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# A family of mixed graphs with large order and diameter 2

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Dedicated to the memory of our college and friend Mirka Miller

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ABSTRACT

A *mixed regular graph* is a connected simple graph in which each vertex has both a fixed outdegree (the same indegree) and a fixed undirected degree. A mixed regular graphs is said to be *optimal* if there is not a mixed regular graph with the same parameters and bigger order.

We present a construction that provides mixed graphs of undirected degree $q$, directed degree $\frac{q-1}{2}$ and order $2q^2$, for $q$ being an odd prime power. Since the Moore bound for a mixed graph with these parameters is equal to $\frac{9q^2-4q+3}{4}$ the defect of these mixed graphs is $(\frac{q-2}{2})^2 - \frac{1}{4}$.

In particular we obtain a known mixed Moore graph of order 18, undirected degree 3 and directed degree 1 called Bosák's graph and a new mixed graph of order 50, undirected degree 5 and directed degree 2, which is proved to be optimal.



## 1. Introduction

In this paper we consider graphs which are finite and mixed, i.e., they may contain (directed edges) arcs as well as undirected ones. The mixed graphs are also called *partially directed graphs*. Bosák [4] investigated those mixed graphs with given degree and given diameter having maximum number of vertices which are called mixed Moore graphs. In some sense, Bosák generalized the concepts of Moore graph and Moore digraph by allowing the existence of both edges and arcs simultaneously. These graphs and digraphs have been very much used to model different kinds of networks (such as a telecommunications, multiprocessor, or local area network, to name just a few). In many real-world networks a mixture of both unidirectional and bidirectional connections may exist (e.g. the World Wide Web network, where pages are nodes and hyperlinks describe the connections). For such networks, mixed graphs provide a perfect modeling framework [14].

Undirected graphs (mixed Moore graphs admitting only edges) with maximum degree $d$ and diameter $k$ are graphs of order $M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$ (undirected Moore bound). There are no Moore graphs of degree $d \geq 3$ and diameter $k \geq 3$, see [3,7,9]. For $k = 1$ and $d \geq 1$ complete graphs $K_{d+1}$ are the only Moore graphs. For $k \geq 3$ and $d = 2$ the cycles $C_{2k+1}$ are the only Moore graphs. For $k = 2$, apart from $C_5 (d = 2)$, Moore graphs exist only when $d = 3$ (Petersen graph), $d = 7$ (the Hoffman–Singleton graph) and possibly $d = 57$. For more details and results concerning Moore graphs see the survey by Miller and Širáň [14].

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Directed Moore graphs (mixed Moore graphs admitting only arcs) with maximum out-degree $d$ and diameter $k$ are digraphs of order $M^*_{d,k} = 1 + d + d^2 + \cdots + d^k$ (directed Moore bound). In [5,17] it was proved that Moore digraphs do not exist for $d > 1$ and $k > 1$. The unique Moore digraphs are directed cycles of length $k+1$, denoted by $\vec{C}_{k+1}$ and complete graphs on $d+1$ vertices.

Directed and undirected Moore graphs are special cases of mixed Moore graphs where the graphs admit only arcs or only edges, respectively [15]. A mixed graph is said to be *proper* if it contains at least one arc and at least one edge. In particular, a *mixed regular graph* is a simple and finite graph $G$ where each vertex $v$ of $G$ is incident with $z$ arcs from it and $r$ edges; $z$ is the directed degree and $r$ is the undirected degree of $v$ and we set $d = r + z$, $d$ being the degree of $v$. If $G$ has diameter equal to $k$, we say that $G$ is a $(z, r; k)$-mixed graph of directed degree $z$, undirected degree $r$ and diameter $k$.

Let $M_{z,r,k}$ denote the largest possible number of vertices of a $(z, r; k)$-mixed graph. A mixed graph that attains this bound is called a $(z, r; k)$-mixed Moore Graph of diameter $k$. Note that $M_{z,r,k} = M_{d,k}$ when $z = 0$ and $M_{z,r,k} = M^*_{d,k}$ when $r = 0$ $(d = z + r)$.

The following theorem was proposed by Bosák in 1979 as a Conjecture (see [4]), and proved in 2007 by Nguyen, Miller and Gimbert (see [16]).

