilon_{ij}, & \text{if } \varepsilon_{ji} > \varepsilon_{ij} \end{cases} = 2p + \varepsilon_{ji} + \min(\varepsilon_{ji}, \varepsilon_{ij}) ,$$

and that

$$d\left(\frac{q+p+1}{v_iv_j}\right) v_iv_j = \begin{cases} 0, & \text{if } \varepsilon_{ji} + \varepsilon_{ij} \le 1 \\ \varepsilon_{ji} + \varepsilon_{ij} - 1, & \text{if } \varepsilon_{ji} + \varepsilon_{ij} > 1 \end{cases} = \max(0, \varepsilon_{ji} + \varepsilon_{ij} - 1) .$$

Therefore,

$$\begin{aligned} \frac{\Pr(A_i \cap A_j)}{2} v_iv_j &= p + \varepsilon_{ji} + \sum_{1 \le k \le p+q+\varepsilon_{ji}+\varepsilon_{ij}} \min(2(p+\varepsilon_{ji}), q + p + \varepsilon_{ji} + \varepsilon_{ij} - k) \\ &= p + \varepsilon_{ji} + \sum_{k=1}^{q-p-1} 2(p + \varepsilon_{ji}) + 2p + \varepsilon_{ji} + \min(\varepsilon_{ji}, \varepsilon_{ij}) \\ &\quad + \sum_{k=q-p+1}^{p+q} (q + p + \varepsilon_{ji} + \varepsilon_{ij} - k) + \max(0, \varepsilon_{ji} + \varepsilon_{ij} - 1) \end{aligned}$$

$$= 2(p + \varepsilon_{ji})(q + \varepsilon_{ij}) + f(\varepsilon_{ji}, \varepsilon_{ij}) \ .$$

Thus,

$$\Pr(A_i \cap A_j) = \frac{2}{v_i v_j} \left(2(p + \varepsilon_{ji})(q + \varepsilon_{ij}) + f(\varepsilon_{ji}, \varepsilon_{ij})\right) = 4\delta^2 + \frac{2f(\varepsilon_{ji}, \varepsilon_{ij})}{v_i v_j} \ .$$

□

Proposition 7 can be easily generalized to pairs of speeds that are not coprime.

Proposition 8. *Let v_i and v_j be positive integers and $0 < \delta < 1/4$. Then*

$$\Pr(A_i \cap A_j) = 4\delta^2 + \frac{2(\gcd(v_i, v_j))^2 f(\varepsilon_{ji}, \varepsilon_{ij})}{v_i v_j} \ .$$

Proof. Consider $v_i' = \frac{v_i}{\gcd(v_i,v_j)}$ and $v_j' = \frac{v_j}{\gcd(v_i,v_j)}$. Define $A_i' = \{t \in [0,1) : \ \|tv_i'\| < \delta\}$ and $A_j' = \{t \in [0,1) : \|tv_j'\| < \delta\}$. Observe that

$$\Pr(A_i \cap A_j) = \Pr(A_i' \cap A_j') \ .$$

The proof follows by applying Proposition 7 to v_i' and v_j', which are coprime. □

Corollary 9. *Let v_i and v_j be positive integers and $0 < \delta < 1/4$. Then,*

$$\Pr(A_i \cap A_j) \geq 2\delta^2. \tag{8}$$

Moreover, if $v_j < v_i$, then

$$\Pr(A_i \cap A_j) \geq \frac{\gcd(v_i, v_j)}{v_i} 2\delta \ . \tag{9}$$

Proof. We observe that, for $x, y \leq 1$, we have that $\min(x, y) \geq xy$ and thus $f(x, y) \geq -xy$. Therefore Proposition 8 leads to the following lower bound,

$$\Pr(A_i \cap A_j) = 4\delta^2 + \frac{2(\gcd(v_i, v_j))^2 f(\varepsilon_{ij}, \varepsilon_{ji})}{v_i v_j} \geq 4\delta^2 - \frac{2(\gcd(v_i, v_j))^2 \varepsilon_{ij}\varepsilon_{ji}}{v_i v_j} \geq 2\delta^2 \ .$$

Moreover, if $v_j < v_i$, from the proof of Proposition 7 with $v_i' = v_i / \gcd(v_i, v_j)$,

$$\Pr(A_i \cap A_j) \geq d(0) = \frac{2\delta}{v_i'} = \frac{2\gcd(v_i, v_j)\delta}{v_i} \ .$$

□

By using (8), we can provide a first lower bound on the second moment of X,

$$\mathbb{E}(X^2) = \sum_{i \neq j} \Pr(A_i \cap A_j) + \sum_{i=1}^{n} \Pr(A_i) \geq 2\delta^2 n(n-1) + 2\delta n \geq 2\delta n(\delta(n-1) + 1) \ . \tag{10}$$

We devote the rest of this section to improve (10). Let us first show for which values is f nonnegative.

Lemma 10. *The function $f(x, y)$ is nonnegative in $[0, 1/2]^2$ and in $[1/2, 1)^2$.*

Proof. If $0 \leq x, y \leq 1/2$, then $\min(x, y) \geq 2xy$, which implies $f(x, y) \geq 0$.

Moreover,

$$\begin{aligned} f(1-x,1-y) &= \min(1-x,1-y) + \max(1-x-y,0) - 2(1-x-y+xy) \\ &= \min(y,x) + \max(0,x+y-1) - 2xy \\ &= f(x,y)\ . \end{aligned}$$

Therefore, we also have $f(x,y) \geq 0$ for all $1/2 \leq x,y < 1$. □

The following lemma shows that the error term of $\Pr(A_i \cap A_j)$ provided in Proposition 8, cannot be too negative if v_i and v_j are either close or far enough from each other.

