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The spectral excess theorem for distance-regular graphs having distance-d graph with fewer distinct eigenvalues

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Abstract

Let Γ be a distance-regular graph with diameter d and Kneser graph $K = \Gamma_d$, the distance-d graph of Γ. We say that Γ is partially antipodal when K has fewer distinct eigenvalues than Γ. In particular, this is the case of antipodal distance-regular graphs (K with only two distinct eigenvalues), and the so-called half-antipodal distance-regular graphs (K with only one negative eigenvalue). We provide a characterization of partially antipodal distance-regular graphs (among regular graphs with $d+1$ distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as a more general version of the so-called spectral excess theorem, which allows us to characterize those distance-regular graphs which are half-antipodal, antipodal, bipartite, or with Kneser graph being strongly regular.

Keywords: Distance-regular graph; Kneser graph; Partial antipodality; Spectrum; Predistance polynomials.

AMS subject classifications: 05C50, 05E30.

1 Preliminaries

Let Γ be a distance-regular graph with adjacency matrix A and $d+1$ distinct eigenvalues. In the recent work of Brouwer and the author [2], we studied the situation where the distance-d graph Γ_d of Γ, or the Kneser graph K of Γ, with adjacency matrix $A_d = p_d(A)$ where p_d is the distance-d polynomial, has fewer distinct eigenvalues than Γ. In this case we say that Γ is *partially antipodal*. Examples are the so-called half antipodal (K with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs (K being disjoint copies of a complete graph). Here we generalize such a study to the case when Γ is a regular graph with $d+1$ distinct eigenvalues. The main result of this paper is a characterization of partially antipodal distance-regular graphs, among regular graphs with $d+1$ distinct eigenvalues, in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as a more general version of the so-called spectral excess theorem, and allows us to characterize those distance-regular graphs which are half antipodal, antipodal, bipartite, or with Kneser graph being strongly regular. Other related characterizations of some of these cases were given by the author in [8, 9, 10]. For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [1], Brouwer and Haemers [3], and Van Dam, Koolen and Tanaka [6].

Let Γ be a regular (connected) graph with degree k, n vertices, and spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0(=k) > \lambda_1 > \cdots > \lambda_d$, and $m_0 = 1$. In this work, we use the following scalar product on the $(d+1)$-dimensional vector space of real polynomials modulo $m(x) = \prod_{i=0}^{d}(x-\lambda_i)$, that is, the minimal polynomial of A.

$$\langle p, q\rangle_\Gamma = \frac{1}{n}\operatorname{tr}(p(A)q(A)) = \frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i)q(\lambda_i), \qquad p,q \in \mathbb{R}_d[x]/(m(x)). \tag{1}$$

This is a special case of the inner product of symmetric $n\times n$ real matrices M, N, defined by $\langle M, N\rangle = \frac{1}{n}\operatorname{tr}(MN)$. The *predistance polynomials* $p_0, p_1, \ldots, p_d$, introduced by the author and Garriga [13], are a sequence of orthogonal polynomials with respect to the inner product (1), normalized in such a way that $\|p_i\|_\Gamma^2 = p_i(k)$ (this makes sense since it is known that $p_i(k) > 0$ for any $i = 0, \ldots, d$).

As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1} \qquad (i = 0, 1, \ldots, d),$$

where the constants β_{i-1}, α_i, and γ_{i+1} are the Fourier coefficients of xp_i in terms of p_{i-1}, p_i, and p_{i+1}, respectively (and $\beta_{-1} = \gamma_{d+1} = 0$), initiated with $p_0 = 1$ and $p_1 = x$.

Then, it is known that Γ is distance-regular if and only if such polynomials satisfy $p_i(A) = A_i$ (the adjacency matrix of the distance-i graph Γ_i) for $i = 0, \ldots, d$, in which case they turn out to be the distance polynomials of Γ. Moreover, as expected, the constants α_i, β_i and γ_i become the intersection numbers a_i, b_i and c_i of Γ.

In fact, we have the following strongest proposition, which is a combination of results in [14, 7].