**Theorem 1.1** (*[16]*). *Let $d \geq 1, k \geq 3$ be two integers. A finite graph $G$ is a mixed Moore graph of degree $d$ and diameter $k$ if and only if either $d = 1$ and $G$ is $\vec{C}_{k+1}$, or $d = 2$ and $G$ is $C_{k+1}$.*

Hence, mixed Moore graphs different from cycles have diameter $k = 2$ and their order attains the following upper bound:

$$M_{z,r,2} = 1 + r + z + r(r - 1 + z) + z(r + z) = (r + z)^2 + z + 1. \tag{1}$$

Bosák in [4] gives for $k = 2$ divisibility conditions for the existence of mixed Moore graphs related with the distribution of undirected and directed edges. He proved that the two parameters $z$ and $r$ must satisfy a tight arithmetic condition. Accordingly, apart from the trivial cases when $r = 0$ and $z = 1$ (graph $\vec{C}_3$), $r = 2$ and $z = 0$ (graph $C_5$), there must exist a positive odd integer $c$ such that

$$c|(4z - 3)(4z + 5) \quad \text{and} \quad r = \frac{1}{4}(c^2 + 3). \tag{2}$$

In the same paper Bosák provides constructions of some of these mixed Moore graphs with all of them being, except the Bosák graph of order $n = 18$, isomorphic to Kautz digraphs $Ka(d, 2)$ with all digons (cycles of length 2) considered as undirected edges (see [11,12]).

In 2007 Nguyen, Miller and Gimbert (see [16]) proved that all mixed Moore graphs of diameter 2 known at that time were unique. However, this is not generally true since Jorgensen recently found (see [10]) two non-isomorphic mixed Moore graphs of diameter 2, out-degree 7, undirected degree 3 and order 108.

Table 1 depicts all values for $n \leq 200$ with the corresponding feasible values of $z$ and $r$, such that a mixed Moore graph of diameter 2 either exists or is not known to exist. Recently, it has been proved the non existence of mixed Moore graphs of orders 40, 50 and 84, see [13]. There are still many values of $r$ and $z$ for which the existence of a mixed Moore graph of diameter 2 has not been settled.

In both the undirected and directed graphs most of the work carried out on this topic has focused on constructing (di)-graphs of diameter $k \geq 2$, degree $d \geq 3$ and a number or vertices as close as possible to the respective Moore bounds. Based on this, we say that a mixed regular graph is *optimal* if there does not exist a mixed regular graph with the same parameters and bigger order.

In this paper we give a construction of mixed graphs of diameter 2, undirected degree $q+2t$, directed degree $(q-1)/2-2t$ and $2q^2$ vertices for $q$ being an odd prime power and either $t \in \{0, \ldots, \frac{q-1}{4}\}$, if $q \equiv 1 \pmod 4$, or $t \in \{0, \ldots, \frac{q-3}{4}\}$, if $q \equiv 3 \pmod 4$. In particular, when $t = \frac{q-1}{4}$ and $q \equiv 1 \pmod 4$, we reobtain the undirected graphs constructed by McKay, Miller and Širáň in 1998 which are currently the largest known graphs with diameter 2 for the corresponding parameters. For $t = 0$ the constructed family of $(\frac{q-1}{2}, q)$-mixed graphs of diameter 2 provide us graphs with large order, since the Moore bound for a mixed Moore graph with these parameters is equal to $\frac{9q^2-4q+3}{4}$, so it follows that the defect of these mixed graphs is $(\frac{q-2}{2})^2 - \frac{1}{4}$.

Furthermore, our construction provides for $q = 3$, a known $(1, 3; 2)$-mixed Moore graph of order 18 called Bosák's graph and a new $(2, 3; 2)$-mixed graph of order 50, which is proved to be optimal. For the rest of the values of $q$ and $t$ our construction produces good lower bounds for the degree/diameter problem.

## 2. Notation and terminology

Let $G$ be a mixed graph with vertex set $V(G)$, edge set $E(G)$ and arc set $A(G)$. The distance from a vertex $u$ to a vertex $v$ is the length of a shortest path from $u$ to $v$. The distance from $u$ to $v$ is infinite, if $v$ is not reachable from $u$. Note that in a directed graph the distance from vertex $u$ to vertex $v$ can differ from the distance from $v$ to $u$. The maximum value $k$ of the distance over all pairs of vertices of $G$ is the diameter of the graph.