Lemma 11. *Let $M \geq 2$ be an integer, $\gamma = M^{-1} > 0$ and $v_j < v_i$. If either $(1-\gamma)v_i \leq v_j$ or $\gamma\delta v_i \geq v_j$, then*

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_i v_j} \geq -\gamma\delta^2\ .$$

Proof. For the sake of simplicity, let us write $v_i/\gcd(v_i,v_j) = k\delta^{-1} + x$ and $v_j/\gcd(v_i,v_j) = l\delta^{-1} + y$ with k and l being nonnegative integers and $0 \leq x,y < \delta^{-1}$. In particular, observe that $\varepsilon_{ij} = x\delta$ and $\varepsilon_{ji} = y\delta$. Moreover, we can assume that v_i and v_j are such that $f(\varepsilon_{ij},\varepsilon_{ji})$ is negative, otherwise, there is nothing to prove.

We split the proof in the two different cases each consisting of some other subcases. Figure 2 illustrates the subcase considered in each situation.

Case A: ($\frac{v_i}{\gcd(v_i,v_j)} \geq (\gamma\delta)^{-1}$): This case covers the case when $\gamma\delta v_i \geq v_j$, since $v_j/\gcd(v_i,v_j) \geq 1$ and also the case when $(1-\gamma)v_i \leq v_j$ and $\frac{v_i}{\gcd(v_i,v_j)} \geq (\gamma\delta)^{-1}$.

Subcase A.1 ($y \leq x$): We have,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_i v_j} \geq \frac{(\gcd(v_i,v_j))^2(\varepsilon_{ji} - 2\varepsilon_{ij}\varepsilon_{ji})}{v_i v_j} = \frac{(\gcd(v_i,v_j))^2(y\delta - 2xy\delta^2)}{v_i v_j} \geq \frac{\gcd(v_i,v_j)(1-2x\delta)}{v_i}\cdot\delta\ ,$$

where the last inequality holds from the fact that $f(\varepsilon_{ij},\varepsilon_{ji}) < 0$ and $y \leq v_j/\gcd(v_i,v_j)$.

Recall that $v_i/\gcd(v_i,v_j) \geq (\gamma\delta)^{-1} = M\delta^{-1}$. Observe also that, since $y \leq x$ and $f(\varepsilon_{ij},\varepsilon_{ji})$ is negative, by Lemma 10 we have $\delta^{-1}/2 \leq x < \delta^{-1}$. Therefore,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_i v_j} \geq \frac{\gcd(v_i,v_j)(1-2x\delta)}{v_i}\delta \geq \frac{1-2x\delta}{M}\delta^2 > -\gamma\delta^2\ .$$

Subcase A.2 ($y > x$): Here,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_i v_j} \geq \frac{(\gcd(v_i,v_j))^2(\varepsilon_{ij} - 2\varepsilon_{ij}\varepsilon_{ji})}{v_i v_j} = \frac{(\gcd(v_i,v_j))^2 x(1-2y\delta)}{v_i v_j}\delta \geq \frac{\gcd(v_i,v_j)(1-2y\delta)}{v_j}\cdot\frac{\delta}{M}\ ,$$

where the last inequality holds from the fact that, in this subcase, $Mx \leq M\delta^{-1} \leq v_i/\gcd(v_i,v_j)$.

As before, since $f(\varepsilon_{ij},\varepsilon_{ji})$ is negative, by Lemma 10 we have $\delta^{-1}/2 \leq y < \delta^{-1}$ and $v_j/\gcd(v_i,v_j) \geq y$. Therefore,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_i v_j} \geq \frac{\gcd(v_i,v_j)(1-2y\delta)}{v_j}\cdot\frac{\delta}{M} \geq \frac{1-2y\delta}{y}\cdot\frac{\delta}{M} > -\gamma\delta^2\ .$$

Case B $\left((1-\gamma)v_i \leq v_j \text{ and } \frac{v_i}{\gcd(v_i,v_j)} \leq (\gamma\delta)^{-1}\right)$**:** By Lemma 10 and since $\frac{v_i}{\gcd(v_i,v_j)} \leq (\gamma\delta)^{-1}$, we can assume that either $k = l$, $y < \delta^{-1}/2$ and $x \geq \delta^{-1}/2$ (Subcases B.1 and B.2) or $k = l + 1$, $y \geq \delta^{-1}/2$ and $x < \delta^{-1}/2$ (Subcases B.3 and B.4). In all these cases, $(1-\gamma)v_i \leq v_j$ implies

$$y \geq (1-\gamma)x - \gamma k\delta^{-1} . \tag{11}$$

Subcase B.1 **(**$k = l$ **and** $x + y \leq \delta^{-1}$**):** Since $x + y \leq \delta^{-1}$, then $\max(0, \varepsilon_{ij} + \varepsilon_{ji} - 1) = 0$.

By using $v_i/\gcd(v_i,v_j) = k\delta^{-1} + x$, $v_j/\gcd(v_i,v_j) = k\delta^{-1} + y \geq y$ and the fact that $f(\varepsilon_{ij}, \varepsilon_{ji}) < 0$ we have

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} = \frac{(\gcd(v_i,v_j))^2(\varepsilon_{ji} - 2\varepsilon_{ij}\varepsilon_{ji})}{v_iv_j} = \frac{(\gcd(v_i,v_j))^2 y(1-2x\delta)}{v_iv_j}\delta \geq \frac{1-2x\delta}{k+x\delta}\delta^2 .$$

By combining (11) with $x + y \leq \delta^{-1}$, we get $x \leq \frac{1+\gamma k}{2-\gamma}\delta^{-1}$. Thus,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} \geq \frac{1-2x\delta}{k+x\delta}\delta^2 \geq \frac{1-\frac{2(1+\gamma k)}{2-\gamma}}{k+\frac{1+\gamma k}{2-\gamma}}\delta^2 = -\gamma\delta^2 ,$$

for each $k \geq 0$.

