On Structures and Linkages

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Abstract

Because this journal has as one of its main concerns the subject of structural rigidity, the question of the instantaneous deformability of a structure has been considered in statical terms rather than kinematic ones. This article is intended to put forward the kinematician's treatment of linkage mobility and to relate it to the internal motion cabability of a structure. We consider the topics of frames (bar-and-joint structures), hinged panel systems and spatial polygons from a kinematic standpoint in such a way that a ready comparison may be made with the statical approach.

Résumé

Parce que cette revue propose parmi ses thèmes privilégiés celui de la rigidité des structures, la question de la déformabilité d'une structure a été considérée en des termes statiques plutôt que cinématiques. Cet article a pour but de montrer la mobilité du chaînon mécanique, vue par le cinématicien, et de l'apparenter aux possibilités de mouvement relatif d'une structure. Nous analysons les sujets de structures (structures à barres et noeuds), systèmes de panneaux à charnières et polygones spatiaux, d'un point de vue cinématique de manière à ce que l'on puisse faire une simple comparaison avec l'approche statique.

Introduction

One of the three stated principal concerns of this journal is that of structural rigidity, which concept may be reinterpreted as the lack of **instantaneous mobility** of a jointed system of rigid bodies. The common ground between this area of study and that of linkage kinematics is thereby immediately established, since the latter field is concerned with **full-cycle**, or **gross**, mobility of a linkage, and full-cycle mobility can be regarded as instantaneous mobility in every configuration of the articulated assemblage of rigid bodies. The civil engineer or architect requires rigidity of a structure, the mechanical engineer normally works with 1 degree of gross mobility of a linkage, but a structure and a linkage, or mechanism, are composed in similar ways.

Because of the fashion in which this journal was set up, the approach to the matter of rigidity / mobility has been taken from the structural viewpoint, rather than the kinematic one. The purpose of this article is to put forward some of the basic tools used by the kinematician in this topic, and to illustrate their applicability in a manner which will allow some comparison with alternative techniques.

Linkages

Although a linkage is composed physically of rigid bodies, the kinematician's attention is focused primarily on the articulations, or kinematic joints, which constrain one member relative to another. The members themselves are replaced by common normals between

successive joint axes. The elements of a **kinematic chain** are illustrated by **Figure 1**, which also includes a representation of the joint variables and linkage parameters. We define these quantities as follows.

$a_{i,i+1}$ is the constant length of the common perpendicular between the axes of successive joints i and $i+1$.

r_i is the variable distance, measured along joint axis i, between successive common perpendiculars; this symbol applies particularly to prismatic and cylindric joints.

R_i is a quantity similar to r_i, but constant, applying especially to revolutes and screws; for a revolute, it is the so-called offset.

h_i is the pitch of screw pair i.

$\alpha_{i,i+1}$ is the angle of skew between the directions of consecutive joint axes i and $i+1$.

Θ_i is the joint angle between the two common normals relating to joint i.

In a single-loop chain, the last link counted coincides with the first.
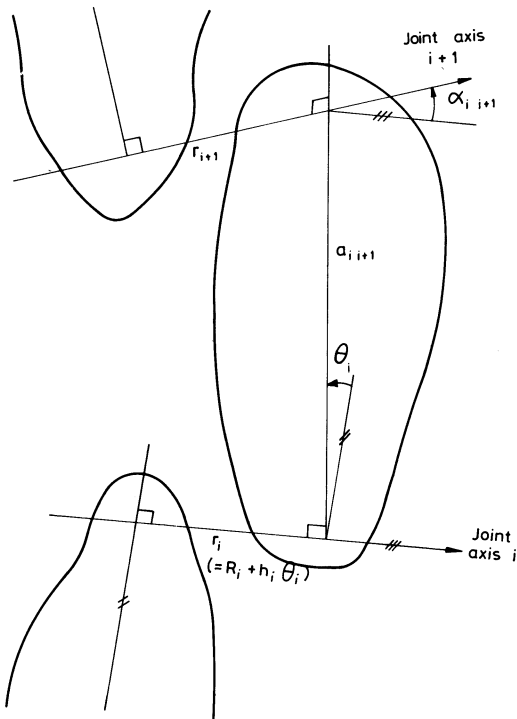

Figure 1

There are six distinct joints, or **kinematic pairs**, which function by virtue of surface contact between adjacent members. They are known as **lower pairs**, and are described below.