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**Table 1**
Feasible values of the parameters for proper mixed Moore graphs of diameter 2 and order up to 200.

| $n$ | $d$ | $z$ | $r$ | Existence | Uniqueness |
|---|---|---|---|---|---|
| 3 | 1 | 1 | 0 | $Z_3$ | YES |
| 5 | 2 | 0 | 2 | $C_5$ | YES |
| 6 | 2 | 1 | 1 | $Ka(2, 2)$ | YES |
| 10 | 3 | 0 | 3 | Petersen graph | YES |
| 12 | 3 | 2 | 1 | $Ka(3, 2)$ | YES |
| 18 | 4 | 1 | 3 | Bosák graph | YES |
| 20 | 4 | 3 | 1 | $Ka(4, 2)$ | YES |
| 30 | 5 | 4 | 1 | $Ka(5, 2)$ | YES |
| 42 | 6 | 5 | 1 | $Ka(6, 2)$ | YES |
| 50 | 7 | 0 | 7 | Hoffman–Singleton graph | YES |
| 56 | 7 | 6 | 1 | $Ka(7, 2)$ | YES |
| 72 | 8 | 7 | 1 | $Ka(8, 2)$ | YES |
| 88 | 9 | 6 | 3 | unknown | unknown |
| 90 | 9 | 8 | 1 | $Ka(9, 2)$ | YES |
| 108 | 10 | 7 | 3 | Jorgensen graph | NO |
| 110 | 10 | 9 | 1 | $Ka(10, 2)$ | YES |
| 132 | 11 | 10 | 1 | $Ka(11, 2)$ | YES |
| 150 | 12 | 5 | 7 | unknown | unknown |
| 156 | 12 | 11 | 1 | $Ka(12, 2)$ | YES |
| 180 | 13 | 10 | 3 | unknown | unknown |
| 182 | 13 | 12 | 1 | $Ka(13, 2)$ | YES |

In our constructions we use the incidence graph of a partial plane. A *partial plane* is defined as two finite sets $\mathcal{P}$ and $\mathcal{L}$ called points and lines respectively, where $\mathcal{L}$ consists of subsets of $\mathcal{P}$, such that any line is incident with at least two points, and two points are incident with at most one line. The *incidence graph* of a partial plane is a bipartite graph with partite sets $\mathcal{P}$ and $\mathcal{L}$ and a point of $\mathcal{P}$ is adjacent to a line of $\mathcal{L}$ if they are incident. Observe that the incidence graph of a partial plane is clearly a bipartite graph with even girth $g \geq 6$. In Remark 2.1 we describe a biaffine plane given in [6].

**Remark 2.1.** Let $\mathbb{F}_q$ be the finite field of order $q$.

(i) Let $\mathcal{L} = \mathbb{F}_q \times \mathbb{F}_q$ and $\mathcal{P} = \mathbb{F}_q \times \mathbb{F}_q$ denoting the elements of $\mathcal{L}$ and $\mathcal{P}$ using "brackets" and "parenthesis", respectively. The following set of $q^2$ lines define a biaffine plane:

$$[m, b] = \{(x, mx + b) : x \in \mathbb{F}_q\} \quad \text{for all } m, b \in \mathbb{F}_q.$$

(ii) The incidence graph of the biaffine plane is a bipartite graph $B_q = (\mathcal{P}, \mathcal{L})$ which is $q$-regular, has order $2q^2$, diameter 4 and girth 6, if $q \geq 3$; and girth 8, if $q = 2$.

(iii) Notice that the vertices mutually at distance 4 are the vertices of the sets $L_m = \{[m, b] : b \in \mathbb{F}_q\}$, and $P_x = \{(x, y) : y \in \mathbb{F}_q\}$ for all $x, m \in \mathbb{F}_q$.