Subcase B.2 **(**$k = l$ **and** $x + y \geq \delta^{-1}$**):**

Now, $\max(0, \varepsilon_{ij} + \varepsilon_{ji} - 1) = (x+y)\delta - 1$. Then,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} = \frac{(\gcd(v_i,v_j))^2(y\delta + (x+y)\delta - 1 - 2xy\delta^2)}{v_iv_j} = -\frac{(\gcd(v_i,v_j))^2(1-2y\delta)(1-x\delta)}{v_iv_j}\delta .$$

It remains to upper bound $g(x,y) = (1-2y\delta)(1-x\delta) = 2\delta^2(\delta^{-1}/2 - y)(\delta^{-1} - x)$ in the corresponding area. Observe that $\partial_x g(x,y) = -2\delta^2(\delta^{-1}/2 - y) \leq 0$ for every $y \leq \delta^{-1}/2$. Thus, the local maximum of $g(x,y)$ is attained in the line $x + y = \delta^{-1}$. Observe that this case is covered by Case $B.1$. Following the steps of the previous case,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} \geq -\gamma\delta^2 .$$

Subcase B.3 **(**$k = l + 1$ **and** $x + y \leq \delta^{-1}$**):**

Now we have $\max(0, \varepsilon_{ij} + \varepsilon_{ji} - 1) = 0$ and $\min(\varepsilon_{ij}, \varepsilon_{ji}) = \varepsilon_{ij}$. Since $v_i/\gcd(v_i,v_j) = k\delta^{-1} + x \geq k\delta^{-1}$ and $v_j/\gcd(v_i,v_j) = (k-1)\delta^{-1} + y \geq y$, $v_j \geq (1-\gamma)v_i$ implies that $y \geq (1-\gamma)x - (\gamma k - 1)\delta^{-1}$.

Then,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} = \frac{(\gcd(v_i,v_j))^2(\varepsilon_{ij} - 2\varepsilon_{ij}\varepsilon_{ji})}{v_iv_j} = -\frac{(\gcd(v_i,v_j))^2 x\delta(2y\delta - 1)}{v_iv_j}$$

We aim to upper bound $g(x,y) = \delta x(2y\delta - 1) = 2\delta^2 x(y - \delta^{-1}/2)$ in the corresponding area. We have that $\partial_y g(x,y) = 2\delta^2 x \geq 0$ for every $x \leq \delta^{-1}/2$. Again, the local maximum of $g(x,y)$ is attained in the line $x+y = \delta^{-1}$. A simple computation gives that $g(x, \delta^{-1} - x)$ attains its maximum in $x_0 = \delta^{-1}/4$.

<u>*Subcase B.3.1*</u> *(*$k \leq \gamma^{-1}/2$*):* In this case, the pair $(x_0, \delta^{-1} - x_0)$ does not lie in the area. Thus, the maximum is attained, when x_0 is minimized, that is in the point where $x + y \leq \delta^{-1}$ and $y = (1-\gamma)x - (\gamma k - 1)\delta^{-1}$ meet.

That is

$$g(x,y) \le g\left(\frac{\gamma k}{2-\gamma}\delta^{-1}, \frac{2-(k+1)\gamma}{2-\gamma}\delta^{-1}\right) = \frac{2-(2k+1)\gamma}{(2-\gamma)^2}\gamma k \ .$$

Since $v_i/\gcd(v_i,v_j) \ge k\delta^{-1}$ and $v_j/\gcd(v_i,v_j) \ge (k-1/2)\delta^{-1}$,

$$\begin{aligned}\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} &\ge -\frac{2-(2k+1)\gamma}{(2-\gamma)^2(k-1/2)}\gamma\delta^2 \\ &\ge -\frac{2(2-3\gamma)}{(2-\gamma)^2}\gamma\delta^2 \ge \gamma\delta^2 \ .\end{aligned}$$

where we used that $k \ge 1$.

Subcase B.3.2 ($k \ge \gamma^{-1}/2$): Using the global maximum of $g(x,\delta^{-1}-x)$ in $x_0 = \delta^{-1}/4$, we have that for any (x,y) in the area

$$g(x,y) \le g(\delta^{-1}/4, 3\delta^{-1}/4) = 1/8 \ .$$

Now $v_i/\gcd(v_i,v_j) \ge k\delta^{-1} \ge (\gamma\delta)^{-1}/2$ and $v_j/\gcd(v_i,v_j) = (k-1)\delta^{-1} + y \ge \delta^{-1}/2$,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} \ge -4\gamma\delta^2 \cdot \frac{1}{8} \ge -\gamma\delta^2 \ .$$

Subcase B.4 ($k = l+1$ and $x+y \ge \delta^{-1}$): We have $\max(0,\varepsilon_{ij}+\varepsilon_{ji}-1) = (x+y)\delta - 1$ and $\min(\varepsilon_{ij},\varepsilon_{ji}) = x$. Then,

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} = -\frac{(\gcd(v_i,v_j))^2(1-2x\delta)(1-y\delta)}{v_iv_j} \ .$$

Now $g(x,y) = \delta(1-2x\delta)(1-y\delta) = 2\delta^2(\delta^{-1}/2 - x)(\delta^{-1}-y)$ and $\partial_x g(x,y) = -4\delta^2(1-y\delta) \le 0$ for every $\delta^{-1}/2 \le y \le \delta^{-1}$. As usual, the local maximum of $g(x,y)$ is attained in the line $x+y=\delta^{-1}$. Since this case is already covered by Case $B.3$, we have

$$\frac{(\gcd(v_i,v_j))^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j} \ge -\gamma\delta^2 \ .$$

□

The following lemma shows that among a large set of positive numbers, there should be a pair of numbers satisfying that they are either close or far enough from each other.

Lemma 12. *For every $c > 1$, $T > c$ and every set $x_1 \ge \cdots \ge x_{m+1} > 0$ of nonnegative numbers, with $m \ge \log_c T$, there is a pair $i,j \in [m+1]$, $i < j$, such that*

$$\text{either} \quad \frac{x_i}{x_j} \le c \quad \text{or} \quad \frac{x_i}{x_j} \ge T \ .$$

Proof. Suppose that for each pair $i < j$ we have $x_i > cx_j$. In particular, for each $i \le m$, we have $x_i > cx_{i+1}$ and $x_1 > c^m x_{m+1} \ge Tx_{m+1}$. Hence the second possibility holds for $i = 1$ and $j = m+1$. □

For any fixed $\varepsilon > 0$, we call a pair $i,j \in [n]$ ε*–good* if $\Pr(A_i \cap A_j) \ge (1-\varepsilon)4\delta^2$. Now we are able to improve the lower bound on the second moment of X given in (10).