Proposition 1. *A regular graph Γ as above is distance-regular if and only if there exists a polynomial p of degree d such that $p(A) = A_d$, in which case $p = p_d$.* □

Many properties of the distance polynomials of distance-regular graphs hold also for the predistance polynomials. For instance, the sum of all predistance polynomials gives the Hoffman polynomial H:

$$H = \sum_{i=0}^{d} p_i = \frac{n}{\prod_{i=1}^{d}(\lambda_0 - \lambda_i)} \prod_{i=1}^{d}(x - \lambda_i),$$

satisfying $H(\lambda_0) = n$ and $H(\lambda_i) = 0$ for $i = 1, \dots, d$. This polynomial characterizes regular graphs by the condition $H(A) = J$, the all-1 matrix [16], and it can be used to show that $\alpha_i + \beta_i + \gamma_i = \lambda_0 = k$ for all $i = 0, \dots, d$.

Also, as in the case of distance-regular graphs, the multiplicities of Γ can be obtained from the values of p_d since,

$$(-1)^i p_d(\lambda_i)\pi_i m_i = p_d(\lambda_0)\pi_0, \qquad i = 1, \dots, d. \tag{2}$$

where $\pi_i = \prod_{j\neq i} |\lambda_i - \lambda_j|$. Indeed, let $L_i(x) = \prod_{j\neq 0,i}(x - \lambda_j)/\prod_{j\neq 0,i}(\lambda_i - \lambda_j)$. Then, since the degree of each L_i is $d - 1$, the equalities in (2) follow from $\langle L_i, p_d\rangle_\Gamma = 0$ for $i = 1, \dots, d$. Some interesting consequences of the above, together with other properties of the predistance polynomials are the following (for more details, see [4]):

- The values of p_d at $\lambda_0, \lambda_1, \dots, \lambda_d$ alternate in sign.
- Using the values of $p_d(\lambda_i)$, $i = 0, \dots, d$, given by (2), in the equality $\|p_d\|_\Gamma^2 = p_d(\lambda_0)$, and solving for $p_d(\lambda_0)$ we get the so-called *spectral excess*

$$p_d(\lambda_0) = n \left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1}. \tag{3}$$

- For every $i = 0, \dots, d$, (any multiple of) the sum polynomial $q_i = p_0 + \cdots + p_i$ maximizes the quotient $r(\lambda_0)/\|r\|_\Gamma$ among the polynomials $r \in \mathbb{R}_i[x]$ (notice that $q_i(\lambda_0)^2/\|q_i\|_\Gamma^2 = q_i(\lambda_0)$), and

$$(1 =)q_0(\lambda_0) < q_1(\lambda_0) < \cdots < q_d(\lambda_0)(= H(\lambda_0) = n).$$

Let Γ have n vertices, $d+1$ distinct eigenvalues, and diameter $D(\leq d)$. For $i = 0, \dots, D$, let $k_i(u)$ be the number of vertices at distance i from vertex u. Let $s_i(u) = k_0(u) + \cdots + k_i(u)$. Of course, $s_0(u) = 1$ and $s_D(u) = n$. The following result can be seen as a version of the spectral excess theorem, due to Garriga and the author [13] (for short proofs, see Van Dam [5], and Fiol, Gago and Garriga [12]):

Theorem 2. *Let Γ be a regular graph with spectrum $\operatorname{sp}\Gamma = \{\lambda_0, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$. Let $\overline{s_i} = \frac{1}{n}\sum_{u\in V} s_i(u)$ be the average number of vertices at distance at most i from every vertex in Γ. Then, for any nonzero polynomial $r \in \mathbb{R}_{d-1}[x]$ we have*

$$\frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} \leq \overline{s_{d-1}}, \tag{4}$$

with equality if and only if Γ is distance-regular and r is a multiple of q_{d-1}.

Proof. Let $S_{d-1} = I + A + \cdots + A_{d-1}$. As $\deg r \le d-1$, $\langle r(A), J\rangle = \langle r(A), S_{d-1}\rangle$. But $\langle r(A), J\rangle = \langle r, H\rangle_\Gamma = r(\lambda_0)$. Thus, Cauchy-Schwarz inequality gives

$$r^2(\lambda_0) \le \|r(A)\|^2 \|S_{d-1}\|^2 = \|r\|_\Gamma^2 \overline{s_{d-1}},$$

whence (4) follows. Besides, in case of equality we have that $r(A) = \alpha S_{d-1}$ for some nonzero constant α. Hence, the polynomial $p = H - (1/\alpha)r$ satisfies $p(A) = J - S_{d-1} = A_d$ and, from Proposition 1, Γ is distance-regular, $p = p_d$, and $r = \alpha q_{d-1}$. The converse is clear from $s_{d-1} = n - k_d = H(\lambda_0) - p_d(\lambda_0) = q_{d-1}(\lambda_0)$. □

In fact, as it was shown in [11], the above result still holds if we change the arithmetic mean of the numbers $s_{d-1}(u)$, $u \in V$, by its harmonic mean.

