SHEAR DESIGN AND ASSESSMENT OF REINFORCED AND PRESTRESSED
CONCRETE BEAMS BASED ON A MECHANICAL MODEL

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Abstract
Safe and economical design and assessment of reinforced (RC) and prestressed concrete (PC) beams requires the availability of accurate but simple formulations which adequately capture the structural response. In this paper, a mechanical model for the prediction of the shear-flexural strength of PC and RC members with rectangular, I or T sections, with and without shear reinforcement, is presented. The model is based on the principles of concrete mechanics and on assumptions supported by the observed experimental behavior and by the results of refined numerical models. Compact, simple and accurate expressions are derived for design and verification of the shear strength, which incorporate the most relevant shear transfer actions. Excellent agreement between the predictions of the model and the results of the recently published ACI-DAfStb databases, including more than 1287 tests on RC and PC beams with and without stirrups, has been observed. The theory behind the model provides consistent explanations for many aspects related to the shear response that are not clearly explained by current code formulations, making it a very helpful tool for daily engineering practice.

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**INTRODUCTION**

The strength of structural concrete members subjected to bending and shear is affected by the existing multi-axial state of stress, the anisotropy induced by diagonal concrete cracking, the bond between concrete and reinforcement and the brittle failure, among other phenomena. Furthermore, prestressing considerably modifies the stresses and strains fields, the cracking load and the crack patterns, affecting not only the structural response under service loads, but also the mode of failure and the ultimate capacity in comparison to reinforced concrete members.

In order to predict such a complex behavior, refined analytical and numerical models have been developed (Bairán and Mari 2006, 2007; Bentz 2000; Ferreira et al. 2013; Mohr et al. 2010; Navarro-Gregori et al. 2007; Petrangeli et al. 1999; Saritas and Filippou 2009; Vecchio and Collins 1986; Vecchio 2000). Particularly, it is possible to account among other aspects for crack-induced anisotropy, which brings in the migration of shear stresses through the cross-section as the cracks develop and the crack width grows. These models have contributed to better understand the evolution of the resisting mechanisms according the experimentally observed behavior.

The applicability of the aforementioned models in daily engineering practice, however, is still limited because of their complexity, time consumption and dependency on the numerous input parameters required. On the opposite, most simple equations used in practice are semi-empirical, without a clear mechanical meaning, present large scatter and bias when compared with databases of experimental results and cannot be easily adapted to the technical advances (new materials, external strengthening, etc.). Furthermore, empirical approaches make the application of performance-based-design difficult.
With the purpose of providing useful formulations for daily engineering practice, simplified models for the shear strength of RC and PC members, based on sound theories have been also developed. The most relevant among them, are those carried out by (Choi and Hong 2007; Collins et al. 2008; Muttoni and Fernández Ruiz 2008; Recupero et al. 2003; Reineck 1991; Tureyen and Frosch 2003; Wolf and Frosch 2007; Zararis and Papadakis, 2001), some of which have been included in structural design codes, such as the CSA A23.3-14 (2014) or the FIB Model Code (2013). On the contrary, other codes have incorporated empirical formulations, such as the ACI-318 Building Code (2011) and the Eurocode 2 (2002) for concrete structures, especially for beams without transverse reinforcement. Nevertheless, the shear strength of structural concrete members is still an open topic: as a matter of fact, no universally accepted simplified and accurate design formulation, which is capable to be adapted or extended to many different situations without the need for adjustments to existing or new experimental results, with similar levels of accuracy in all cases, is yet available.

For prestressed concrete members the debate is even more open, despite the extensive research carried out during more than six decades, oriented to identify the governing parameters, their influence on the shear strength and the failure modes. Relevant works were the early tests performed Zwoyer and Siess (1954), by Evans and Schumacher (1963), by Olesen et al. (1967), and by Kar (1968), on pre-tensioned concrete beams. Variables included in their studies were the level of prestressing (P/A), the longitudinal and shear reinforcement ratios, the shape of the cross section, the shear span a and the type of curing. Among the different modes of failure identified, Muguruma et al. (1983), studied the web crushing of prestressed concrete beams, the main variables being the ratio of yield force in the prestressed steel to yield force in the non-prestressed steel and the area of vertical stirrups. With respect to the concrete strength, Elzanaty et al. (1986) studied the shear capacity of prestressed concrete beams using high strength concrete and Choulli et al. (2008) tested high strength self-
compacting concrete beams. In both cases, a reduction of the shear strength was found with respect to that of conventional concrete, so that a limitation in the concrete compressive strength of about 65 MPa was recommended when computing the shear strength.

In order to assess existing post-tensioned concrete bridges, Rupf et al. (2013), tested post-tensioned I beams with draped tendons and low amounts of shear reinforcement. They found that the shear strength and the failure mode of prestressed concrete girders are significantly influenced by the amount of shear reinforcement, the level of post-tensioning force, and the presence of flanges. De Silva et al. (2006), studied experimentally the influence of longitudinal and transverse reinforcement ratios, prestressing level and concrete strength on the crack width and spacing in partially prestressed concrete members.

Related to shear design models, Wolf and Frosch (2007) made relevant contributions proposing a simplified model based on the shear transfer capacity of the un-cracked concrete region, and including the contribution of the flanges in I beams. Park et al. (2013) extended their strain-based strength model to prestressed concrete structures by performing a layered analysis of the compressed zone. Zhang et al. (2014a, 2014b) developed a shear strength numerical model based on plasticity concepts, bond slip, crack widening and shear-flexure interaction. One of their conclusions was, as also reported by Mari et al. (2015), that the stirrups enhance the shear transfer capacity of the un-cracked concrete chord.

Furthermore, some aspects related to the shear behavior and shear strength are not clearly explained by the existing simplified formulations. Some examples are: the influence of the load level, of the amount of transverse reinforcement or of the presence of axial forces, on the concrete contribution, $V_c$; the contribution of the flanges in beams with T or I cross-section to the shear strength; the influence of the eccentricity of the post-tensioning tendons on the shear strength; and the reason why the shear strength is influenced by the longitudinal reinforcement and the position of the critical shear crack.
In order to answer these and other similar questions in daily engineering practice, simple but accurate models supported by a theory capable to capture the physics of the shear response are needed.

Recently, a mechanical model for the prediction of the shear strength of reinforced concrete beams with rectangular, I or T shaped sections, with or without transverse reinforcement, has been developed by Cladera et al. (2015) and Marí et al. (2015). The predictions of the model were compared with a large database of tests results on RC beams, already published by Reineck et al. (2013, 2014), showing small bias and dispersion. This model has also been extended to FRP reinforced concrete beams (Marí et al. 2014; Oller et al. 2015).

In this paper, the above mentioned shear model is generalized to the case of prestressed concrete members. For this purpose, the structural effects of prestressing on the shear response and on the different modes of failure are identified and accounted for in the derivation of the equations governing the model. The model predictions are compared with those of a large database of prestressed concrete beams with and without stirrups obtaining, in general, very good results. In addition, the influence of some relevant parameters on the shear response is studied and compared with the experimentally observed tendencies. A design example, based on a beam tested by Oh and Kim (2004), is fully developed to show the practical applicability of the model. Finally, conclusions are drawn about the capability of the model to provide rational explanations for the observed behavior.

**MECHANISM OF SHEAR RESISTANCE**

According to ASCE-ACI Committee 445 (1998), the shear strength in a slender RC beam is provided by: a) the shear resisted by the uncracked concrete chord; b) the friction and residual tensile forces developed along the crack; c) the shear strength provided by the transverse reinforcement; and d) the dowel action developed by the longitudinal
reinforcement. Under incremental loading, inclined flexural cracks develop along the web, forming the first branch of the critical crack (see Figs. 1 and 2). During a relatively large portion of the load history, a significant fraction of the shear force is transferred through the inclined cracks by means of aggregate interlock and residual tensile stresses, both produced by the bridging of stresses through the aggregates and the uncracked concrete at the meso-scale. However, while loading increases, the critical crack width increases, reducing the shear transferred along the crack. Then, a redistribution of shear stresses to the advantage of other resisting actions takes place, as is needed to satisfy the internal equilibrium conditions. In general, the shear stresses in the compression chord, which has remained uncracked during most part of the loading, tend to increase. Therefore, the compression chord will be subjected to a multi-axial stress state consisting in, at least, axial compression and shear.