## 3. An infinite family of mixed graphs with large order

### 3.1. Basic construction

Let $q$ be an odd prime power. Let $\mathbb{F}_q$ be the finite field of order $q$ and let $M \subseteq \mathbb{F}_q - 0$, $|M| = (q-1)/2$, be such that for all $u, v \in M$, $u + v \neq 0$. Therefore $|-M| = (q-1)/2$, $M \cap (-M) = \emptyset$ and $M \cup (-M) = \mathbb{F}_q - 0$. Let $T \subseteq M$ with $|T| = 2t$ be such that $t \in \{0, \dots, \frac{q-1}{4}\}$ if $q \equiv 1 \pmod 4$, or $t \in \{0, \dots, \frac{q-3}{4}\}$ if $q \equiv 3 \pmod 4$.

Let $-T = \{t' \in -M : t + t' = 0, \text{ for } t \in T\}$ and consider the following two sets:

$$T_1 = \{t_i, t_i', i = 1, \dots, t : t_i + t_i' = 0\} \subseteq T \cup (-T) \quad \text{and}$$
$$T_2 = \{t_i, t_i', i = t+1, \dots, 2t : t_i + t_i' = 0\} \subseteq T \cup (-T).$$

Let $S = M - T$ and $(-S) = (-M) - (-T)$.

**Construction 3.1.** *Let $q \geq 3$ be an odd prime power. Using the aforementioned defined sets we construct a mixed graph $G_{q,t}$ for any $t \in \{0, \dots, \frac{q-1}{4}\}$, if $q \equiv 1 \pmod 4$ or $t \in \{0, \dots, \frac{q-3}{4}\}$ if $q \equiv 3 \pmod 4$ as follows: Let $B_q$ be the bipartite graph given in Remark 2.1. The mixed graph $G_{q,t}$ is defined as follows:*
$V(G_{q,t}) = V(B_q)$,
$E(G_{q,t}) = E(B_q) \cup \{([m, b], [m, b + i]) : i \in T_1\} \cup \{((x, y), (x, y + j)) : j \in T_2\}$ *and*
$A(G_{q,t}) = \{([m, b], [m, b + i]) : i \in S\} \cup \{((x, y), (x, y + j)) : j \in -S\}$.

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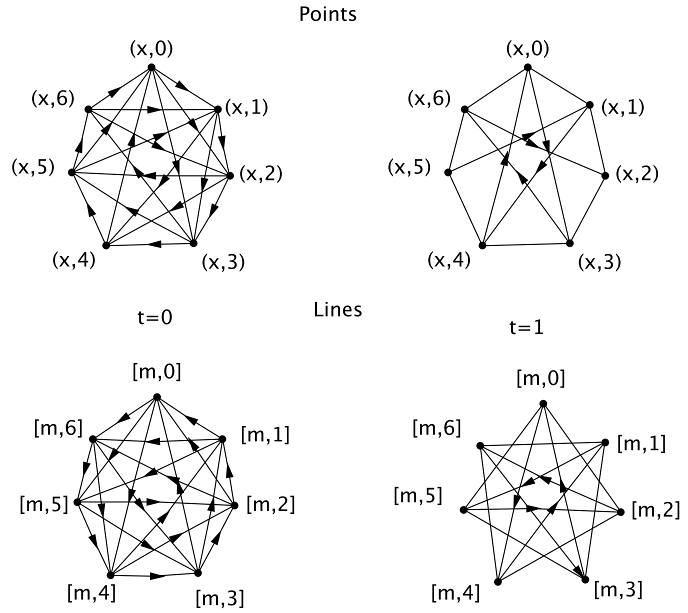


**Fig. 1.** The induced mixed subgraphs of $G_{7,t}$ for points and lines for $t = 0$ (on the left) and $t = 1$ (on the right).

In Fig. 1 we exhibit, for $q = 7$, the induced mixed subgraphs of $G_{7,t}$ for points and lines when $t = 0$ (on the left) and $t = 1$ (on the right).