2 The results

As commented above, in [2] we studied the situation where the distance-d graph Γ_d, of a distance-regular graph Γ with diameter d, has fewer than $d+1$ distinct eigenvalues. Now, we are interested in the case when Γ is regular and with $d+1$ distinct eigenvalues. In this context, p_d is the highest degree predistance polynomial and, as $p_d(A)$ is not necessarily the distance-d matrix A_d (usually not even a 0-1 matrix), we consider the distinct eigenvalues of $p_d(A)$ vs. those of A. More precisely, given a set $\mathcal{I} \subset \{0, \ldots, d\}$, we give conditions for all $p_d(\lambda_i)$ with $i \in \mathcal{I}$ taking the same value. Notice that, because the values of p_d at the mesh $\lambda_0, \lambda_1, \ldots, \lambda_d$ alternate in sign, the feasible sets $\mathcal{I}$ must consist of either even or odd numbers.

2.1 The case $0 \notin \mathcal{I}$

We first study the more common case when $0 \notin \mathcal{I}$. For $i = 1, \ldots, d$, let $\phi_i(x) = \prod_{j \ne 0,i}(x - \lambda_j)$, and consider again the Lagrange interpolating polynomial $L_i(x) = \phi_i(x)/\phi_i(\lambda_i)$, satisfying $L_i(\lambda_j) = \delta_{ij}$ for $j \ne 0$, and $L_i(\lambda_0) = (-1)^{i+1}\frac{\pi_0}{\pi_i}$, where $\pi_i = |\phi_i(\lambda_i)|$.

Theorem 3. *Let Γ be a regular graph with degree k, n vertices, and spectrum $\operatorname{sp}\Gamma = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0(=k) > \lambda_1 > \cdots > \lambda_d$. Let $\mathcal{I} \subset \{1, \ldots, d\}$. For every $i = 0, \ldots, d$, let $\pi_i = \prod_{j \ne i}|\lambda_i - \lambda_j|$. Let $\overline{k_d} = \frac{1}{n}\sum_{u\in V} k_d(u)$ be the average number of vertices at distance d from every vertex in Γ. Then,*

$$\overline{k_d} \le \frac{n\sum_{i\in\mathcal{I}} m_i}{\left(\sum_{i\in\mathcal{I}} \frac{\pi_0}{\pi_i}\right)^2 + \sum_{i\notin\mathcal{I}} \frac{\pi_0^2}{m_i\pi_i^2}\sum_{i\in\mathcal{I}} m_i}, \tag{5}$$

and equality holds if and only if Γ is a distance-regular graph with $k_d(u) = k_d$ for each $u \in V$, and constant

$$P_{id} = p_d(\lambda_i) = k_d\frac{\sum_{i\in\mathcal{I}}(-1)^{i+1}\frac{\pi_0}{\pi_i}}{\sum_{i\in\mathcal{I}} m_i} \qquad \textit{for every } i \in \mathcal{I}. \tag{6}$$

Proof. The clue is to apply Theorem 2 with a polynomial $r \in \mathbb{R}_{d-1}[x]$ having the desired properties of q_{d-1}. To this end, let us assume that $p_d(\lambda_i) = t$ for any $i \in \mathcal{I}$, where t is a constant number. Moreover, as $q_{d-1} = H - p_d$, we have $q_{d-1}(\lambda_i) = -p_d(\lambda_i)$ for any $i \neq 0$. Thus, we take the polynomial r with values $r(\lambda_i) = -t$ for $i \in \mathcal{I}$, and $r(\lambda_i) = -p_d(\lambda_i)$ for $i \notin \mathcal{I}$, $i \neq 0$. Then, using (2),

$$\begin{aligned} r(x) &= -t\sum_{i\in\mathcal{I}} L_i(x) - \sum_{i\notin\mathcal{I}, i\neq 0} p_d(\lambda_i)L_i(x), \\ r(\lambda_0) &= -t\sum_{i\in\mathcal{I}}(-1)^{i+1}\frac{\pi_0}{\pi_i} - \sum_{i\notin\mathcal{I}, i\neq 0} p_d(\lambda_i)(-1)^{i+1}\frac{\pi_0}{\pi_i} \\ &= -t\sum_{i\in\mathcal{I}}(-1)^{i+1}\frac{\pi_0}{\pi_i} + p_d(\lambda_0)\sum_{i\notin\mathcal{I}, i\neq 0}\frac{\pi_0^2}{m_i\pi_i^2}, \\ n\|r\|_\Gamma^2 &= r(\lambda_0)^2 + t^2\sum_{i\in\mathcal{I}} m_i + \sum_{i\notin\mathcal{I}, i\neq 0} m_i p_d(\lambda_i)^2. \end{aligned}$$