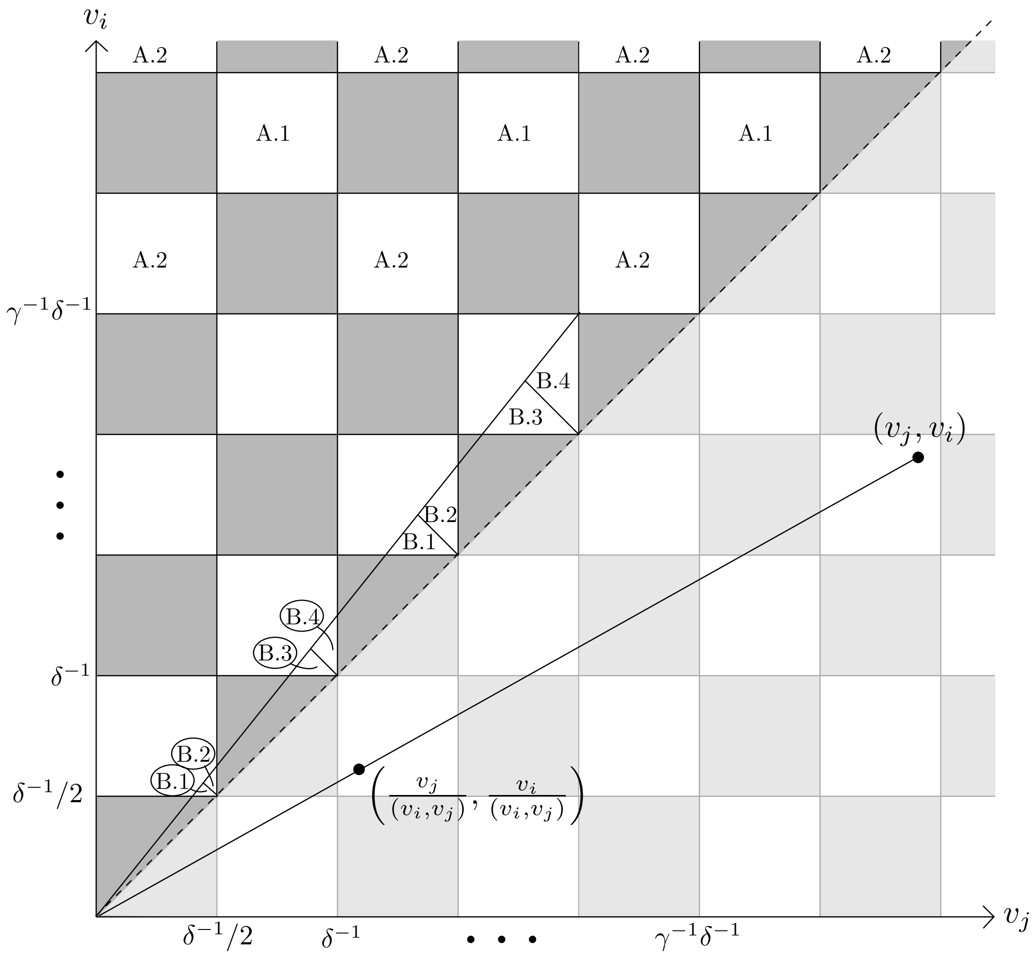

Figure 2: Different cases in the proof of Lemma 11. Grey areas correspond to positive values of $f(\varepsilon_{ij}, \varepsilon_{ji})$ according to Lemma 10.

Proof of Proposition 2. Recall that by (8), for any pair $i, j \in [n]$, we have $\Pr(A_i \cap A_j) \geq 2\delta^2$. We will show that at least a $\Omega\left(\frac{1}{\log \delta^{-1}}\right)$ fraction of the pairs are ε–good.

Lemma 11 with $M = \gamma^{-1} = \lceil (2\varepsilon)^{-1} \rceil$ implies that every pair $v_j < v_i$ with either $v_i/v_j \leq (1-\gamma)^{-1}$ or $v_i/v_j \geq (\gamma\delta)^{-1}$ is a $(\gamma/2)$–good pair, and thus, also an ε–good pair.

Consider the graph H on the vertex set $V(H) = [n]$, where ij is an edge if and only if ij is ε–good. Using Lemma 12, with $c = (1-\gamma)^{-1}$ and $T = (\gamma\delta)^{-1}$ we know that there are no independent sets of size larger than $m = \lceil \log_c(T) \rceil = \lceil \frac{\log \delta^{-1}}{\log c} + \log_c \gamma^{-1} \rceil$. Since $\delta \to 0$ when $n \to +\infty$, if n is large enough, there exists some constant c'_ε that depends only on ε such that $m \geq \frac{\log \delta^{-1}}{c'_\varepsilon} + 1$. Thus, the complement of H, $\overline{H}$, has no clique of size m. By the Erdős–Stone theorem (see [?]), $|E(\overline{H})| \leq (1+o(1))\frac{m-2}{m-1}\frac{n^2}{2}$, which implies that there are

$$|E(H)| \geq (1+o(1))\frac{n^2}{2(m-1)} = (1+o(1))\frac{c'_\varepsilon n^2}{2\log \delta^{-1}} \ ,$$

ε–good unordered pairs.

Now, we are able to give a lower bound on the second moment,

$$\begin{aligned}\mathbb{E}(X^2) &= \sum_{ij\ \varepsilon\text{–good}} \Pr(A_i\cap A_j) + \sum_{ij\ \text{non}\ \varepsilon\text{–good}} \Pr(A_i\cap A_j) + \sum_{i=1}^{n}\Pr(A_i)\\ &\geq (1-\varepsilon)4\delta^2\frac{c'_\varepsilon n^2}{\log\delta^{-1}} + 2\delta^2\left(n(n-1) - \frac{c'_\varepsilon n^2}{\log\delta^{-1}}\right) + 2\delta n\\ &= (1-\varepsilon)4\delta^2\frac{c'_\varepsilon n^2}{\log\delta^{-1}} + 2\delta^2\left(1 - \frac{c'_\varepsilon}{\log\delta^{-1}} - \frac{1}{n}\right)n^2 + 2\delta n\\ &\geq 2\delta n\left(\delta\left(1+\frac{c_\varepsilon}{\log\delta^{-1}}\right)n+1\right)\ ,\end{aligned}$$

for some c_ε that depends only on ε.

