Questions on the Rigidity of Structures

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Abstract

Every graph has a typical behavior which allows it to be classified as either **generically rigid** or **generically non-rigid**, that is, as having none or at least one finite motion, respectively, when placed in general position in a given Euclidean space. For strut and cable structures, a more restrictive notion of general position must be introduced.

Structures which are rigid but not infinitesimally rigid are constructed by placing bars between nodes when they are at a minimum distance with respect to some finite motion. Energy functions may be used to evaluate the stability of equilibria for structures in special positions.

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Résumé

Chaque graphe a un comportement typique et peut ainsi être classifié comme étant génériquement rigide ou génériquement non rigide, respectivement, lorsque placé en position générale dans un espace euclidien donné. Pour des structures autotendues, il faut introduire une notion plus restrictive de la position générale.

Les structures qui sont rigides mais non rigides infinitésimalement sont réalisables en plaçant des barres entre les noeuds lorsqu'ils se trouvent à une distance minimale quant au mouvement fini. On peut se servir de fonctions énergétiques pour évaluer la stabilité de l'équilibre structural.

1. Introduction

For the last five years, I have been involved in the study of structural rigidity, focusing primarily on **bar and joint structures** and recently **strut and cable structures**. Initially, I was interested in the notion of rigidity which means the absence of finite (rather than just infinitesimal) motions. At the time this struck me as the most natural concept (and my ignorance of engineering left me sheltered from the influence of competing notions). Thus, in this article, a structure is said to be **rigid** if it is not a mechanism. A structure which does not admit even infinitesimal motions will here be called **infinitesimally rigid**. These definitions are not in agreement with those employed by Henry Crapo in his article *Structural Rigidity* in the first issue of this journal, but this terminology does seem to be gaining acceptance in the literature. (However, the meaning of all other terms used here will agree with Crapo's usage.)

Figure 1 consists of examples of bar and joint structures in the plane which are (a) rigid and infinitesimally rigid, (b) rigid but not infinitesimally rigid, and (c) neither rigid nor infinitesimally rigid. As our definitions suggest, infinitesimal rigidity does imply rigidity and thus infinitesimally rigid structures which are mechanisms do not exist.

Of course, from a structural point of view the bar and joint structures in **Figure 1** are of little interest — they are just particularly simple illustrations of the concepts. It is certainly possible to demonstrate the same ideas by examples which do arise in engineering

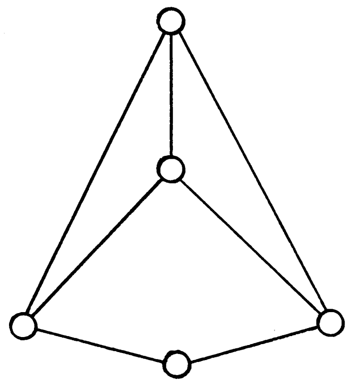

Figure 1a

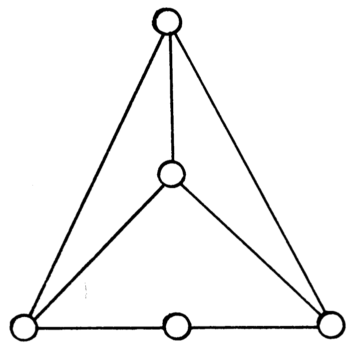

Figure 1b

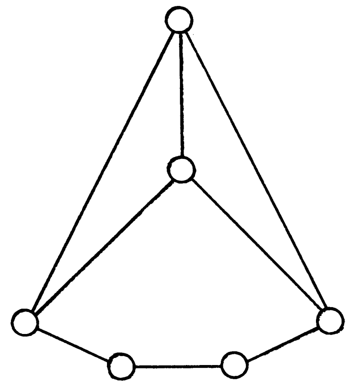

Figure 1c

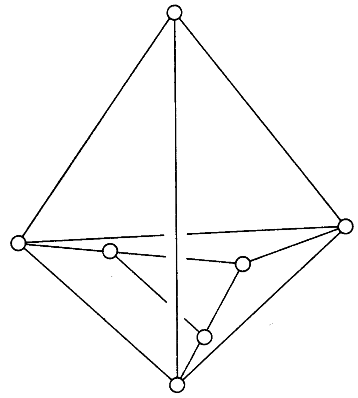

Figure 2

and architectural practice, although such examples tend to be much more complicated.