**Theorem 3.2.** *The mixed graph $G_{q,t}$ defined in Construction 3.1 is a mixed graph of diameter 2 with parameters $r = q+2t$, $z = (q-1)/2 - 2t$ and $2q^2$ vertices.*

**Proof.** It is immediate that the parameters of $G_{q,t}$ are $r = q+2t$, $z = (q-1)/2 - 2t$ since $B_q$ is a $q$-regular graph and we have added to each vertex $(q-1)/2 - 2t$ outing arcs. Let us show that the diameter is 2. First, observe that given any two vertices $[m, b]$ and $[m, b']$, the set of their adjacent vertices in $B_q$ are by definition $(x, mx+b)$ and $(x, mx+b')$ for all $x \in \mathbb{F}_q$, respectively, and we have four possibilities in $G_{q,t}$:

(i) If $b' - b \in T_1$ then $[m, b]$ and $[m, b']$ are adjacent and $(x, mx+b)$ and $(x, mx+b')$ are not adjacent.
(ii) If $b' - b \in T_2$ then $[m, b]$ and $[m, b']$ are not adjacent and $(x, mx+b)$ and $(x, mx+b')$ are adjacent.
(iii) If $b' - b \in S$ then there exists an arc from $[m, b]$ to $[m, b']$ and an arc to $(x, mx+b)$ from $(x, mx+b')$.
(iv) If $b' - b \in (-S)$ then there exists an arc to $[m, b]$ from $[m, b']$ and an arc from $(x, mx+b)$ to $(x, mx+b')$.

Let us check the distance $d_{G_{q,t}}([m, b], (x, y))$ for any pair of vertices $[m, b]$, $(x, y)$ assuming that they are not adjacent. Suppose that $[m, b']$ is adjacent to $(x, y)$ in $B_q$; if $b' - b \in T_1 \cup S$, then by (i) and (iii), $[m, b]$, $[m, b']$, $(x, y)$ is a path of length two in $G_{q,t}$ and $d_{G_{q,t}}([m, b], (x, y)) = 2$.

Now, if $b' - b \in T_2 \cup (-S)$ since $[m, b']$ is adjacent to $(x, y)$ in $B_q$ we have $(x, y) = (x, mx+b')$ and also $[m, b]$ is adjacent to $(x, mx+b)$ in $B_q$, then by (ii) and (iv) it follows that $[m, b]$, $(x, mx+b)$, $(x, y)$ is a path of length two in $G_{q,t}$ and $d_{G_{q,t}}([m, b], (x, y)) = 2$.

Consequently we conclude that $d_{G_{q,t}}([m, b], (x, y)) \le 2$ for any pair of vertices $\{[m, b], (x, y)\}$.

Let us check the distance $d_{G_{q,t}}([m, b], [m, b'])$. If $b' - b \in T_1 \cup S$ then $d_{G_{q,t}}([m, b], [m, b']) = 1$. Therefore we assume that $s = b' - b \in T_2 \cup (-S)$, that is, either $[m, b]$ and $[m, b']$ are not adjacent or there is an arc to $[m, b]$ from $[m, b']$. Observe that the set $A_1 = \{[m, b+s] : s \in T_1 \cup (S)\}$ has $(q-1)/2$ vertices. Moreover the set $A_2 = \{[m, b'-s] : s \in T_2 \cup (-S)\}$ has $(q-1)/2$ vertices. If $A_1 \cap A_2 = \emptyset$ then the set $V_m = \{[m, b] : b \in \mathbb{F}_q\} = A_1 \cup A_2 \cup \{[m, b], [m, b']\}$, implying that $|V_m| = q+1$, which is a contradiction because $|V_m| = q$. Thus $A_1 \cap A_2 \neq \emptyset$ yielding that there exists some $s \in S$ such that $[m, b]$, $[m, b'-s]$, $[m, b']$ is a path of length two in $G_{q,t}$. Thus in either case $d_{G_{q,t}}([m, b], [m, b']) \le 2$.

Analogously, it is proved that $d_{G_{q,t}}((x, y), (x, y')) \le 2$. Hence we can conclude that the diameter of $G_{q,t}$ is 2. ■

From Theorem 3.2 the following corollaries are immediate.

**Corollary 3.1.** *For $q$ being an odd prime power and $t = 0$ the graph, called for simplicity $G_q$, given in Theorem 3.2 is a mixed graph of diameter 2 with parameters $r = q$, $z = (q-1)/2$ and $2q^2$ vertices.*

In this case $S = M$ and $(-S) = (-M)$; and $E(G_q) = E(B_q)$, $A(G_q) = \{([m, b], [m, b+i]) : i \in M\} \cup \{((x, y), (x, y+j)) : j \in -M\}$. Fig. 2 depicts $G_5$.