R   revolute, or turning pair — 1 degree of freedom,

P   prismatic or sliding pair — 1 degree of freedom,

H   helical or screw joint — 1 degree of freedom,

C   cylindric joint — 2 degrees of freedom

S   spherical or global pair — 3 degrees of freedom,

E   planar pair — 3 degrees of freedom

There are also higher pairs, in which the contacting elements have only a line or a point in common. In general kinematic studies, all lower pairs are usually considered, but, in the present context, only two of them are relevant, namely: the **spherical**, or **universal**, or **global** joint and the **revolute**, or **turning pair**, or **hinge**.

Frames

In the investigation of mobility (or rigidity) of frames, such as the spatial bar-and-joint systems treated briefly by (Crapo 1979) and at greater length in Calladine's excellent paper (Calladine 1978), the joint in question, which coincides multiply with each node, must be a spherical one. Let us see how a kinematician would determine the expected degree of mobility of such a frame, in contrast to the approach explicit or implicit in the just-mentioned references, which relies on relative freedom between nodes. Our method, instead, is based on relative freedom between rigid bodies.

A general rigid body requires 6 coordinates for its specification, 2 more than for a straight line in space. But a rigid bar requires no specification for rotation about its own axis, since it is regarded as having no lateral dimension — as a mechanical component, it can only be stressed in direct tension or compression, not by flexure. Thus, a rigid bar unconstrained in space has 5 degrees of freedom.

Of the ten degrees of freedom of two such rods, four are relative degrees of freedom. A spherical joint between the two bars reduces the number of relative degrees of freedom by 3. The remaining relative motion of the bars can be prevented by adding a

single bar. Fastening the prame to some spatial reference system reduces the total number of degrees of freedom of the assembly by 6.

At a node n where B_n bars meet, the number of spherical joints J_n coinciding with the node is given by $J_n = B_n - 1$, since each of these joints can be considered as connecting only 2 bars at that node. If the total number of nodes is N, bars B and joints J, we may write

$$J = \sum_{n=1}^{N} (B_n - 1) = 2B - N.$$

From the preceding paragraph, if the total number of degrees of freedom (mobility) of the frame is M, we have that, in the absence of special geometrical properties,

$$M = 5B - 6 - 3J = 3N - 6 - B.$$

For the case of simple rigidity, $M = 0$. We then have the result $B = 3N - 6$, in accordance with that given by (Calladine 1978). For a standard mobility-1 linkage, we should have instead that $B = 3N - 7$.

For a planar frame, instead of the above equations, we can put

$$M = 3B - 3 - 2J = 3B - 3 - 2(2B - N) = 2N - 3 - B.$$

In the case of simple rigidity, we find, again in accordance with the result stated by Calladine, that $B = 2N - 3$.

In fact, however, the kinematician should be aware of the probable importance of metrical properties in what he sees as a **multiloop** linkage / structure, and so, rather than the simplistic tally carried out above, he should proceed by means of an appropriate algorithm, such as that outlined in (Baker 1980b, 1981). A multiloop linkage, as the name implies, is one the graph of which has a cyclomatic number (the number of independent circuits) in excess of 1. (In graphing a linkage, the nodes represent the links and the edges depict the joints.) The addition of independent loops to a linkage / structure increases the possibility of special geometrical properties which may give rise to unexpected mobility.

A word of warning is sounded here to model-builders to ensure that the joint freedoms assumed in analysis

comply with the actual motion capabilities of the articulations. Care is required especially in the case of a frame / linkage, because of the manner in which flexible junctions are commonly provided.

Panel systems

We now turn to hinged panel structures, given some attention by (Crapo 1979), and looked at in more detail by (Baracs 1975). The type of joint used for such systems is, of course, the revolute. A panel, or **plate**, is a true rigid body and requires 6 coordinates for its specification. A hinge allows only 1 degree of relative freedom between two adjacent panels. For a hinged panel system, then, without regard to possible special dimensional properties, we may write

$$M = 6P-6-5J,$$

where P is the number of panels. In the case of one continuous loop, or ring, of plates, for 1 degree of mobility, since P = J, we must have that P = 7. For such a ring to be a simple structure, P = 6. These results are given in (Crapo 1979).

Again, special geometrical properties are very important, and there are many more singularities than is suggested by Crapo's summary. Probably, the best contemporary general list of the various sets of singular conditions is given by (Hunt 1978). It is not necessary to be aware of the geometrical significance of such a speciality, however. The algebraic approach of (Baker 1980b, 1981) is capable of taking into account any special dimensional conditions. It should also be mentioned that, despite the qualitative value of (Baracs 1975), some quantitative conclusions therein we find difficult to reconcile with the kinematic approach. We direct the reader to (Calladine 1978) and to the linkage literature for a rigorous approach.