□

Next, we show some applications of our bounds that extend some known results.

2.1 Improving the gap of loneliness

In this subsection we show how to use the result of Proposition 8 on the pairwise join probabilities to prove Theorem 3. To this end we will use the following Bonferroni–type inequality due to Hunter [?] (see also Galambos and Simonelli [?]) that slightly improves the union bound in the case where the events are not pairwise disjoint.

Lemma 13 (Hunter [?])**.** *For any tree T with vertex set $V(T)=[n]$, we have*

$$\Pr\left(\bigcap_{i=1}^{n}\overline{A_i}\right) \geq 1 - \sum_{i=1}^{n}\Pr(A_i) + \sum_{ij\in E(T)}\Pr(A_i\cap A_j)\ . \tag{12}$$

As we have already mentioned, $\Pr(A_i)=2\delta$. Thus, it remains to select a tree T that maximizes $\sum_{ij\in E(T)}\Pr(A_i\cap A_j)$.

Lemma 14. *For every $\varepsilon'>0$ and $0<\delta<1/4$, there exists a tree T on the set of vertices $[n]$ such that*

$$\sum_{ij\in E(T)}\Pr(A_i\cap A_j) \geq (1-\varepsilon')4\delta^2 n\ .$$

Proof. Recall that Proposition 8 states that if $0<\delta<1/4$, then

$$\Pr(A_i\cap A_j) = 4\delta^2 + \frac{2(v_i,v_j)^2 f(\varepsilon_{ij},\varepsilon_{ji})}{v_iv_j}.$$

Set M to be the largest integer satisfying $M=\gamma^{-1}<\lceil(2\varepsilon')^{-1}\rceil$. We will construct a large forest F on the set of vertices $[n]$, where all the edges $ij\in E(F)$ are ε'–good. In particular they will satisfy,

$$\Pr(A_i\cap A_j) \geq (4-2\gamma)\delta^2 \geq (1-\varepsilon')4\delta^2\ .$$

Let us show how to select the edges of the forest by a procedure. Set $S_0=[n]$ and $E_0=\emptyset$. In the k-th step, we select different $i,j\in S_{k-1}$ such that either $v_i/v_j\leq(1-\gamma)^{-1}$ or $v_i/v_j\geq(\gamma\delta)^{-1}$, and set $E_k=E_{k-1}\cup\{ij\}$, $S_k=S_{k-1}\setminus\{i\}$. If no such pair exists, we stop the procedure.

Let τ be the number of steps that the procedure runs before being halted. By Lemma 12 with $c = (1-\gamma)^{-1}$ and $T = (\gamma\delta)^{-1}$ we can always find such an edge ij, provided that the set S_k has size at least $\log_c(\gamma\delta^{-1})$. Thus $\tau \geq n - \log_c T$. Since the size of the sets E_k increases exactly by one at each step, we have $|E_\tau| \geq n - \log_c T = n - O(\log \delta^{-1}) = (1-o(1))n$. Besides, by construction E_τ is an acyclic set of edges: since we delete one of the endpoints of each selected edge from the set S_k, E_τ induces a 1-degenerate graph or equivalently, a forest.

By Lemma 11, for each edge ij in E_τ we have

$$\Pr(A_i \cap A_j) \geq (4-2\gamma)\delta^2 \ .$$

Therefore we can construct a spanning tree T on the vertex set $[n]$, that contains the forest F and thus satisfies

$$\sum_{ij\in E(T)} \Pr(A_i \cap A_j) \geq (1-o(1))\,(4-2\gamma)\delta^2 n \geq (1-\varepsilon')4\delta^2 n \ ,$$

if n is large enough.

□

Let us proceed to prove Theorem 3.

Proof of Theorem 3. Given an $\varepsilon > 0$, by Lemma 12 and Lemma 14 with $\varepsilon' = \varepsilon/2$ and $0 < \delta < 1/4$, we have

$$\begin{aligned}\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) &\geq 1 - \sum_{i=1}^{n} \Pr(A_i) + \sum_{ij\in E(T)} \Pr(A_i \cap A_j) \\ &\geq 1 - 2\delta n(1 - 2(1-\varepsilon')\delta) \ .\end{aligned}$$

The expression above is strictly positive for

$$\delta \leq \frac{1}{2n-2+2\varepsilon'} = \frac{1}{2n-2+\varepsilon} \ ,$$

and the theorem follows if n is large enough. □

Theorem 4 follows from the following Corollary.

Corollary 15. *For every $c > 0$, every sufficiently large n and every set of nonzero speeds $v_1, \dots, v_n$ for which there exists a tree T satisfying*

$$\sum_{ij\in E(T)} \frac{\gcd(v_i, v_j)}{\max(v_i, v_j)} \geq c \ , \tag{13}$$

then there exists a time $t \in [0,1)$ such that

$$\|tv_i\| \geq \frac{1}{2(n-c)} \ ,$$

for every $i \in [n]$. In particular, if $\sum_{i=2}^{n} \frac{1}{v_i} = c$ the same conclusion follows.

Proof. By using inequality (9) we have

$$\sum_{ij\in E(T)} \Pr(A_i \cap A_j) \geq 2\delta \sum_{ij\in E(T)} \frac{\gcd(v_i, v_j)}{\max(v_i, v_j)} \geq 2c\delta$$

Using Lemma 12 in a similar way as in the proof of Theorem 3, we obtain that $\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$ for every $\delta < \frac{1}{2(n-c)}$.

The last part of the corollary follows by considering T to be the star with center in the smallest of the speeds. □

3 Weaker conjectures and interval graphs

In this section we give a proof for Theorem 6. The following weaker conjecture has been proposed by Spencer[1].