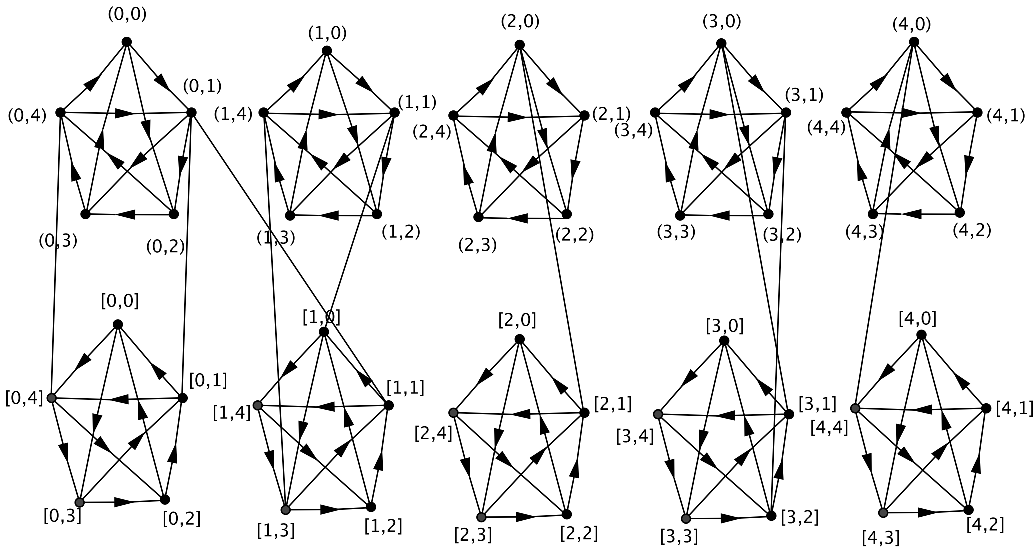


**Fig. 2.** Graph $G_5$ (by clarity of the picture not all the edges of $B_q$ are included).

**Corollary 3.2.** *Let $q \equiv 1 \pmod 4$ be an odd prime power and $t = \frac{q-1}{4}$. Then the mixed graph $G_{q,\frac{q-1}{4}}$ given in Theorem 3.2 has diameter 2 and parameters $r = \frac{3q-1}{2}$, $z = 0$ and $2q^2$ vertices.*

The graphs $G_{q,\frac{q-1}{4}}$ have already been given by various authors using different techniques. The first construction was made by McKay, Miller and Širáň in [14] using lifts and voltage graph. Afterwards, it was constructed by Hafner [8] and by Araujo, Noy and Serra in [1] using geometrical techniques. It is important to take into consideration that this family of graphs $G_{q,\frac{q-1}{4}}$ is the largest known until now related to the Moore bound for given parameters and diameter 2 (see [14] or [18]). Moreover, the graphs constructed by these authors are vertex transitive, in particular the graph $G_{q,\frac{q-1}{4}}$ for any $q \equiv 1 \pmod 4$ odd prime power is vertex transitive. Furthermore, in [2] there is a complete discussion devoted to symmetries and automorphism groups of these graphs and other related issues.

## 4. Vertex transitivity of $G_q$

In this section we prove that graph $G_q := G_{q,0}$ of Corollary 3.1 is vertex transitive. In addition, we provide a short remark explaining why $G_{q,t}$ is not vertex transitive for $1 \le t \le \frac{q-1}{4} - 1$ with $q \equiv 1 \pmod 4$ and for $1 \le t \le \frac{q-3}{4}$ with $q \equiv 3 \pmod 4$. Consequently, $G_{q,t}$ is vertex transitive if either $t = 0$ or $t = \frac{q-1}{4}$ and $q \equiv 1 \pmod 4$ is an odd prime power.

**Theorem 4.1.** *Let $q$ be an odd prime power then $G_q$ is a vertex transitive mixed graph.*

**Proof.** Note that the function $\theta$ that interchanges the line $[m, b]$ at point $(-m, -b)$, and similarly point $(x, y)$ at line $[-x, -y]$, is an automorphism of $B_q$ which exchanges stable sets of the graph.