Skew polygons

Now, there is a close relationship (often an equivalence) between hinged polyhedra and what have been called in the classical literature skew polygons. (Bricard 1879) was possibly the first to recognize this fact; his discovery of deformable octahedra led him to establish the concomitant existence of certain six-revolute linkages of a very special type. Bricard himself seems to have been somewhat confused over skew polygons, however, as is reported at length in (Baker 1980a). The apparent reason for this misunderstanding is that he was involved both with some polygons which have zero offsets (Bricard 1927), namely, his basic plane-symmetric and line-symmetric loops, as well as his trihedral linkage, and other, derived from his octahedra, which have zero link-lengths. This misapprehension and others have continued to be promulgated over the years, by researchers who do not fully understand the criteria for linkage mobility.

The hinges of a spatial panel system will be the revolutes of an associated linkage. Whenever they have non-zero length, the linkage will have non-zero offsets. Whenever hinges intersect adjacent ones, the link-lengths of the associated kinematic chain will be zero. We should therefore have a skew polygon, in which every edge is a joint axis. Mobility or otherwise of a linkage of non-standard type (an **overconstrained loop**) is usually established by a lengthy geometrical or algebraic process, but, for such skew polygons, we may often proceed much faster because of some well-known results.

Mobility / rigidity

Let us consider a question related to that posed by (Baracs 1975) concerning, in our case, two different circuits of joint axes about a hinged cube (**Figures 2**). In the first instance, we assume that we are dealing only with skew polygons; that is, the plates are not present, the polygons being defined solely by the hinges. In the diagrams, the arrows on the hinges are conventional, but necessary in order to write down relationships among joint offsets. The other arrows indicate selected directions for the link-lengths, which have zero magnitude. They are required for the determination of joint angles.

For both **a** and **b** of **Figures 2**, we may write

$$R_1 = R_2 = R_3 = R_4 = R_5 = R_6$$
$$\alpha_{i/i+1} = 3\pi/2, \text{ for all i}$$
$$a_{i/i+1} = 0, \text{ for all i.}$$

In the case of **a**, we see that

$$\Theta_1 = \Theta_4 = = 0, \Theta_2 = \Theta_5 = \pi/2, \Theta_3 = \Theta_6 = 3\pi/2,$$

a

b

Figures 2 a and b

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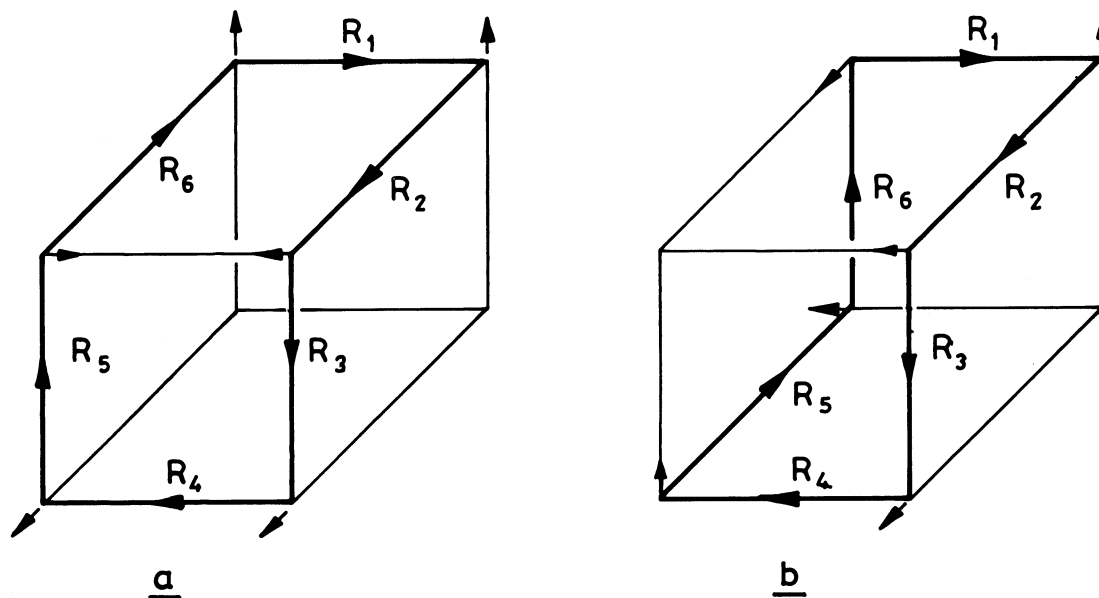
and so the assemblage is **line-symmetric**, whence we can establish mobility — the reader is referred to (Bricard 1927, Waldron 1969, Baker 1979). On the other hand, for **b**, we have

$$\Theta_1 = \Theta_3 = \Theta_5 = 3\pi/2, \Theta_2 = \Theta_4 = \Theta_6 = \pi/2,$$

which is not line-symmetric, and the polygon is a rigid structure. We say that **a** and **b** are two different **closures** of the same loop. In fact, this very chain, in a different guise, is used in an example in (Baker 1979).