Conjecture 16 (Weak Lonely Runner Conjecture)**.** For every $n \geq 1$ and every set of different speeds $v_1, \dots, v_n$, there exist a time $t \in \mathbb{R}$ and a runner $j \in [n]$, such that

$$\|t(v_i - v_j)\| \geq \frac{1}{n}$$

for every $i \neq j$.

For every set $S \subseteq [n]$, we say that S is *isolated at time t* if,

$$\|t(v_i - v_j)\| \geq \frac{1}{n} \qquad \text{for each } i \in S,\ j \in V \setminus S. \tag{14}$$

Observe that $S = \{i\}$ is isolated at time t, if and only if, v_i is lonely at time t.

To study the appearance of isolated sets, it is convenient to define a dynamic graph $G(t)$, whose connected components are sets of isolated runners at time t. For each $1 \leq i \leq n$ and $t \in [0, 1)$, define the following dynamic interval in $[0, 1)$ associated to the i-th runner,

$$I_i(t) = \left\{ x \in [0,1)\ : \{x - tv_i\} < \frac{1}{n} \right\} .$$

In other words, $I_i(t)$ is the interval that starts at the position of the i-th runner at time t and has length $\frac{1}{n}$.

Now we can define the following dynamic circular interval graph $G(t) = (V(t), E(t))$. The vertex set $V(t)$ is composed by n vertices u_i that correspond to the set of runners, and two vertices u_i and u_j are connected if and only if $I_i(t) \cap I_j(t) \neq \emptyset$ (see Figure 3).

Observation 17. *The graph $G(t)$ satisfies the following properties,*

1. *$G(0) = K_n$.*
2. *Each connected component of $G(t)$ corresponds to an isolated set of runners at time t.*
3. *If u_i is isolated in $G(t)$, then v_i is alone at time t.*
4. *All the intervals have the same size, $|I_i(t)| = 1/n$, and thus, $G(t)$ is a unit circular interval graph.*

We can restate the Lonely Runner Conjecture in terms of the dynamic interval graph $G(t)$.

Conjecture 18 (Lonely Runner Conjecture)**.** For every $i \leq n$ there exists a time t such that u_i is isolated in $G(t)$.

[1] Transmitted to the authors by Jarek Grytczuk.

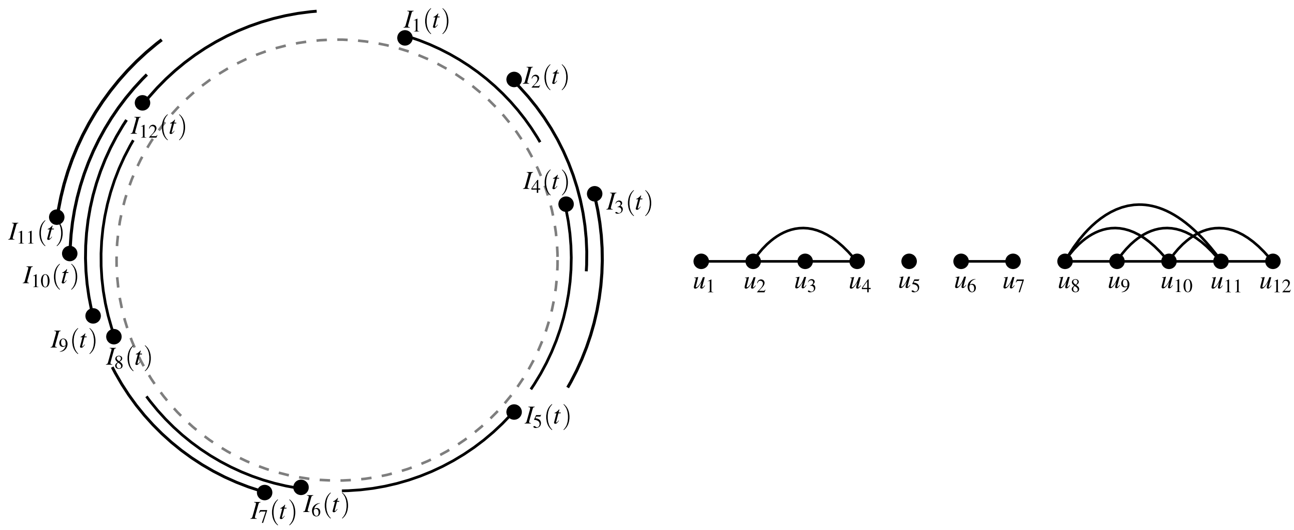

Figure 3: An instance of the graph $G(t)$.

Let μ be the uniform measure in the unit circle. For every subgraph $H \subseteq G(t)$ we define $\mu(H) = \mu(\cup_{u_i \in V(H)} I_i(t))$, the length of the arc occupied by the intervals corresponding to H. Notice that, if H contains an edge, then

$$\mu(H) < \frac{|V(H)|}{n}\,, \tag{15}$$

since the intervals $I_i(t)$ are closed in one extreme but open in the other one. If H consists of isolated vertices, then (15) does not hold.

The dynamic interval graph $G(t)$ allows us to prove a very weak version of the conjecture. Let us assume that $v_1 < v_2 < \cdots < v_n$.

Proposition 19. *There exist a time $t \in \mathbb{R}$ and a nonempty subset $S \subset [n]$ such that S is isolated at time t.*

Proof. Let t be the number for which the equation $tv_n - 1 = tv_1 - 1/n$ holds. This is the first time that the slowest runner v_1 is at distance exactly $1/n$ ahead from the fastest runner v_n.

For the sake of contradiction, assume that there is just one connected component of order n. Note that $u_1u_n \notin E(G(t))$ and since $G(t)$ is connected, there exists a path in $G(t)$ connecting u_1 and u_n. By (15), we have $\mu(G) < 1$. Thus, there is a point $x \in [0,1)$ such that $x \notin I_i(t)$ for every $i \in [n]$.