Eventually, a new branch of the critical crack develops above the neutral axis through the compression chord, at the point where the concrete strength under multiaxial stresses is first reached. At the same time, there is an increment of force in the longitudinal reinforcement due to inclined cracks in the web. Near the supports, this fact, jointly with the dowel action, may produce longitudinal bond cracks in the concrete in contact with the tensile flexural reinforcement. If premature bond failure does not take place, failure of the element will be controlled by the shear capacity of the compression chord as it is the last element that typically initiates softening, reducing its capacity as the crack propagates. Resistance of the compression chord is assumed in this work to be governed by Kupfer’s biaxial failure envelope (Kupfer and Gerstle 1973) (Fig. 3), mainly in the compression-tension branch, when vertical compression stresses are negligible or small in comparison to longitudinal ones. This failure criterion is in accordance with that proposed by Zwoyer and Siess more than 60 years ago, when the concept that shear failures could be considered as a type of premature compression failure (Zwoyer and Siess 1954) was developed, and with the work of Kar in
prestressed concrete beams related to the internal stress redistribution and the strain
concentration in the concrete chord (Kar 1968).

Prestressing introduces a set of self-balanced forces on concrete structures which modify
the stresses, internal forces, strains, deflections and failure modes. The most important effects
of prestressing on the shear response, which will be accounted for in the proposed model, are
briefly recalled next.

In partially prestressed members, in which flexural cracks may develop at service under
certain load combinations, a shear-flexure failure mechanism may take place, as it occurs in
many reinforced concrete members. In such cases the cracks inclination, the neutral axis
depth, the stress levels and, consequently, the shear transferred by the uncracked concrete
chord and by the stirrups, are affected by the level of the prestressing force. Fig. 4 shows a
prestressed specimen tested at TU Dortmund in the course of a research project funded by the
German Federal Highway Research Institute (BASt) (Maurer et al. 2014). Initial flexural
cracks appeared at $P=700$ kN, first shear-flexural cracks were observed at $P=1200$ KN and
failure took place at $P=1890$ kN. It can be observed how the angle of inclination of the cracks
near the centroid of the beam section is very small (about 22°), because the effect of the
compressive stresses introduced by the prestressing force.

In totally prestressed members, mostly without shear reinforcement, inclined cracks may
develop and propagate along the web, without the formation of flexural cracks, thus
preventing the formation of a compression chord and the development of a flexure-shear
resisting mechanism. In addition, in members with thin webs the compression stresses due to
prestressing may increase the risk of concrete crushing. These two failure modes will be dealt
with later in the paper. Finally, in the case of inclined tendons, the vertical component of the
prestressing force, $V_p = P \cdot \sin \alpha$ reduces the external shear force.
DESCRIPTION OF THE MODEL

Basic assumptions

In the present model, it is considered that the total shear resistance, Eq. (1), is the sum of the shear resisted by concrete and by the transverse reinforcement \( V_s \). The concrete contribution is explicitly separated into the following components, whose importance is considered to be variable as damage propagates: shear resisted in the uncracked compression chord \( V_c \), shear transferred across web cracks \( V_w \) and the contribution of the longitudinal reinforcement \( V_l \). Note that here \( V_c \) represents the shear in the compression chord of the beam, not the total concrete contribution to shear.

\[
V = V_c + V_w + V_l + V_s = f_{ct} \cdot b \cdot d \cdot (v_c + v_w + v_l + v_s)
\]  

(1)

Lower case variables \( v_c, v_w, v_l \) and \( v_s \) are the dimensionless values of the shear transfer actions and \( f_{ct} \) is the concrete tensile strength defined as given by Eq. (2), and not greater than 4.60 MPa, as for high-strength concrete the shear strength does not increase significantly due to the fracture of the aggregates (Cladera and Marí 2004, 2005).

\[
f_{ct} = 0.30 \cdot \sqrt[2]{f_{c}^{2}} \leq 4.60 \text{ MPa}
\]  

(2)

As the crack opens, the aggregate interlock and residual tension decrease, and an increase in the shear transferred by the compression concrete chord takes place. Hence, it will be assumed that at the limit state, previous to incipient failure, the shear stress distribution in the critical section is similar to the one represented in Fig. 5, where the approximated distribution of each contributing action is also indicated, \( c \) is the neutral axis depth and \( d \) the effective depth of the cross-section. Note that in Fig. 5, the stage just prior to the formation of the crack in the compression chord is considered. This stress profile is a qualitative distribution of the stresses in a section close to that of the tip of the first branch of the critical crack, and is not affected by the local state of stress around the crack tip.
In developing the design formulation, some additional simplifications are also considered and described in the following.

**Neutral axis depth**

Neutral axis depth \( (c) \) and height of uncracked zone are treated as equivalent, for design purposes. Furthermore, in the case of shear failure prior to flexural yielding, longitudinal reinforcement is assumed to remain in the elastic range. The distribution of normal compressive stresses in the uncracked zone will be then considered linear; hence, the position of the neutral axis can be computed as that of a cracked section in the elastic range. A methodology to compute the neutral axis depth in prestressed concrete members is explained next. Fig. 6 shows a typical cross section of a prestressed concrete beam subjected to a bending moment \( M \) and a prestressing force \( P \) with an eccentricity \( e \). To take into account the contribution of the different types of reinforcement that may exist in the cross section (only mild reinforcement, \( A_s \), only active reinforcement, \( A_p \), or both), the effective depth will be defined as:

\[
d = \frac{A_s d_s + A_p d_p}{A_s + A_p}
\]

(3)

where \( d_s \) and \( d_p \) are the distances between the maximum compressed concrete fiber and the centroid of the mild steel tensile reinforcement and the prestressing reinforcement placed in the tensile zone of the cross-section, respectively.

In order to accurately calculate the neutral axis depth in a cracked prestressed concrete section, the equilibrium equations between the internal forces (axial forces and bending moments) and the stresses in the concrete and in the mild and prestressing reinforcements must be solved, considering also the strain compatibility provided by the plane section assumption. Due to the presence of the axial load, both equilibrium equations are coupled and, as a consequence, the neutral axis depth depends on the mild and prestressing reinforcement
amounts, on the bending moment, on the prestressing force and on its eccentricity. Therefore, its computation is not straightforward, requiring an iterative procedure to solve the non-linear set of equations (Saqan and Rasheed 2011).

For practical purposes, it is proposed to obtain \( c \) by means of a linear interpolation between the neutral axis depth \( c_0 \) obtained for the same reinforcement amounts, considering \( P=0 \), and the total depth \( c=h \), corresponding to the decompression prestressing force, i.e. the force applied at the considered eccentricity, which, in combination with an external moment \( M \), produces zero stress at the most tensioned fiber (see Fig. 6). The interpolation equation for the relative neutral axis depth, \( c/d \) becomes:

\[
\frac{c}{d} = \frac{c_0}{d} + \left( \frac{h}{c_0} - \frac{h}{d} \right) \frac{P k_t}{M - P e} \tag{4}
\]

where \( k_t = I_c/(A_c \cdot v_{bot}) \) is the upper limit of the central kern, \( I_c \) and \( A_c \) are the second moment of inertia and the area of the cross section, respectively, and \( v_{bot} \) is the distance between the centroid and the most tensioned fiber of the section.

The neutral axis depth for \( P=0 \) (reinforced concrete) can be obtained by means of the following approximate expression:

\[
\frac{c_0}{d} = n \rho_t \left( 1 + \sqrt{1 + \frac{2}{n \rho_t}} \right) \tag{5}
\]

where \( \rho_t \) is the equivalent longitudinal tensile reinforcement ratio and \( n \) is the modular ratio.