Now, if the directions of R_2, R_4, R_6 are reversed in both **a** and **b** of **Figures 2**, a certain kind of **plane-symmetry** is noted in each case. Although some varieties of plane-symmetry in a loop can be used to establish mobility, this is not one of them, and our conclusions are unchanged.

We now reinstate the plates and look at the implications. We need consider **b** no further, as we have already found it to be rigid. In the case of **a**, we see that there are two panels each of which contains three of the joint axes. Each of the panels imposes an additional constraint between the two alternate hinges, effectively locking the central one. That is, R_2 and R_6 are no longer free to move relative to each other; nor are R_3 and R_5. So, although the polygon is mobile, the polyhedron is not. It is clear that, in a hinged polyhedron, we cannot have three consecutive hinges of a loop on the same face and maintain mobility, because the joint angle of the middle hinge will be fixed. Bricard's octahedra were so assembled that each triangular face had only two consecutive hinges. Each type of octahedron was a multiloop linkage, from which he was able to derive four distinct single-loop chains of six members.

We turn now to the actual matter raised by Baracs, that of two different circuits of faces. One of the circuits is the structure shown in **b** of **Figures 2**. The other is illustrated in **Figure 3** and has the following dimensional properties.

$$R_1 = R_4 = 0, R_2 = R_3 = R_5 = R_6$$
$$a_{12} = a_{23} = a_{45} = a_{56} = 0, a_{34} = a_{61}$$
$$\alpha_{12} = \alpha_{23} = \alpha_{45} = \alpha_{56} = \pi/2, \alpha_{34} = \alpha_{61} = 0$$

For the configuration depicted, we also see that

$$\Theta_1 = \Theta_3 = \Theta_4 = \Theta_6 = 0, \Theta_2 = \Theta_5 = 3\pi/2.$$

We therefore find that the chain is line-symmetric and, consequently, mobile.

Closing remarks

There is a wealth of classical and contemporary literature in the separate (or separated) fields of static and kinematic analysis. They have become over-specialised to the point where a worker in one area is quite ignorant of what is known in the other. We have the opportunity, particularly in the pages of this journal, to remedy this unfortunate situation. The matter need not rest with mutual understanding and cross-fertilisation between these two fields, however. There is reason to believe that applications for theoretical results from linkage / polyhedron research abound in the studies of structural chemistry and molecular biology, plate tectonics and the theory of plasticity, kinesiology and robotics, as well as others. In addition, seemingly unrelated topics in mathematics, such as graph theory, can be brought to bear on the joint disciplines. It would be a pity to allow the chance to pass us by.
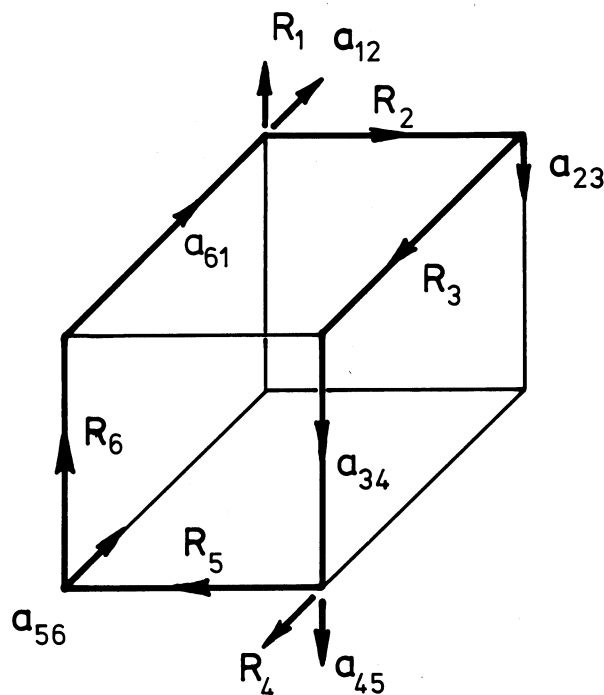

Figure 3

Bibliography

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