Observe that, at time t, all the intervals $I_i(t)$ are sorted in increasing order around $[0,1)$. Let $\ell \in [n]$ be such that $x > \{tv_\ell\}$ and $x < \{tv_{\ell+1}\}$. Then, $\{u_1, \dots, u_\ell\}$ and $\{u_{\ell+1}, \dots, u_n\}$ are in different connected components since $u_1u_n, u_\ell u_{\ell+1} \notin E(G(t))$. □

Observe that, if one of the parts in Proposition 19 consists of a singleton, say $S = \{i\}$, then Conjecture 16 would be true.

Let us show how to use the dynamic graph to prove an invisible lonely runner theorem, similar to Theorem 5.

Proposition 20. *There exists $t \in [0,1)$ such that $G(t)$ has either at least one isolated vertex or at least four vertices of degree one.*

Proof. Define $Y : [0,1) \to \mathbb{N}$ by,

$$Y(t) = |E(G(t))| .$$

Let $t \in [0,1)$ be chosen uniformly at random. Then $Y(t)$ is a random variable over $\left\{0,1,\dots,\binom{n}{2}\right\}$. We will show that $\mathbb{E}(Y(t)) = n-1$. If we are able to do so, since $Y(t)$ is not constant, by a first moment argument, we know that there exists a time t_0 for which $Y(t_0) \leq n-2$. Then, denoting by d_i the degree of u_i, we have

$$\sum_{i=1}^{n} d_i \leq 2(n-2) .$$

Suppose that there are no isolated vertices. Then $d_i > 0$ for all i and this ensures the existence of at least 4 vertices of degree one, concluding the proof of the proposition.

Now, let us show that $\mathbb{E}(Y(t)) = (n-1)$. We can write $Y(t) = \sum_{i<j} Y_{ij}(t)$, where $Y_{ij}(t) = 1$ if u_i and u_j are connected at time t and $Y_{ij}(t) = 0$ otherwise. Then $\mathbb{E}(Y(t)) = \sum_{i<j} \mathbb{E}(Y_{ij}) = \sum_{i<j} \Pr(I_i(t) \cap I_j(t) \neq \emptyset)$. For the sake of simplicity when computing $\Pr(I_i(t) \cap I_j(t) \neq \emptyset)$, we can assume that $v_i = 0$. Since the intervals are half open, half closed, we have $\Pr(I_i(t) \cap I_j(t) \neq \emptyset) = 2/n$, no matter the value of v_j.

Finally,

$$\mathbb{E}(Y(t)) = \sum_{i<j} \frac{2}{n} = \binom{n}{2}\frac{2}{n} = n-1 .$$

□

In the dynamic interval graph setting, an invisible runner is equivalent to a vertex u with a neighbor of degree one, say v. If u is removed, then v becomes isolated in $G(t)$ and thus, alone in the runner setting. Thus, Theorem 6 is a direct corollary of Proposition 20.

4 Concluding remarks and open questions

1. In Proposition 2 we gave a lower bound for $\mathbb{E}(X^2)$. We believe that this proof can be adapted to show that $\mathbb{E}(X^2)$ is larger.

Conjecture 21. For every set of different speeds $v_1, \dots, v_n$, and every $\delta < 1$, we have

$$\mathbb{E}(X^2) \geq (1+o(1))4\delta^2 n^2 + 2\delta n .$$

The proof of this conjecture relies on showing that either most pairs are ε–good or the contribution of the positive error terms is larger than the contribution of the negative ones. On the other hand, notice that it is not true that $\mathbb{E}(X^2)$ is $O(\delta^2 n^2)$. For the set of speeds in (1), Cilleruelo [**?**] has shown that

$$\mathbb{E}(X^2) = (1+o(1))\frac{12}{\pi^2}\delta n \log n , \tag{16}$$

which is a logarithmic factor away from the lower bound in Proposition 2, when $\delta = \Theta(n^{-1})$. It is an open question whether (16) also holds as an upper bound for $\mathbb{E}(X^2)$.

2. Ideally, we would like to estimate the probabilities $\Pr(\cap_{i\in S} A_i)$, for every set $S \subseteq [n]$ of size k. In general, it is not easy to compute such probability. As in (16), the sum of the k–wise join probabilities are not always of the form $O(\delta^k n^k)$. However it seems reasonable to think that, for every set S, we have

$$\Pr(\cap_{i\in S} A_i) \geq c_k \delta^k \ ,$$

where c_k depends only on k. Moreover, we know that $c_k \leq 2^k$, since this is the case when the speeds $\{v_i\}_{i\in S}$ are rationally independent and the conjecture holds (see e.g. Horvat and Stoffregen [?].)

The inequality (8) shows that $c_2 = 2$. In general, we also conjecture that

$$c_k = (1 + o_k(1))2^k \ .$$

3. Below we give a short proof of the result by Chen and Cusick [?] improving the bound on the lonely runner problem with n runners when $p = 2n-3$ is a prime number.

Proposition 22. *If $2n-3$ is a prime number then, for every set of n speeds, there exists a time $t \in \mathbb{R}$ such that*

$$\|tv_i\| \geq \frac{1}{2n-3} \ ,$$

for every $i \in [n]$.

Proof. Let $p = 2n-3$. We may assume $p > 7$. For a positive integer x let $\nu_p(x)$ denote the smallest power of p in the p–adic expansion of x. Set $m = \max_i \nu_p(v_i)$ and $N = p^{m+1}$. We consider the problem in the cyclic group $\mathbb{Z}_N$. We will show that there is $t \in \mathbb{Z}_N$ such that $\|tv_i\|_N := \min\{tv_i \pmod N, -tv_i \pmod N\}$, that is, the circular distance in $\mathbb{Z}_N$ from tv_i to 0, satisfies $\|tv_i\|_N \geq p^m = N/p$. This clearly implies that $\|tv_i\| \geq 1/p$ and proves the Proposition.

If $m = 0$ then we have $\|v_i\|_p \geq 1$ for each i and there is nothing to prove, so assume $m > 0$.