For members with mild steel reinforcement and tendons, \( n \rho_t \) can be determined as follows:

\[
n \rho_t = n_s \rho_s + n_p \rho_p \quad ; \quad n_s \rho_s = \frac{E_s}{E_c} \frac{A_s}{b d} \quad ; \quad n_p \rho_p = \frac{E_p}{E_c} \frac{A_p}{b d} \tag{6}
\]

where \( b \) is the width of the cross-section, which is equal to the flexural effective compression flange width, for T or I-shaped sections; \( E_c \) is the secant modulus of elasticity of concrete according to Eurocode 2 (see Eq. (7)), and not greater than 39000 MPa for ULS shear calculations.
\[ E_c = 22000 \left( \frac{f_{cu}}{10} \right)^{0.3} \leq 39000 \text{ MPa} \]  

As will be seen later, it is conservatively assumed in this model that the critical shear crack develops from the last flexural crack, considering that the bending moment at ULS at the initiation of the crack equals to the cracking moment. Therefore, when \( M \) is the cracking moment in Eq. (4), it can be expressed as Eq. (8)

\[
\frac{c}{d} = \frac{c_p}{d} + \left( \frac{h - c_p}{h} \right) \left( \frac{d}{h} \right) \frac{\sigma_{sp}}{\sigma_{ct} + f_{ct}}
\]

where \( \sigma_{sp} = P/A_c \) is the concrete compression stress at the centroid of the section produced by the prestressing force. Eq. (8) has been compared with the theoretical solution by solving the non-linear system of coupled equations, for a large number of cases covering different levels of prestressing \((\sigma_{sp}/f_{ct})\), eccentricities \((e/c)\) and non-prestressed neutral axis depth \(c_0\), obtaining good correlation for usual cases. Therefore, the straightforward Eq. (8) will be adopted in this study to obtain \(c/d\) for prestressed elements.

**Critical crack inclination**

Based on experimental and numerical observations of RC beams, the mean crack inclination is approximated by a function of the relative neutral axis depth \((c/d_s)\), by Eq. (9).

\[
\cot \theta = 0.85 \left( \frac{c}{d_s} \right) = \frac{0.85d_s}{(d_s - c)} \leq 2.5
\]

For the case of I or T shaped sections, there is a change of crack inclination in case that the crack reaches the flanges, see Fig. 7. After parametric studies, however, it has been observed that the influence of this fact on the shear strength is small so, for the sake of simplicity, it will not be taken into account in the formulation, thus obtaining a slightly conservative solution (Cladera et al. 2015).
The longitudinal normal stresses introduced by prestressing modify the angle of the principal stresses and, therefore, the angle of inclination of cracks. Since in prestressed concrete members the neutral axis depth increases, the change produced in the angle $\theta$ is automatically incorporated in the model by keeping the horizontal projection of the crack between its initiation and the neutral axis depth to $0.85 \cdot d_s$. The cot of the critical shear crack angle for prestressed concrete members, given by Eq. (9) will be, therefore, higher than in reinforced concrete members without axial load. In prestressed members without longitudinal mild reinforcement ($A_s=0$), cot$\theta$ can be computed adopting $d_s = d_p$ in Eq (9).

**Position of the critical crack**

In the present model, the weakest section in front of a combined shear-bending failure is considered to be placed at the tip of the first branch of the critical crack for beams with constant geometry and reinforcement (Fig. 8). Any other section closer to the zero bending moment point has a larger depth of the compression chord, produced by the inclination of the crack, and will resist a larger shear force. Any other section placed farther from the support will have the same depth of the compression chord, but will be subjected to higher normal stresses and, therefore, will have a larger shear transfer capacity.

When the load is increasingly applied, flexural cracks successively appear as the bending moment increases. The crack spacing depends basically on tensile concrete and bond properties (Fig. 9). It is assumed that the critical crack is the closest crack to the zero bending moment point and that it starts where the bending moment diagram at failure reaches the cracking moment of the section. As a result, $s_{cr} = \frac{M_{cr}}{V_u}$, which is a conservative assumption.

In prestressed concrete, the increment of cracking moment modifies the position where the critical crack initiates and the position of the critical section, which are displaced farther from the zero bending moment point than in RC structures (Fig. 9). Thus, the bending
moment at the critical section increases and, consequently, also the shear resisted by the compressed concrete chord.

**Vertical concrete stresses produced by transverse reinforcement**

When stirrups are anchored in the compression zone, they contribute to the strength of the compression chord by producing a confining vertical compression ($\sigma_y$) at a depth larger than the concrete cover ($d'$), see Fig. 10, of value given by Eq. (10):

$$
\sigma_y = \frac{A_{yw}f_{yw}}{b} = \rho_w f_{yw} = \frac{v_s f_{ct}}{0.85}
$$

(10)

where $v_s$ is the dimensionless contribution of the transverse reinforcement to the shear strength, given by Eq. (11).

**Contribution of transverse reinforcement ($V_s$)**

The contribution of shear reinforcement, Eq. (11) is taken as the sum of the forces in the stirrups that intersect the inclined crack up to a height of ($d-c$), as shown in Fig. 5, and assuming that the transverse reinforcement is yielded along the entire crack height.

$$
v_s = \frac{V}{f_{ct} \cdot b \cdot d} = \frac{0.85 d_x A_{sw} \cdot f_{yw}}{f_{ct} \cdot b \cdot d}
$$

(11)

**Contribution of cracked concrete web ($V_w$)**

The shear resistance of cracked concrete in the web is considered as the residual tensile stress of cracked concrete. The mean tensile stress of the stress-strain curve is considered distributed along the depth $c_w$ of the cracked zone of the cross-section where the tensile $\sigma$-$\varepsilon$ curve reaches zero tension, see Fig. 11. A linear softening branch of the $\sigma$-$\varepsilon$ curve has been assumed, which is consistently dependent on the fracture energy in mode I ($G_f$).

By enforcing the compliance between the crack opening at the level of the longitudinal reinforcement and at the point of the crack where the residual tensile stress is zero, and relating the crack opening to the reinforcement longitudinal strain $\varepsilon_l$ and the crack spacing, the shear transferred along the closest part of the crack can be expressed as a function of $\varepsilon_l$. At
ULS a strain in the reinforcement close to 0.0009 has been assumed, resulting the same expression for reinforced concrete and prestressed concrete members to simplify the calculation procedure, see Eq. (12). The complete derivation of this equation can be found in Marí et al. (2015).

\[
v_w = \frac{V_w}{f_{ct} \cdot b \cdot d} = 167 \frac{f_{ct}}{E} \frac{b}{b} \left(1 + \frac{2 \cdot G_f \cdot E_c}{f_{ct}^2 \cdot d_0}ight)
\]

(12)

where \(b_w\) and \(b\) are the web and the flange widths, respectively; \(f_{ct}\) is the concrete tensile strength evaluated as the mean tensile strength according to Eurocode 2 (2002), but limiting the concrete compression strength to 60 MPa in order to account for the possible fracture of the aggregate in high-strength concrete; \(G_f\) is the fracture energy of concrete, which depends on the concrete strength and the aggregate size, computed as shown in Eq. (13) (Marí et al. 2015); and \(d_0\) is the effective depth, \(d\), but not lesser than 100 mm.

\[
G_f = 0.028 \cdot f_{cm}^{0.18} \cdot d_{max}^{0.32}
\]

(13)

where \(f_{cm}\) is the mean concrete compressive strength and \(d_{max}\) is the maximum aggregate size.

In the case of T- or I-shaped sections, it is accepted that the shear transferred along the crack takes place mainly in the web, so the web width \(b_w\) has been used in Eq. (12) when computing \(v_w\).