Thus, (4) yields

$$\Phi(t) = \frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} = \frac{n(\alpha t + \beta)^2}{(\alpha t + \beta)^2 + \sigma t^2 + \gamma} \leq \overline{s_{d-1}} \tag{7}$$

where

$$\alpha = \sum_{i\in\mathcal{I}}(-1)^{i+1}\frac{\pi_0}{\pi_i}, \qquad \beta = -p_d(\lambda_0)\sum_{i\notin\mathcal{I}, i\neq 0}\frac{\pi_0^2}{m_i\pi_i^2}, \tag{8}$$

$$\gamma = \sum_{i\notin\mathcal{I}, i\neq 0} m_i p_d(\lambda_i)^2 = \sum_{i\notin\mathcal{I}, i\neq 0}\frac{p_d(\lambda_0)^2}{m_i}\frac{\pi_0^2}{\pi_i^2} = -p_d(\lambda_0)\beta, \qquad \sigma = \sum_{i\in\mathcal{I}} m_i. \tag{9}$$

Now, to have the best result in (7) (and since we are mostly interested in the case of equality), we have to find the maximum of the function $\Phi(t)$, which is attained at $t_0 = \alpha\gamma/\beta\sigma$. Then,

$$\Phi_{\max} = \Phi(t_0) = \frac{n(\alpha^2\gamma + \beta^2\sigma)}{\alpha^2\gamma + \beta^2\sigma + \gamma\sigma} \leq \overline{s_{d-1}} = n - \overline{k_d}.$$

Thus, using (8)–(9) and simplifying we get (5). In case of equality, we know, by Theorem 2, that Γ is distance-regular with $r(x) = \alpha q_{d-1}(x)$ for some constant α. If $i \notin \mathcal{I}, i \neq 0$, $r(\lambda_i) = -p_d(\lambda_i) = \alpha q_{d-1}(\lambda_i) = -\alpha p_d(\lambda_i)$, so that $\alpha = 1$ since $p_d(\lambda_i) \neq 0$. Then, for every $i \in \mathcal{I}$, we get

$$P_{id} = p_d(\lambda_i) = H(\lambda_i) - q_{d-1}(\lambda_i) = -r(\lambda_i) = t_0.$$

Conversely, if Γ is distance-regular, we have that $\overline{k_d} = k_d$, and, if P_{id} is a constant, say, τ for every $i \in \mathcal{I}$, we obtain, from (2), that $\sigma = \frac{k_d}{\tau}\sum_{i\in\mathcal{I}}(-1)^i\frac{\pi_0}{\pi_i} = -\frac{k_d}{\tau}\alpha$, whence $\tau = -k_d\frac{\alpha}{\sigma}$, which corresponds to (6). Moreover,

$$nk_d = \|p_d\|_\Gamma^2 = \sum_{i\notin\mathcal{I}} m_i p_d(\lambda_i)^2 + \sum_{i\in\mathcal{I}} m_i\tau^2 = k_d^2\sum_{i\notin\mathcal{I}}\frac{\pi_0^2}{m_i\pi_i^2} + k_d^2\frac{\left(\sum_{i\in\mathcal{I}}\frac{\pi_0}{\pi_i}\right)^2}{\sum_{i\in\mathcal{I}} m_i},$$

and equality in (5) holds. □

As mentioned above, when Γ is already a distance-regular graph, Brouwer and the author [2] gave parameter conditions for partial antipodality, and surveyed known examples. The examples listed here are taken from [2].