For each positive integer x we denote by $\pi_j(x)$ the coefficient of p^j in the p–adic expansion of the representative in $\{0, 1, \ldots, N-1\}$ of x modulo N. We seek a certain t such that, for each i, $\pi_m(tv_i)$ does not belong to $\{0, p-1\}$. This implies that $\|tv_i\|_N \geq p^m$ for each i, which is our goal.

Partition the set $V = \{v_1, \ldots, v_n\}$ of speeds into the sets $V_j = \{v_i : \nu_p(v_i) = j\}$, $j = 0, 1, \ldots, m$. Since $\gcd(V) = 1$ we have $V_0 \neq \emptyset$. By the definition of m, we also have $V_m \neq \emptyset$. We consider two cases:

Case 1. $|V_0| < n-1$. This implies $|V_j| < n-1$ for each j.

For each $\lambda \in \{1, \ldots, p-1\}$ and each $v_i \in V_m$ we have $\pi_m(\lambda v_i) = \lambda \pi_m(v_i) \pmod p$, which is nonzero. By pigeonhole, there is λ such that $\pi_m(\lambda v_i) \notin \{0, p-1\}$ for each $v_i \in V_m$. (Actually, for speeds in V_m it is enough that $\pi_m(\lambda v_i) \neq 0$.)

Suppose that there is $\lambda_{j+1} \in \{0, 1, \ldots, p-1\}$ such that $\pi_m(\lambda_{j+1} v_i) \notin \{0, p-1\}$ for each $v_i \in V_{j+1} \cup V_{j+2} \cup \cdots \cup V_m$ and some $j+1 \leq m$. Since $|V_j| < n-1$, there is $\mu \in \{0, 1, \ldots, p-1\}$ such that

$$\pi_m((1 + \mu p^{m-j})v_i) = \pi_m(v_i) + \mu \pi_j(v_i) \pmod p$$

does not belong to $\{0, p-1\}$ for each $v_i \in V_j$. Moreover, for each $v_i \in \cup_{l>j} V_l$,

$$\pi_m((1 + \mu p^{m-j})\lambda_{j+1} v_i) = \pi_m(\lambda_{j+1} v_i).$$

Therefore, by setting $\lambda_j = (1 + \mu p^{m-j})\lambda_{j+1}$, we have $\pi_m(\lambda_j v_i)$ not in $\{0, p-1\}$ for each $v_i \in \cup_{l\geq j} V_l$. Hence the sought multiplier is $t = \lambda_0$.

Case 2. $|V_0| = n-1$. Thus $V = V_0 \cup V_m$ with $|V_m| = 1$, say $V_m = \{v_n\}$. For each $\lambda \in \mathbb{Z}_N^*$ we have $\|\lambda v_n\|_N \geq p^m$. We may assume that $v_1 = 1 \pmod N$. Let

$$A_i = \{\lambda \in \mathbb{Z}_N^* : \|\lambda v_i\|_N < p^m\}$$

denote the set of bad multipliers for v_i. By choosing a multiplier in $\mathbb{Z}_N^*$ uniformly at random we have

$$\Pr(A_i) = 2p^{m-1}(p-1)/p^m(p-1) = 2/p.$$

If we show that $\Pr(A_1 \cap A_i) > 2/p(p-1)$ for each $i = 2, \dots, n-1$ then, by using Hunter's inequality 13,

$$\Pr(\cap_{i=1}^{n-1}\overline{A_i}) \geq 1 - \Pr(\cup_{i=1}^{n} A_i) \geq 1 - \left(\sum_{i=1}^{n-1}\Pr(A_i) - \sum_{i=2}^{n-1}\Pr(A_1 \cap A_j)\right) > 1 - \left(1 + \frac{1}{p} - \frac{p-1}{2}\frac{2}{p(p-1)}\right) = 0,$$

which implies that there is $\lambda \in \mathbb{Z}_N^*$ such that $\|\lambda v_i\|_N \geq p^m$ for all i, concluding the proof.

Suppose $m \geq 2$. Consider the set $C = [0, p^m] \cap \mathbb{Z}_N^*$, so that $\lambda \in A_i$ if and only if $\lambda v_i \in C \cup (-C)$. The pairwise disjoint sets

$$C_j = \{jp+1, 2(jp+1), \dots, (p-1)(jp+1)\},\ j = 0, 1, \dots, p^{m-2}+1,$$

satisfy

$$C_j \subset C,\ \ (C_j - C_j) \setminus \{0\} = C_j \cup (-C_j)\ \text{ and }\ (C_j - C_j) \cap (C_{j'} - C_{j'}) = \{0\}, j \neq j'.$$

The first inclusion holds because no two elements in C_j are congruent modulo p and the largest element in C_j is at most $(p-1)(p^{m-1}+p+1) = p^m - p^{m-1} + p^2 - 1 < p^m$ if $m \geq 3$ and $(p-1)(p+1) = p^2 - 1$ if $m = 2$. The last two equalities clearly hold.

Fix $i \in \{2, \dots n-1\}$. We have $\lambda v_i \in \mathbb{Z}_N^*$ for each $\lambda \in C_j$ and each C_j. By pigeonhole, if $\lambda v_i \notin C \cup (-C)$ for each $\lambda \in C_j$ then there are distinct $\lambda, \lambda' \in C_j$ such that $\lambda v_i, \lambda' v_i \in C + kp^m$ for some $k \in \{1, 2, \dots, p-2\}$. Therefore $(\lambda - \lambda')v_i \in C \cup (-C)$. Thus, for each j there is $\mu \in (C_j - C_j) \setminus \{0\}$ such that both $\mu v_i, -\mu v_i \in C \cup (-C)$. Since $(C_j - C_j) \cap (C_{j'} - C_{j'}) = \{0\}$, we have

$$\Pr(B_1 \cap B_i) \geq \frac{2(p^{m-2}+2)}{p^{m-1}(p-1)} > \frac{2}{p(p-1)}.$$

It remains to consider the case $m = 1$. In this case the same argument as above with an only $C_0 = \{1, 2, \dots, p-1\}$ suffices to give $\Pr(B_1 \cap B_i) \geq 1/(p-1)$. □

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