**Contribution of the longitudinal reinforcement (Dowel action, \(V_d\))**

Contribution of longitudinal reinforcement, or dowel action, is considered by the model only when there are stirrups, as they provide a constraint to the vertical movement of the longitudinal bars, enabling them to transfer shear. In the case of prestressed concrete members, only mild reinforcement or prestressing bars supported by the stirrups are considered to produce dowel action. In order to evaluate such contribution, it is considered that the longitudinal bars are doubly fixed at the two stirrups adjacent to the crack, and subjected to bending due to a relative imposed displacement between the two extremities.
This relative vertical displacement is caused by the opening of the critical crack and by the shear deformation of the compression chord. This contribution clearly depends on the tensile steel ratio, that is implicitly represented by means of $c/d$. A simplified expression is presented in Eq. (14). A more detailed derivation of the equation is carried out in Mari et al. (2015).

$$v_y > 0 \Rightarrow v_i = \frac{V_i}{f_{ce} \cdot b \cdot d} \approx 0.23 \cdot \frac{n \rho_l}{1 - \frac{c}{d}}$$

(14)

**Contribution of the compression chord ($V_c$)**

As previously mentioned, the shear transferred by the compression chord starts softening when the most stressed fiber reaches Kupfer’s failure envelope. The stresses at any point of the uncracked zone are obtained assuming a linear distribution of longitudinal flexural stresses and a constant transversal confining stresses provided by the existing stirrups. The position of the failure point depends on the ratio between bending moment and shear force ($M/Vd$) at the considered critical section. A study performed by Mari et al. (2015) shows that for values of $M/Vd$ less than 3 (which is usually the position of the critical shear section), the point where failure initiates is placed at a distance from the neutral axis around $\lambda = 0.425c$, being $c$ the depth of the compressed zone.

By means of a Mohr’s circle analysis, the shear stress $\tau_\lambda$ at the point where failure initiates, can be related to the principal and normal stresses by Eq. (15)

$$\tau_\lambda = \sigma_1 \sqrt{1 - \left(\frac{\sigma_x + \sigma_y}{\sigma_1}\right)^2 - \frac{\sigma_x \sigma_y}{\sigma_1^2}}$$

(15)

Once the value of the shear stress at three points is known, $\tau(0)=0$, $\tau(c)=0$ and $\tau(\lambda)=\tau_\lambda$, the equation for the assumed parabolic shear stress distribution (Fig. 5) is obtained and, by integration, the shear force transferred by the compression chord is derived, Eq. (16):

$$V_c = \int_0^c \tau(y) \cdot b \cdot dy \quad ; \quad v_c = \frac{V_c}{f_{ce} \cdot b \cdot d} = \frac{c}{b} \cdot K_\lambda R_c \cdot b_{y,eff} \cdot R_c \cdot b \cdot c \cdot \left(1 \cdot \frac{\sigma_x + \sigma_y}{\sigma_1} \right)$$

(16)
where $\sigma$ is the normal stress in the most stressed fiber, located at a distance $\lambda c$ from the neutral axis; $K\lambda$ is a parameter relating the mean shear stress in the compression chord with the shear stress in the most stressed fiber, and $b_{v_{,\text{eff}}}$ is an effective width, to take into account that for T or I sections the shear stresses along the depth of the compression chord are concentrated in the portion of the flanges near the web:

\begin{align}
  c \leq h_f & \quad b_{v_{,\text{eff}}} = b_v = b_w + 2h_f \leq b \\
  c > h_f & \quad b_{v_{,\text{eff}}} = b_v \eta + b_w (1 - \eta) \quad ; \quad \eta = 3 \left( \frac{h_f}{c} \right)^2 - 2 \left( \frac{h_f}{c} \right)^3
\end{align}

where $b$ and $b_w$ are the width of the flange and the web respectively; $h_f$ is the flange depth, as defined in Fig. 6. The neutral axis depth $c$ must be calculated accounting for the geometry of the T or I section, as is proposed in Cladera et al (2015). For rectangular sections, $b = b_w$ and $b_{v_{,\text{eff}}}/b = 1$.

In order to obtain the concrete contribution to the shear strength in PC members, the prestressing force, with its inclination and eccentricity, must be included in the equilibrium of forces and moments. Fig 12 shows the internal forces acting on a portion of beam placed over the critical shear crack.

$\Delta P$ in Fig. 12 is the increment of force in the active reinforcement due to the crack opening, which will be neglected in this work assuming in a simplified manner the same shear strength for members with bonded and unbonded tendons. Then, the equilibrium equations are:

\begin{align}
  C &= T + V_w \tan \theta + P \cos \alpha \\
  V &= V_c + V_w + V_s + P \sin \alpha \\
  C \cdot z_s &= M_{Ed} + V_c 0.85 d_s + V_w \beta_w d_s + 0.425 \cdot V_s d_s + P (d_s - d_p) \cos \alpha
\end{align}
where equilibrium of moments is taken with respect to the point where the crack is crossed by
the longitudinal reinforcement.

The compressive normal stress at the failure point of the concrete chord (at a distance \(\lambda c\)
from the neutral axis) is given by:

\[
\sigma_x = \frac{2\lambda c}{bc} \left( M_{Ed} + P(d_s - d_p) \cos \alpha + V_c \cdot 0.85d_s + V_{cw} \cdot d_s + 0.425 \cdot V_s \cdot d_s \right) \tag{21}
\]

The confining stresses (\(\sigma_c\)) are given by Eq. (10) as a function of the contribution of the
transverse reinforcement, \(v_s\), (Eq. 11) as \(\sigma_{cy} = f_{ct} v_s / 0.85\).

By setting the equilibrium equations between the internal forces and the stress resultants, shown in Fig. 12, Eq. (16) can be rewritten as

\[
v_c = \zeta K R_i \frac{c b_{\text{eff}}}{d b} \sqrt{\frac{2\lambda}{1 - \frac{c}{d} \left(1 - \frac{c}{3d_s}\right) R_i} \left( \frac{V_s}{0.85R_i} - 1 \right) - \frac{V_s}{0.85R_i}} \tag{22}
\]

where \(\mu\) is the non-dimensional moment at shear failure at the crack initiation given by Eq.
(23) (Fig. 9); \(\mu_p\) is the non-dimensional moment with respect to the mild reinforcement, due
to prestressing given by Eq. (24); \(z_w\) is the lever arm of \(v_w\) with respect to point A (Fig. 12) (a
conservative value of \(z_w=0.425d\) can be adopted without much error, since both \(v_w\) and the
extension of residual stresses along the crack, at failure, are usually small); \(\zeta\) is the parameter
accounting for size effect in the compression chord, which can be assimilated to that of a
splitting test, as proposed by Zararis and Papadakis (2001), and can be computed by Eq. (25).

\[
\mu = \frac{M_{Ed}}{f_{ct} b d^2} \tag{23}
\]

\[
\mu_p = \frac{P \cos \alpha (d_s - d_p)}{f_{ct} b d^2} \tag{24}
\]

\[
\zeta = 1.2 - 0.2 a \geq 0.65 \quad (a \text{ in meters}) \tag{25}
\]
It is observed that \( v_c \) depends on \( v_s \) and that Eq. (22) is a recurrent equation, since \( R_t \) (strength reduction factor due to the biaxial stress state) is not known a priori; hence, in general, an iterative procedure is required to find \( v_c \). After solving Eq. (22), however, it was found that \( v_c \) is almost a linear function of \( c/d \), so for practical purposes, Eq. (26) is proposed as a linearization of Eq. (22).

\[
v_c = \zeta \left[ \left. 0.88 + 0.20 + 0.50 \frac{b}{b_w} \right] v_s \frac{c}{d} + 0.02 \right] \frac{b_{v,\text{eff}}}{b} \left[ 0.94 + 0.3(\mu + \mu_p) \right]
\]  

(26)

Term 0.94+0.3(\mu+\mu_p) of Eq. (26) accounts for the influence of the bending moment on the concrete contribution, \( V_c \). Adopting, as a conservative assumption, that the bending moment at shear failure is equal to the cracking moment, given by Eq. (27)

\[
\mu = \mu_{cr} = \frac{1}{f_{ct}b_d^2} \left[ I_{ct} \frac{f_{ct}}{y_t} + P(e + k_t) \right]
\]  

(27)

taking into account that \( d_p = \nu_{top} + e \) (see Fig. 12) and after some mathematical arrangements, Eq. (26) becomes:

\[
v_c = \zeta \left[ \left. 0.88 + 0.20 + 0.50 \frac{b}{b_w} \right] v_s \frac{c}{d} + 0.02 \right] \frac{b_{v,\text{eff}}}{b} \left[ 1 + 0.3 \frac{P \cos \alpha (e + d_s - d_p)}{f_{ct}b_d^2} \right]
\]  

(28)

The right bracket in Eq. (28) will be called \( K_p \), a factor which takes into account the higher cracking moment in a prestressed concrete section, with respect to a reinforced concrete section. In case of reinforced concrete, \( P=0 \), the non-dimensional cracking moment (Eq. 27) becomes \( \mu_{cr}=0.2 \) and term 0.94+0.3(\mu+\mu_p) becomes equal to 1.