Example 4. The Odd graph $\Gamma = O_5$, with $n = 126$ vertices and diameter $d = 4$, has intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$, so that $k_d = 60$, and spectrum $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$. Then, with $\mathcal{I} = \{2, 4\}$, the function $\Phi(t)$ is depicted in Fig. 1. Its maximum is attained for $t_0 = 6$, and its value is $\Phi(6) = 66 = s_{d-1}$. Then, $P_{24} = P_{44}$. Indeed, its distance-4 polynomial is $p_4(x) = \frac{1}{4}(x^4 - 17x^2 + 40)$ with values $p_4(5) = 60$, $p_4(3) = -8$, $p_4(1) = 6$, $p_4(-2) = -3$, and $p_4(-4) = 6$. Hence, the spectrum of Γ_4 is $60^1, 6^{50}, -3^{48}, -8^{27}$.

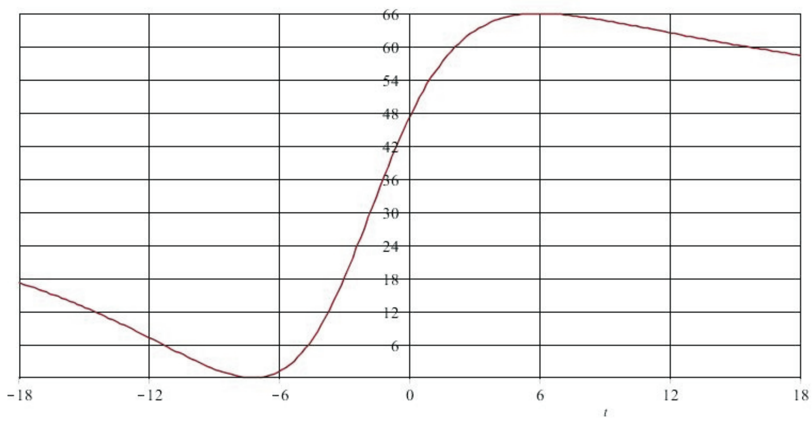

Figure 1: The function $\Phi(t)$ for O_5 with $\mathcal{I} = \{2, 4\}$.

Notice that if, in the above result, $\mathcal{I}$ is a singleton, there is no restriction for the values of p_d, and then we get the so-called spectral excess theorem (originally proved by Garriga and the author [13]).

Corollary 5 (The spectral excess theorem)**.** *Let* Γ *be a regular graph with spectrum* $\operatorname{sp}\Gamma$ *and average number* $\overline{k_d}$ *as above. Then* Γ *is distance-regular if and only if*

$$\overline{k_d} = p_d(\lambda_0) = n\left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2}\right)^{-1}.$$

Proof. Take $\mathcal{I} = \{i\}$ for some $i \neq 0$ in Theorem 3. □

As mentioned before, in [2, Th. 9–10] a distance-regular graph Γ was said to be *half antipodal* if the distance-d graph has only one negative eigenvalue (i.e., P_{id} is a constant for every $i = 1, 3, \ldots$). Then, a direct consequence of Theorem 3 by taking $\mathcal{I} = \mathcal{I}_{\text{odd}} = \{1, 3, \ldots\}$ is the following characterization of half antipodality.

Corollary 6. *A regular graph* Γ *as above is a half antipodal distance-regular graph if and only if the following equality holds:*

$$\overline{k_d} = \frac{n \sum_{i\ odd} m_i}{\left(\sum_{i\ odd} \frac{\pi_0}{\pi_i} \right)^2 + \sum_{i\ even} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\ odd} m_i}. \tag{10}$$

□

Example 7. The Coxeter graph $\Gamma = C$, on $n = 28$ vertices, has diameter $d = 4$, intersection array $\{3,2,2,1;1,1,1,2\}$, $k_4 = 6$, and spectrum $3^1, 2^8, (\sqrt{2}-1)^6, -1^7, (-1-\sqrt{2})^6$. Then, with $\mathcal{I} = \{1,3\}$, the equality in (10) holds and, then $P_{14} = P_{34}$. In fact, the distance-4 polynomial is $p_4(x) = \frac{1}{2}(x^4 - x^3 - 7x^2 + 5x + 6)$ with values $p_4(3) = 6$, $p_4(2) = -2$, $p_4(\sqrt{2}-1) = 2+\sqrt{2}$, $p_4(-1) = -2$, and $p_4(-1-\sqrt{2}) = 2-\sqrt{2}$. Thus, Γ is half antipodal since the spectrum of Γ_4 is $6^1, (2+\sqrt{2})^6, (2-\sqrt{2})^6, -2^{15}$.