We will prove that $\theta$ is an automorphism of $G_q$. Let $v \in V(G_q)$, we will prove that $N^+(\theta(v)) = \theta(N^+(v))$ where $N^+(v)$ denotes the exneighborhood of $v$ (the set of vertices that receives an arc from $v$). Let $[m, b] \in \mathcal{L}$ be a line of $G_q$. Since $N^+([m, b]) = \{[m, b + i] : i \in M\}$ it follows that

$$\theta(N^+([m, b])) = \{(-m, -b - i) : i \in M\} = \{(-m, -b + j) : j \in -M\} = N^+(\theta[m, b]).$$

A similar argument is used to prove that $\theta(N^+((x, y))) = N^+(\theta(x, y))$ for $(x, y) \in \mathcal{P}$ a point of $G_q$.

Let us define the function $\Psi_{(a,t)} : V(G_q) \longrightarrow V(G_q)$ by

$$\Psi_{a,t}([m, b]) = [-m, b + am + t];$$
$$\Psi_{a,t}((x, y)) = (-x + a, y + t).$$

Next, we prove that $\Psi_{a,t}$ is an automorphism of $G_q$. First, we check that:

$$N(\Psi_{a,t}[m, b]) \cup N^+(\Psi_{a,t}[m, b]) = \Psi_{a,t}(N([m, b]) \cup N^+([m, b])).$$

Note that

$$N([m, b]) \cup N^+([m, b]) = \{(x, mx + b) : x \in \mathbb{F}_q\} \cup \{[m, b + i] : i \in M\}.$$

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Then

$$\begin{aligned} N(\Psi_{a,t}[m,b]) \cup N^+(\Psi_{a,t}[m,b]) &= N([-m, b+am+t]) \cup N^+([-m, b+am+t]) \\ &= \{(x', -mx'+b+am+t) : x' \in \mathbb{F}_q\} \cup \{[-m, b+am+t+i] : i \in M\}. \end{aligned}$$

Note that if $x' = -x+a$, then $-m(-x+a)+b+am+t = mx+b+t$ and

$$\begin{aligned} N(\Psi_{a,t}[m,b]) \cup N^+(\Psi_{a,t}[m,b]) &= \{(-x+a, mx+b+t) : x \in \mathbb{F}_q\} \cup \{[-m, b+am+t+i] : i \in M\} \\ &= \Psi_{a,t}(N([m,b]) \cup N^+([m,b])). \end{aligned}$$

Let us also prove that:

$$N(\Psi_{a,t}(x,y)) \cup N^+(\Psi_{a,t}(x,y)) = \Psi_{a,t}(N(x,y) \cup N^+(x,y)).$$

We have $N(x,y) \cup N^+(x,y) = \{[m, y-mx] : m \in \mathbb{F}_q\} \cup \{(x, y+i) : i \in -M\}$. Then

$$\begin{aligned} N(\Psi_{a,t}(x,y)) \cup N^+(\Psi_{a,t}(x,y)) &= N(-x+a, y+t) \cup N^+(-x+a, y+t) \\ &= \{[m', y+t-m'(-x+a)] : m' \in \mathbb{F}_q\} \cup \{(-x+a, y+t+i) : i \in -M\}. \end{aligned}$$

Note that if $m' = -m$, then

$$\begin{aligned} N(\Psi_{a,t}(x,y) \cup N^+(\Psi_{a,t}(x,y)) &= \{[-m, y+t-mx+ma)] : m \in \mathbb{F}_q\} \cup \{(-x+a, y+t+i) : i \in -M\} \\ &= \{[-m, b+am+t] : m \in \mathbb{F}_q\} \cup \{(-x+a, y+t+i) : i \in -M\} \\ &= \Psi_{a,t}(N(x,y) \cup N^+(x,y)). \end{aligned}$$

Now, if we compose $\theta$ with $\Psi_{a,t}$ we obtain an automorphism which exchanges any pair of elements of $G_q$, consequently, $G_q$ is vertex-transitive. Indeed, let $[m,t]$ and $(x,y)$ be two vertices of $G_q$. We have $\Psi_{x-m,y+b} \circ \theta([m,t]) = (x,y)$. Thus, there always exists a composition that sends any element to another. ■

**Remark 4.1.** The graph $G_{q,t}$ for any $t \in \{1, \ldots, \frac{q-1}{4} - 1\}$ if $q \equiv 1 \pmod 4$ or $t \in \{1, \ldots, \frac{q-3}{4}\}$ if $q \equiv 3 \pmod 4$ is not vertex transitive.