**Shear failure in regions without flexural cracks**

In case of highly prestressed simply supported concrete beams, such as some T or I beams, with minimum or no shear reinforcement, no flexural cracks take place near the supports even at high loading levels. In these regions, the beam web (usually thin) is subjected to high shear stresses produced by the applied shear force, combined with normal
compressive stresses produced by prestressing, generating a biaxial compression-tension state of stress. When, at the most stressed point of the web, see Fig. 13, the principal stresses reach the biaxial failure envelope, (see Fig. 3), a diagonal crack develops through the entire beam height (Fig. 14). It has been experimentally observed that the cracking load turns out to be almost equal to the ultimate load (Evans and Schumacher 1963; Elzanaty et al. 1986); because of the eccentricity of prestressing, the top compression stresses are not capable to control the diagonal crack propagation (see Fig. 14).

According to Mohr’s circle of stresses, the shear producing the principal stress $\sigma_l$ at the beam centroid is:

$$V_u = \frac{I_c}{S_c} \sigma_l \sqrt{1 + \alpha_t \frac{\sigma_{cp}}{\sigma}} = \frac{I_c}{S_c} R_t f_{ct} \sqrt{1 + \alpha_t \frac{\sigma_{cp}}{R_t f_{ct}}}$$  \hspace{1cm} (29)$$

where $I_c$ is the second moment of area, $S_c$ is the first moment of the area above and about the centroid, $\alpha_t$ is the factor which takes into account the portion of prestressing force transferred to concrete at the considered section, which is $\leq 1.0$ for pretensioned tendons, and is equal to 1.0 for other types of prestressing, $\sigma_{cp}$ is the concrete compressive stress at the centroid axis due to prestressing and $R_t$ is a tensile strength reduction factor due to the biaxial stress state, given by

$$\sigma_l = \left(1 - 0.8 \frac{\sigma_{cp}}{f_{ct}}\right) f_{ct} = R_t f_{ct}$$  \hspace{1cm} (30)$$

Usually $R_t$ ranges from 0.77 to 0.95, so a value of $R_t = 0.8$ outside the square root and $R_t = 1$ inside the square root will be conservatively adopted.

In the case of beams with shear reinforcement, once the web cracks, stirrups start working and a shear force higher than the cracking shear can be resisted, as observed by Elzanaty et al. (1986). As a consequence, higher bending moments take place near the supports which, in most cases, produce flexural cracks. Therefore, in beams with transverse
reinforcement, the shear strength will be calculated accepting flexural cracks, by means of the
model previously described in the previous sections which will provide a lower bound value.

**Concrete compression crushing in regions with shear cracks**

The beam failure may be governed by the concrete struts in beams with shear
reinforcement and relatively thin webs. In this case, the concrete diagonals would crush prior
to the yielding of the transverse reinforcement, resulting in a brittle failure (Aparicio et al.
1998; Rangan 1991). To check the maximum capacity of the struts, Eq. (31), this proposal
adopts the same formulation of the current Eurocode 2 (2002), with the angle of the
compression strut given by Eq. (9):

\[
V_{rd,\text{max}} = \alpha_{cw} b_d d_v f'_c \frac{\cot \theta}{1 + \cot^2 \theta}
\]

(31)

where \( \nu_1 \) is a strength reduction factor for concrete cracked in shear, taking into account the
transversal tensile stresses. The values given by Eq. (32) are adopted:

\[
\nu_1 = 0.6 \quad \text{for} \quad f'_c \leq 60 \, \text{MPa} \tag{32a}
\]

\[
\nu_1 = 0.9 - \frac{f'_c}{200} \quad \text{for} \quad f'_c > 60 \, \text{MPa} \tag{32b}
\]

and \( \alpha_{cw} \) is a coefficient taking account the state of stress in the struts, Eq. (33)

\[
\alpha_{cw} = 1 \quad \text{for non–prestressed structures} \tag{33a}
\]

\[
\alpha_{cw} = 1 + \frac{\sigma_{cp}}{f'_c} \quad \text{for} \quad 0 \leq \sigma_{cp} \leq 0.25 f'_c \tag{33b}
\]

\[
\alpha_{cw} = 1.25 \quad \text{for} \quad 0.25 f'_c \leq \sigma_{cp} \leq 0.50 f'_c \tag{33c}
\]

\[
\alpha_{cw} = 2.5 \left( 1 - \frac{\sigma_{cp}}{f'_c} \right) \quad \text{for} \quad 0.5 f'_c \leq \sigma_{cp} \leq f'_c \tag{33d}
\]

**Summary of the steps following the proposed general model**

The described model provides the contribution of each shear transfer action to the global
shear strength. In summary, to obtain the shear strength or to calculate the stirrups needed, it
is necessary to use Eq. (1) for the total shear strength, and Eqs. (11), (12), (14) and (28) for
the different non-dimensional contributions. Moreover, to calculate the neutral axis depth, Eq. (5) shall be used for RC structures. For prestressed elements, Eq. (8) is needed to obtain \( c/d \) taking into account the stress due to prestressing. Finally, the effective width of the concrete chord, in case of beams with I or T cross-sections, is computed by Eq. (17). Additionally, the maximum shear strength given by the struts should be checked, according to Eqs. (31)-(33).

**Simplified Equations in a Code-Type Format**

For design purposes, some simplifications are still necessary in order to make the model easier to use in daily engineering practice. Taking into account that when shear-flexure failure takes place, both the aggregate interlock and the dowel action are small compared to the shear resisted by the uncracked zone, \( v_w \) and \( v_l \) have been incorporated into \( v_c \). For this purpose, the following average values have been adopted: \( v_w = 0.035 \), \( v_l = 0.025 \), \( v_s = 0.25 \) and \( c/d = 0.35 \). Then, term \((v_w+0.02)/0.35 = 0.157\) has been added to 0.88 and the dowel action term \( v_l/(0.35 \cdot 0.25)\), which only exists when \( A_{st} > 0 \), has been added to the factor multiplying \( v_s \) in Eq. (28), resulting in the following compact equation:

\[
V_{cu} = \frac{c}{d} \cdot K_p \left[ 0.30 \frac{f_c^{2/3}}{\gamma_c} + 0.5 \left( 1 + \frac{b}{b_w} \right) \frac{V_s}{b_d} \right] b_{v,eff} d
\]  

(34)

where the design tensile strength of concrete, \( f_{cd} \), is expressed in terms of the concrete compressive strength, \( f'_{c} \), reduced by the corresponding partial safety coefficient, and \( K_p \) is the factor which takes into account the effects of prestressing, whose value is:

\[
K_p = 1 + 0.24 \frac{P \cdot y_t}{f'_{c} b_d^2}
\]  

(35)

where \( y_t \) is the distance from the centroid of the section to the most stressed fiber in tension.