Recall that a regular graph Γ is strongly regular if and only if it has, either three (when Γ is connected), or two (when Γ is the disjoint union of several copies of a complete graph) distinct eigenvalues (see e.g. Godsil [15]). Then, we have the following characterization of those distance-regular graphs having strongly regular distance-d graph.

Corollary 8. *A regular graph* Γ *as above is distance-regular with strongly regular distance-*d *graph* Γ_d *if and only if the following equality holds:*

$$\overline{k_d} = \frac{n(n-1)}{\left(\sum_{\substack{i\ even \\ i\neq 0}} \frac{\pi_0}{\pi_i} \right)^2 + \left(\sum_{i\ odd} \frac{\pi_0}{\pi_i} \right)^2 + \left(1 + \sum_{\substack{i\ even \\ i\neq 0}} \frac{\pi_0^2}{m_i \pi_i^2} \right) \sum_{i\ odd} m_i + \left(1 + \sum_{i\ odd} \frac{\pi_0^2}{m_i \pi_i^2} \right) \sum_{\substack{i\ even \\ i\neq 0}} m_i}. \tag{11}$$

Proof. Apply Theorem 3 with equality for $\mathcal{I}_{\text{even}} = \{2,4,\ldots\}$, and $\mathcal{I}_{\text{odd}} = \{1,3,\ldots\}$, to obtain the values of $\sum_{i \text{ odd}} m_i$ and $\sum_{\substack{i \text{ even} \\ i\neq 0}} m_i$, add up both equalities and solve for $\overline{k_d}$. □

Example 9. The Wells graph $\Gamma = W$, on $n = 32$ vertices, has intersection array $\{5,4,1,1;1,1,4,5\}$ and spectrum $5^1, \sqrt{5}^8, 1^{10}, -\sqrt{5}^8, -3^5$. This graph is 2-antipodal, so that $k_d = 1$. Then, Fig. 2 shows the functions $\Phi_0(t)$ with $\mathcal{I}_0 = \{2,4\}$, and $\Phi_1(t)$ with $\mathcal{I}_1 = \{1,3\}$. Their (common) maximum value is attained for $t_0 = 1$ and $t_1 = -1$, respectively, and it is $\Phi_0(1) = \Phi_1(-1) = 31 = s_{d-1}$. Then, $P_{24} = P_{44}$ and $P_{14} = P_{34}$. Indeed, the distance-4 polynomial is $p_4(x) = \frac{1}{20}(x^4 - 3x^3 - 13x^2 + 15x + 20)$ with values $p_4(5) = 1$, $p_4(\sqrt{5}) = -1$, $p_4(1) = 1$, $p_4(-\sqrt{5}) = -1$, and $p_4(-3) = 1$. Hence, the spectrum of Γ_4 is $1^{16}, -1^{16}$ since it is constituted by 16 disjoint copies of K_2.

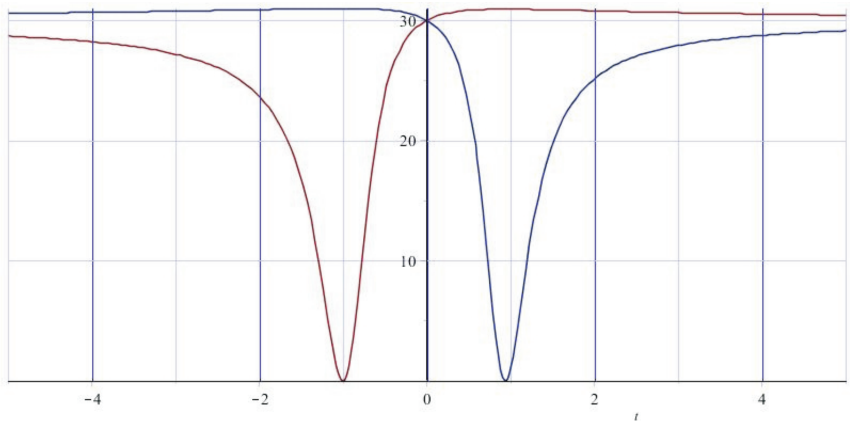

Figure 2: The functions $\Phi_0(t)$ (in red) with $\mathcal{I}_0 = \{2,4\}$, and $\Phi_1(t)$ (in blue) with $\mathcal{I}_1 = \{1,3\}$ of the Wells graph.