Notice that, if $G_{q,t}$ is vertex transitive, then the automorphism (or composition of automorphisms) should exchange stable sets (points and lines). Moreover, it should send sets of vertices $L_m$ for $m \in \mathbb{F}_q$ to sets of vertices $P_x$ for $x \in \mathbb{F}_q$. Next, we will prove that the automorphism that preserving arcs between any pair of sets $L_m$ and $P_x$ do not preserve adjacencies:

Suppose that $\Phi$ is an automorphism of $G_{q,t}$ that sends $[m,b]$ to $(-m,-b)$, for $[m,b]$ and $(-m,-b)$ a line and a point of $G_{q,t}$, respectively. As $N^+([m,b]) = \{[m, b+i] : i \in S\}$, and $N^+((-m,-b)) = \{(-m, -b+i) : i \in (-S)\}$, the arcs of $G_{q,t}$ need to be preserved, that is, $\Phi([m, b+i]) = (-m, -b+j)$ for some $i, -j \in S$. On the other hand, the vertex $(x,y)$ is adjacent to $[m,b]$ in $B_q$ if and only if $[-x,-y]$ is adjacent to $(-m,-b)$. Hence, to preserve the adjacencies of $B_q$ the requirement $j = -i$ is necessary, and $\Phi([m,b]) = (-m,-b)$, $\Phi((x,y)) = [-x,-y]$ for all the lines and points of $G_{q,t}$. However, by definition of $G_{q,t}$, this automorphism does not preserve the edges between stable sets.

## 5. Optimal mixed graphs

In this section we provide some results related to the family constructed in Corollary 3.1. The following lemma is used in what follows.

**Lemma 5.1.** *Let $r \geq 1$ be an odd integer. Then there is not a $(z, r; k)$-mixed graph of odd order.*

**Proof.** Suppose that there is a $(z, r; k)$-mixed graph of odd order with $z, r \geq 1$ and $r$ odd. Deleting the directions of the arcs we obtain a regular graph of odd degree $2z + r$ and odd order, which is a contradiction. ■

**Remark 5.1.** We construct a family of $(\frac{q-1}{2}, q)$-mixed graphs of large order and diameter 2. Since the Moore bound for a mixed Moore graph with these parameters is equal to $\frac{9q^2-4q+3}{4}$ the defect of these mixed graphs is $(\frac{q-2}{2})^2 - \frac{1}{4}$.

**Theorem 5.1.** *For $q = 3$, $G_3$ is a mixed Moore graph and for $q = 5$, $G_5$ is an optimal mixed graph.*

**Proof.** For $q = 3$ it turns out that $G_3$ has 18 vertices and parameters $r = 3$ and $z = 1$. Since Bosák's graph is unique, see [16], we obtain that $G_3$ given in Construction 3.1 is isomorphic to Bosák's graph.

For $q = 5$ it turns out that $G_5$ has 50 vertices and parameters $r = 5$ and $z = 2$. By (1) the upper bound on the number of vertices for this particular case is 52. Let us show that a mixed Moore graph with 52 vertices and parameters $r = 5$ and $z = 2$ cannot exist. Otherwise, by (2) an odd integer $c$ dividing $(4z-3)(4z+5) = 65$ exists, such that $r = 5 = \frac{1}{4}(c^2+3)$. But then $c = \sqrt{17}$ which implies that $c$ is not an integer. Therefore the upper bound on the number of vertices must be at most 51. However, from Lemma 5.1 it follows that there is no graph of order 51, and we conclude that the upper bound is 50, yielding that $G_5$ is an optimal mixed graph. ■

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In Fig. 2 we show the optimal (2, 5; 2)-mixed graph.

To conclude the paper we pose a problem which could be studied in the future.

**Problem.** For $q = 7$, $r = 7$ and $z = 3$ the Moore Bound is equal to 104. Our construction provides a (3, 7; 2)-mixed graph on 98 vertices. By Bosák's condition we know that the Moore bound is not attainable, thus we need to know whether there is or there is not a (3, 7; 2)-mixed graph on either 100 or 102 vertices.

Note that for these parameters, by Lemma 5.1, graphs with odd order are not possible.

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