In Eq. (35) a coefficient 0.24 has been used in spite of the original value of 0.30 (right parenthesis in Eq. 26), to take into account that the neutral axis depth in prestressed concrete sections (see Eqs. 5 to 8) is higher than the one assumed \((c/d = 0.35)\) to merge the different components into a single concrete contribution \( V_{cu} \).
As previously discussed (see Fig. 9), in the case of prestressed elements, the flexural cracking is displaced farther away from the support axis with respect to a similar element with $P = 0$. Consequently, the control cross-section also moves accordingly. To take this fact into account, in design, shear strength shall be checked at least at a distance $s_{cr} = d_s(1+0.4\frac{\sigma_{cp}}{f_{ct}})$ from the support axis, where $\sigma_{cp} = P/A_c$ is the mean concrete normal stress due to prestressing.

Then, the shear strength verification can be carried out as follows:

$$V_{Ed} = V_{Ed0} - P \sin \alpha \leq V_u = V_{cu} + V_{su} \quad \text{or alternatively} \quad V_{Ed} \leq V_u = V_{cu} + V_{su} + P \sin \alpha \quad (36)$$

where $V_{Ed0}$ is the design shear force acting at the control section, $V_{cu}$ is given by Eq. (29) and $V_{su}$ is:

$$V_{su} = (d_s - c)\cot \theta A_{nc} \frac{f_{syd}}{f_{ct}}$$

(37)

where $\theta$ is the mean crack inclination given by Eq. (9).

Shear reinforcement is necessary when the shear design force exceeds the shear resisted by the concrete, without transverse reinforcement, $V_{cu0}$.

$$V_{cu0} = 0.30 \frac{f_{ct}^{2/3}}{\gamma_c} \frac{c}{d} K_p b_{v,\text{eff}} d$$

(38)

In that case, the required strength contribution provided by the shear reinforcement is given by Eq. (39), and the area of shear reinforcement will be obtained using Eq. (37).

$$V_{su} = \frac{V_{Ed} - V_{cu0}}{1 + 0.5 \frac{\zeta}{d} K_p \left(1 + \frac{b}{b_w}\right) \frac{b_{v,\text{eff}}}{b}}$$

(39)

It is observed that due to the confinement of the uncracked concrete zone produced by the stirrups, the amount of transverse reinforcement is reduced, as the denominator in Eq. (39) is always larger than 1.
The four complete databases developed by ACI-DafStb for reinforced and prestressed concrete beams (Reineck et al. 2013, 2014; ACI-DafStb 617, 2015) have been used to verify the proposed model and to compare its predictions with those of three current international codes ACI318.11 (2011), CSA A23.3-14 (2014) and Eurocode 2 (2002). The results are shown in Table 1 and in Fig. 15. All beams included in the four databases have been used for this evaluation. The ranges of the variables for the different tests included in the databases are presented in Table 2. All explicit partial safety factors have been removed from the original formulations. Moreover, the mean value of the materials strength has been used for all calculations. Therefore, the predictions compared are not exactly the real predictions of the different models as $f_{ck}$ or $f'_{c}$ values should be used depending on the formulation employed.

As can be seen in Table 1 and Fig. 15, the proposed model, both in its general or simplified formulations, correlates better with the results of the four databases than any of the three considered code formulations. Summing up, for the 1285 tested beams, the average of the $V_{test}/V_{pred}$ ratio, parameter directly related to accuracy, is 1.04 for the general equations and 1.14 for the simplified equations. For the ACI318-11 provisions the ratio equals 1.44, 1.26 for EC2 and 1.33 for CSA A23.3-14. The coefficient of variation, a measure of precision, is clearly lower for the proposed models than for the current codes of practice. The two most precise models for the four databases are the general model (CoV = 18.4%) and the simplified model (CoV = 19.3%). For ACI318-04 and EC-2 predictions, the CoV is very close to 35%, almost doubling the CoV of the general model. For CSA A23.3-14 the CoV equals 26.9%. As observed in Table 1, with respect to the current codes of practice, the CSA A23.3-14 is the one that provides least scatter for the four groups of beams.

The predictions obtained by means of the proposed formulation and the three codes considered are compared in Fig. 16 with some selected series of tests extracted from the ACI-
DafStb databases for prestressed elements (De Silva et al. 2006; Elzanaty et al. 1986; Muguruma et al. 1983; Olesen et al. 1967). In Fig. 16 the influence of the following parameters is shown: the compression stresses due to prestressing for beams without stirrups (Fig. 16a); the concrete compression strength for beams without stirrups (Fig. 16b); the influence of the amount of stirrups in I-beams (Fig. 16c); and the influence of the slenderness, $a/d$, for I-beams with stirrups (Fig. 16d). It is clearly shown that the proposed equations follow the trends given by the tested beams in all studied situations. It must be highlighted that the proposed model predicts higher shear strength for 3 out of 4 test series shown in Figure 16. For the 3 experimental tests by Elzanaty et al. (1986) shown in Figure 16b the predicted shear strength by EC-2 is unconservatively larger than the strength predicted by the model proposed in this paper. In fact, these 3 beams collapsed due to strut crushing. The model proposed uses the same equations than EC-2 for this type of collapse, as previously commented, but the angle of the concrete struts is fixed in this model.

**APPLICATION EXAMPLE**

The shear strength of a simply supported post-tensioned I-shaped bridge girder tested by Oh and Kim (2004) (girder 1) is predicted using the proposed code format simplified equations. The beam has a total length of 10.6 m, a span length of 10 m, a total height 1.20 m and is subjected to a concentrated load placed at 3.6 m of the right support and 6.4 m of the left support. The cross-section dimensions and reinforcement arrangements are shown in Fig. 17. For calculation purposes, the top and bottom flanges have been considered constant in depth, resulting: $b_{ftop}=800 \text{ mm}$, $h_{ftop}=175 \text{ mm}$, $b_w=180 \text{ mm}$, $h_w=850 \text{ mm}$, $b_{fbot}=600 \text{ mm}$, $h_{fbot}=175 \text{ mm}$. The bottom longitudinal reinforcement consists of 16 bars of 19.1 mm diameter ($A_s =4584.3 \text{ mm}^2$, effective depth $d_s = 1100 \text{ mm}$). No safety coefficients are used since the objective is to predict the experimental ultimate load ($P_u =2303 \text{ kN}$).

The beam is post-tensioned with three tendons of parabolic layout, each composed by 6
seven-wire strands, with a nominal diameter of 98.71 mm$^2$ ($A_{p,tendon} = 6 \times 98.71 = 592.3$ mm$^2$),
placed inside a 50 mm diameter sheath, subsequently injected with cement grout. The
equivalent tendon has a steel area $A_{p,tot} = 3 \times 592.3 = 1776.8$ mm$^2$ and a parabolic layout with
zero eccentricities at the support axes and a maximum eccentricity of 410 mm at the center
span (see Figure 17), resulting Eq. 40, where $s$ is the distance to the support (in m):

$$e(s) = 0.052 + 0.188s - 0.0188s^2; \quad e'(s) = 0.0376(5 - s)$$  \hspace{1cm} (40)

The tendon is stressed from both ends and the prestressing force after losses, which can
be considered constant along the whole tendon length, is $P = 1790.8$ kN ($\sigma_{cp} = 1007.9$ N/mm$^2$).

The stirrups, of 12.9 mm nominal diameter, are double legged with a loop extending out
of the web of the girder. The stirrups spacing was 200 mm ($A_{sw}/S_t = 1.307$ mm$^2$/mm) at the
right hand side and 400mm ($A_{sw}/S_t = 0.653$ mm$^2$/mm) at the left hand side.

Concrete was made with 19 mm crushed granite aggregate. The compressive strength
(derived from 100 x 200 cylinder tests at the time of shear testing) was $f_{cm} = 42.8$ MPa, the
tensile strength $f_{ct} = 3.67$ MPa and the modulus of elasticity $E_c = 34000$ MPa. Yield strength
and ultimate strengths of longitudinal and stirrup bars were 345 MPa and 540 MPa,
respectively, obtained from tests in the laboratory. Ultimate and yield strengths of prestressing
steel were $f_{pmax} = 1860$ MPa and $f_{py} = 1670$ MPa; and $E_p = 195000$ MPa.