In fact, the above expression can be simplified because $\sum_{i \text{ even}} m_i + \sum_{i \text{ odd}} m_i = n$ (with $m_0 = 1$), $\sum_{i \text{ even}} \frac{\pi_0}{\pi_i} = \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i}$ (see [9]), and, from (3), $\sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} + \sum_{i \text{ odd}} \frac{\pi_0^2}{m_i \pi_i^2} = n/p_d(\lambda_0)$. Anyway, we have written (11) as it is to emphasize the 'symmetries' between even and odd terms.

As in the case of Theorem 3, the equalities in Corollaries 6 and 8 also hold as inequalities, but, as one of the referees pointed out, the best inequality for general graphs is the one that would come with Corollary 5. Namely $\overline{k_d} \le p_d(\lambda_0)$, where $p_d(\lambda_0)$ is the spectral excess given by (3). This follows from the mentioned property that $q_{d-1}(x) = n - p_d(x)$ is the polynomial $r \in \boldsymbol{R}_{d-1}[x]$ that maximizes the quotient $r(\lambda_0)/\|r\|_\Gamma$.

2.2 The case $0 \in \mathcal{I}$

To deal with this case, we could proceed as above by defining conveniently a degree $d-1$ polynomial r. Then the proof is similar to the one for Theorem 3. If $0 \in \mathcal{I}$ then $p(\lambda_i) = p(\lambda_0)$ for any $i \in \mathcal{I}$. Moreover, the odd indexes, cannot belong to $\mathcal{I}$. In particular $1 \notin \mathcal{I}$. For instance, a possible choice for $r \in \mathbb{R}_{d-1}[x]$ is:

- $r(\lambda_0) = n - p_d(\lambda_0)$, $r(\lambda_i) = -p_d(\lambda_0)$ for $i \in \mathcal{I}$, $i \neq 0$.
- $r(\lambda_i) = -tp_d(\lambda_i)$ for $i \notin \mathcal{I}$, $i \neq 1$,

However, we can follow a more direct approach by using (7). First, the following result was proved in [2]:

Proposition 10 ([2, Prop. 8]). *Let Γ be a distance-regular graph with diameter d. If $P_{0d} = P_{id}$ then i is even. Let $i > 0$ be even. Then $P_{0d} = P_{id}$ if and only if Γ is antipodal, or $i = d$ and Γ is bipartite.* □

Notice that, in this case, the Kneser graph is disconnected. Thus, the above proposition can be seen as a spectral characterization of the so-called *imprimitive* distance-regular graphs (see Smith [17]).

Theorem 11. *Let* Γ *be a regular graph with* n *vertices, spectrum* $\mathrm{sp}\,\Gamma$ *as above, and mean excess* $\overline{k_d}$*. Then, for every* $i = 1, \ldots, d$,

$$\overline{k_d} \le \frac{n\left(m_i + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j \pi_j^2}\right)}{\left(\frac{\pi_0}{\pi_i} + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j \pi_j^2}\right)^2 + m_i + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j \pi_j^2}}. \tag{12}$$

Moreover:

(a) *Equality holds for some* $i \neq d$ *if and only if it holds for any* $i = 1, \ldots, d$ *and* Γ *is an antipodal distance-regular graph.*

(b) *Equality holds only for* $i = d$ *if and only if* Γ *is a bipartite, but not antipodal, distance-regular graph.*

Proof. The inequality (12) follows from (7) by taking $\mathcal{I} = \{i\}$ for some even $i \neq 0$, and choosing $t = p_d(\lambda_0)$. Then, in case of equality, Theorem 3 tells us that Γ is distance-regular. Then, Γ_d is a regular graph with equal eigenvalues P_{0d} and P_{id}. So, the result follows from Proposition 10. □

Example 12. For the Wells graph the right hand expression of (12) gives $1(= k_4)$ for any $i = 1, \ldots, 4$, in concordance with its antipodal character. In contrast, the folded 10-cube FQ_{10}, on $n = 512$ vertices, has intersection array $\{10, 9, 8, 7, 6; 1, 2, 3, 4, 10\}$ and spectrum $10^1, 6^{45}, 2^{210}, -2^{210}, -6^{45}, -10^1$. Then, the right hand expression of (12) gives 234.16, 293.36, 293.36, 234.16 for $i = 1, 2, 3, 4$, respectively, and $126(= k_5)$ for $i = 5$, showing that FQ_{10} is a bipartite distance-regular graph, but not antipodal.

Another characterization of antipodal distance-regular graphs was given by the author in [8] by assuming that the distance-d graph of a regular graph is already a disjoint union of cliques.

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