Gross section characteristics: area $A_c = 0.398$ m$^2$, first moment of inertia $S_c = 0.0903$ m$^3$;
second moment of inertia $I_c = 0.01046$ m$^4$, and the distances from the c.o.g to the most
compressed and tensile fibers are, respectively, $y_c = 0.555$m, $y_t = -0.645$m.

The distance from the control section to the support axis is:

$$\sigma_{cp} = \frac{P}{A_c} = \frac{1790816}{398000} = 4.5 \text{ MPa}; \quad s_{cr} = d_s \left(1 + 0.4 \frac{\sigma_{cp}}{f_{ct}}\right) = 1639 \text{ mm}$$  \hspace{1cm} (41)

The critical section will be placed in the short span, since the shear force is higher and an
arch effect is not expected because $a/d > 2.5$. 

...
The eccentricity and slope of the equivalent tendon at the control section \((s_{cr} = 1639\) mm), obtained from Eq. (40), are \(e = 0.309\) m and \(e' = 0.126\). The effective depth of the tendon at such section is \(d_p = e + y_c = 864\) mm.

Preliminary calculations, using Eqs. (3) and (5):

\[
n = \frac{E_s}{E_c} = 5.88; \quad n_p = \frac{E_p}{E_c} = 5.73; \quad d = \frac{d_s A_s + d_p A_p}{A_s + A_p} = 1035mm; \quad b = b_{jup} = 800mm
\]  

\[
\rho_s = \frac{A_s}{br_d} = 0.00554; \quad \rho_p = \frac{A_p}{br_d} = 0.00364; \quad n\rho_t = n_s\rho_s + n_p\rho_p = 0.05343
\]  

\[
\frac{c_0}{d} = n\rho_t \left(-1 + \sqrt{1 + \frac{2}{n\rho_t}}\right) = 0.278; \quad c_0 = 287mm
\]  

Neutral axis depth including the prestressing effects (Eq. 8):

\[
\frac{c}{d} = \frac{c_0}{d} + \left(\frac{h}{d} - \frac{c_0}{d}\right)\left(\frac{d}{h}\right)\frac{\sigma_{cp}}{\sigma_{cp} + f_{ct}} = 0.278 + 0.8 \left(\frac{1200}{1031} - 0.278\right)\left(\frac{1031}{1200}\right) = 0.613; \quad c = 634mm > h_f
\]  

The shear span of the right hand side is \(a=3.6\) m, and the size effect (Eq. 25) factor results to be:

\[
\zeta = 1.2 - 0.2a = 1.2 - 0.2 \times 3.6 \geq 0.65 \Rightarrow \zeta = 0.65
\]  

The effective shear width (Eq. 17), taking into account that \(c > h_f\) (\(h_f/c=0.276\)) is:

\[
b_v = b_u + 2h_f = 530; \quad \eta = 3\left(\frac{h_f}{c}\right)^2 - 2\left(\frac{h_f}{c}\right)^3 = 0.186; \quad b_{v,\text{eff}} = b_v\eta + b_u(1-\eta) = 245mm
\]  

Prestressing factor (Eq. 35):

\[
K_p = 1 + 0.24\frac{P \cdot y_i}{f_{ct} \cdot b \cdot d^2} = 1 + 0.24\frac{1790816 \cdot 645}{3.67 \cdot 800 \cdot 1035^2} = 1.088
\]  

Angle of inclination of the struts (Eq. 9):

\[
\cot \theta = \frac{0.85 \cdot d_s}{d_s - c} = \frac{0.85 \cdot 1100}{1100 - 634} = 2.00 \leq 2.5 \quad ; \quad \theta = 26.5^\circ
\]
Shear resisted by the stirrups at the short shear span (Eq. 37):

\[
V_{su} = (d_s - c) \cot \theta \frac{A_{sw}}{s_y} f_{ywd} = 0.85 d_s \frac{A_{sw}}{s_y} f_{ywd} = 0.85 \cdot 100 \cdot 1.307 \cdot 540 = 659904 \text{N} \quad (50)
\]

The shear resisted by the concrete (Eq. 34), including the confinement of the stirrups, is:

\[
V_{cu} = \frac{c}{d} \frac{f_c^{2/3}}{\gamma_c} + 0.5 \left( 1 + \frac{b}{b_u} \right) \frac{V_s}{bd} b_{v,off} d =
\]

\[
= 0.65 \cdot 0.6131 \cdot 0.88 \left[ 3.65 + 0.5 \left( 1 + \frac{800}{180} \right) \frac{659904}{800 \cdot 1035} \right] 245 \cdot 1035 = 639735 \text{N} \quad (51)
\]

Vertical component of the prestressing force at the control section:

\[
V_p = P \sin \alpha = 1790816 \cdot 0.1256 = 224926 \text{N} \quad (52)
\]

Total shear resisted (Eq. 36), at the critical section, including the prestressing contribution \( V_p \):

\[
V_u = V_{cu} + V_{su} + V_p = 639735 + 659904 + 224922 = 1524561 \text{N} \quad (53)
\]

Part of the shear is due to the beam’s self weight \( V_g \). Therefore, the shear caused by the applied point load which can be resisted \( (V_{u,q}) \) is:

\[
V_{u,q} = V_u - V_g = 1524561 - g \left( \frac{1}{2 \cdot s_{cr}} \right) = 1524561 - 9.95(5000 - 1639) = 1491119 \text{N} \quad (54)
\]

and the load which produces such shear is \( Q = V_{u,q}/0.64 = 2330 \text{ kN} \), which is 1.1% higher than the load resisted by the tested girder 1 (2303 kN).

The maximum shear that can be resisted (the shear producing crushing of the web, Eq. (31)) is:

\[
V_{rd,max} = \alpha_{cv} b_u d_s f_{c'} \cot \theta + \cot \alpha \frac{2}{1 + \cot^2 \theta} = 0.6 \cdot 155 \cdot 1100 \cdot 1.105 \cdot 42.8 \cdot 1.5 = 1931084 \text{N} > V_u \quad (55)
\]

Where \( \alpha_{cv} = 0.6 \), \( v_l = 1 + \sigma_{up}/f_{c'} = 1.105 \) and \( b_u = 180 - 0.5 \cdot 50 = 155 \text{ mm} \) (50 mm diameter grouted sheath). Therefore, shear-flexure takes place, for a concentrated load of \( Q_u = 2330 \text{ kN} \).
Oh and Kim (2004) tested a second beam (girder 2) identical to girder 1 except that its concrete compressive strength was 62.1 MPa in spite of 42.8 MPa. The ultimate load obtained for girder 2 was 2312 kN, which was practically the same than girder 1 (2302 kN), even though the compressive strength of girder 2 was almost 50% higher.

Repeating all calculations for girder 2, using the code format equations proposed in this paper, the following results are obtained: $f_{ct}=4.60$ MPa, $\alpha_{cp}=4.50$ MPa, $\alpha_{cp}/f_{ct}=0.978$, $s_{ct}=1531$ mm, $d=1031$ mm, $\alpha_{p}=0.0478$, $c/d=0.571$, $b_{v,ef}=254.4$ mm, $\sin\alpha=0.13$, $V_{su}=659.9$ kN, $V_{cu}=702.0$ kN, $P \cdot \sin\alpha=232.3$ kN, $V_{u}=V_{cu}+V_{su}+P \cdot \sin\alpha=1594.2$ kN, $V_{uq}=1559.7$ kN and $Q_{u}=2434$ kN, that is 5.4% higher than the experimental value and only 4.6% higher than the ultimate load computed with the model for girder 1. Such a small difference is due to several reasons:

1) The concrete contribution to shear strength depends on the tensile strength $f_{ct}=0.3f_{c}^{2/3}$, rather than on the compressive strength. Thus, increasing the compressive strength from 42.8 to 62.1 MPa, and provided that $f'_{c}$ must be limited to 60 MPa, results in an increment of 24.9 % of the tensile strength.

2) If $f_{ct}$ increases, the relative compressive stress introduced by prestressing $\alpha_{cp}/f_{ct}$ decreases, resulting in a reduction of $c/d$ from 0.613 to 0.571 (6.9% smaller).

3) The contribution of the stirrups $V_{su}$ is independent of $f'_{c}$, since the horizontal projection of the first branch of the critical crack is constant and equal to $0.85d_{s}$.

4) The vertical concrete stresses due to confinement provided by the stirrups, given by term $0.5\left(1+\frac{b}{b_{w}}\right)\frac{V}{bd}$ in Eq. (34), are comparable to the concrete tensile strength and, therefore, the influence of $f'_{c}$ on $V_{cu}$ becomes diluted.

**CONCLUSIONS**

The following conclusions can be drawn from the model capacities and results:
1. A general model has been developed for the prediction of the shear-flexural strength of reinforced and prestressed concrete members with and without transverse reinforcement, with I, T or rectangular cross sections, which explicitly accounts for the most significant shear transfer actions. The expressions obtained can be particularized for reinforced concrete members, rectangular sections and sections without transverse reinforcement in a straightforward manner.

2. After some justified simplifications, simple and direct expressions that include the most relevant parameters governing the structural behavior, have been derived. The proposed formulae are valid both for design of the transverse reinforcement and for assessment, without the need of performing iterations.

3. The prestressing force increases the shear strength of the compressed concrete chord by increasing the neutral axis depth, the cracking moment and the normal stresses. These effects have been incorporated into the model.

4. A simple expression to obtain the neutral axis depth in prestressed concrete sections cracked in bending has been derived, which results very useful for computing the shear strength of prestressed concrete members.

5. It is assumed by the model that the critical crack initiates at the section where the bending moment, at failure, equals the cracking moment. Therefore, the critical crack in PC members is farther from the zero bending moment point than in RC members.

6. The predictions of the present model are in very good agreement with the experimental results of the four ACI-DAfStb databases consisting of 1285 shear tests on slender reinforced and prestressed concrete beams with and without stirrups. Summing up, for the 1285 tested beams, the average of the \( \frac{V_{\text{test}}}{V_{\text{pred}}} \) ratio is 1.04 for the general equations and 1.14 for the simplified equations. For the ACI318-11 provisions the ratio equals 1.44, 1.26 for EC2 and 1.33 for CSA A23.3-14. The CoV is 18.4% for the general model and 19.3% for the...
simplified model. For ACI318-04, EC-2 and CSA A23.3-14 the CoV equals 35.4%, 34.5% and 26.9% respectively. This fact indicates that the proposed model exhibits less scattering and is more accurate than the provisions of the current codes. Moreover, according to the studies of tendency made, the model predicts very well the experimentally observed influence of the different parameters involved.

7. The ultimate capacity of two simply supported post-tensioned I beams tested by Oh and Kim (2004) has been computed in detail, to show the practical applicability of the model. The ultimate capacities predicted by the model are very similar to those experimentally measured ($\frac{V_{\text{pred}}}{V_{\text{exp}}}=1.01$ and 1.05 for girders 1 and 2, respectively). The model is capable of capturing the small influence of an increment in the value of $f'_{c}$ on the shear strength.

**ACKNOWLEDGEMENTS**

The present work has been developed under the framework of research projects BIA2012-36848, BIA2012-31432, funded by the Spanish Ministry of Economy and Competitiveness (MINECO) and the Europeans Funds for Regional Development (ERDF), and under the financial help of Infraestructures de Catalunya (ICAT).

**REFERENCES**


**Table 1.** Verification of the proposed model for different databases: mean value and Coefficient of Variation (%) for $V_{\text{test}}/V_{\text{pred}}$ ratio.

<table>
<thead>
<tr>
<th>Database original source</th>
<th>Comments</th>
<th>No. beams</th>
<th>General eqs. (Eq. 1)</th>
<th>Simplified eqs. (Eq. 30)</th>
<th>ACI318-11</th>
<th>EC-2</th>
<th>CSA A23.3-14</th>
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<tr>
<td>(Reineck et al. 2013)</td>
<td>RC beams w/o stirrups</td>
<td>784</td>
<td>Mean: 1.04, CoV: 18.9</td>
<td>Mean: 1.16, CoV: 18.9</td>
<td>Mean: 1.42, CoV: 38.3</td>
<td>Mean: 1.10, CoV: 27.9</td>
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<td>(Reineck et al. 2014)</td>
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<td>Mean: 1.05, CoV: 17.0</td>
<td>Mean: 1.12, CoV: 16.5</td>
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<td>Mean: 1.44, CoV: 29.6</td>
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<td>(ACI-DAfStb 617 2015)</td>
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<td>Mean: 1.02, CoV: 19.6</td>
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Table 2. Range of variables in the used databases.

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<th>$a/d$</th>
<th>$f_c$ (MPa)</th>
<th>$n\phi_c$ Eq.6</th>
<th>$\rho_{eff}$ (MPa)</th>
<th>$\sigma_y$ (MPa)</th>
<th>$V_{cor}$ (kN)</th>
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</table>
Figure 15a

ACI318-11

- RC w/o stirrups
- RC with stirrups
- PC w/o stirrups
- PC with stirrups

$V_{test}/V_{pred}$ vs. $d$ (mm)

Click here to download Figure 15a.pdf
Eurocode 2

- ♦ RC w/o stirrups
- □ RC with stirrups
- △ PC w/o stirrups
- × PC with stirrups

Figure 15b

Click here to download Figure Figure 15b.pdf
General equations

- RC w/o stirrups
- RC with stirrups
- PC w/o stirrups
- PC with stirrups

$V_{test}/V_{pred}$ vs $d$ (mm)
Simplified equations

- \( \text{RC w/o stirrups} \)
- \( \text{RC with stirrups} \)
- \( \text{PC w/o stirrups} \)
- \( \text{PC with stirrups} \)

### Figure 15e

![Graph showing test results compared to predictions](Figure Figure 15e.pdf)
Figure 16a
Figure 16b
Figure 16c
Fig. 1. Critical crack evolution under shear loading

Fig. 2. Numerical prediction of shear stresses under shear loading

Fig. 3. Adopted failure envelope for concrete under a biaxial stress state (Kupfer and Gerstle 1973)

Fig. 4. Shear-flexural crack patterns at failure in partially prestressed concrete beams (Maurer et al. 2014)

Fig. 5. Qualitative distribution of shear stresses at imminent shear failure and distribution

Fig. 6. Interpolation of neutral axis depth between reinforced and fully prestressed concrete

Fig. 7. Change of crack inclination when entering the flange

Fig. 8. Position of the shear critical section in the beam

Fig. 9. Position of the shear critical crack in reinforced and in prestressed concrete members

Fig. 10. Confinement stresses introduced by the stirrups in the un-cracked concrete zone

Fig. 11. Contribution of cracked concrete to shear resistance

Fig. 12. Forces acting on a rigid body part of a beam placed over the critical shear crack

Fig. 13. Normal and shear stresses and damage in a section of a prestressed beam

Fig. 14. Crack pattern at failure in a prestressed concrete girder without flexural cracks (Choulli et al. 2008; Choulli 2005)

Fig. 15. Correlation between the predictions and the experimental results as a function of the effective depth, \( d \), for the 1285 beams included in the four ACI-DafStb databases

Fig. 16. Correlation between the predictions and experimental results for PC beams: a) influence of the amount of prestressing for rectangular beams w/o stirrups; b) influence of the concrete compression strength for beams w/o stirrups failing due to diagonal cracking; c) influence of the amount of stirrups in I-beams; d) Influence of the slenderness, \( a/d \), for I-beams with stirrups

Fig. 17. Prestressed simply supported beam studied: prestressing layout and mid-span cross section