Before continuing, it might be useful to look at a typical instance of the mathematician's perspective on the rigidity of structures. In even a cursory examination of the literature, one is likely to encounter the assertion that in general a bar and joint structure with j joints must have at least 3j-6 bars in order to be rigid. There is abundant evidence in support of this assertion, but, alas, there are also exceptions (as suggested by the presence of the phrase «in general»). One such example, consisting of a tetrahedron with a triangle in the plane of its base, is shown in **Figure 2**. Typically, the mathematician is much more interested in understanding the nature and extent of the exceptions than in appreciating the fact that the formula is usually correct. This characteristic perverseness of the mathematician will certainly be evident throughout my article.

2. Generic properties of structures

Abstractly (or topologically), a bar and joint structure is given by a graph consisting of a finite set of **nodes**, certain pairs of which are distinguished by being joined by **bars**. One then obtains a bar and joint structure in some Euclidean space by specifying the position of the nodes of the graph in this Euclidean space.

For example, the bar and joint structures in the plane shown in (a) and (b) of **Figure 1** are both obtained from the same graph by specifying different positions for one of the nodes of the graph. By specifying other locations for some of the nodes of this graph (in which two of the nodes and three pairs of the bars lie on top of one another), one can even create the mechanism shown in **Figure 3(a)**. Thus rigidity is not an intrinsic property of the underlying graph; the position of the nodes must also be taken into account. Similarly, the bar and joint structure shown in **Figure 3(b)** is rigid even though it arises from the same graph as the structure shown in **Figure 1(c)**. However, it appears that the positions of the nodes in **Figure 3** are rather carefully contrived (satisfying certain projective geometric conditions) while, in some sense, (a) and (c) of **Figure 1** represent the typical behaviors of structures given by these graphs. Thus the structures given by the graph in **Figure 1(a)** are typically rigid, while those given by the graph in **Figure 1(c)** are typically mechanisms.

These observations lead to a number of questions. Does every graph have such a typical behavior which allows it to be classified as either **generically rigid** or **generically nonrigid** in a given Euclidean space? If so, how can one predict this generic classification? Finally, what can be said about the exceptions to the typical behavior? Building on ideas in (Gluck 1975), Asimow and I (Asimow 1978) show that every graph can be classified as generically rigid or generically nonrigid in any Euclidean space. Indeed, this classification can be determined from the maximum value of the rank of the various structures given by locating the nodes of the graph in all possible positions.

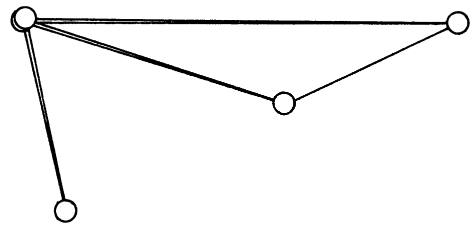

Figure 3a

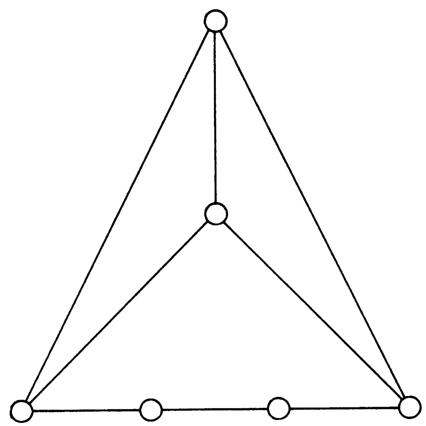

Figure 3b

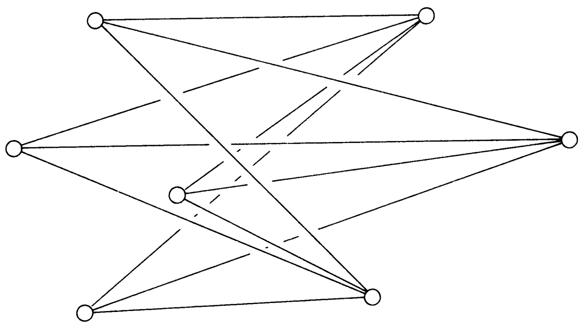

Figure 4

The positions of the nodes for which this maximum rank is attained form a dense open set of well-behaved locations for the nodes. For example, for these positions of the nodes (called **regular points**), the structure is rigid if and only if it is infinitesimally rigid. Thus the relationship between rigidity and infinitesimal rigidity is clarified by these results, an expository account of which appears in (Roth 1979). Incidentally, these ideas also shed light on the assertion involving 3j-6 mentioned in the introduction.

One concrete illustration of these results deals with the role that triangles play in rigid structures. A brief look at the commonplace rigid structures (bridges, cranes, towers, etc.) suggests that triangles are at least a pervasive element in the design of structures. But are they essential? Do there exist rigid or infinitesimally rigid bar and joint structures in space with no triangles?

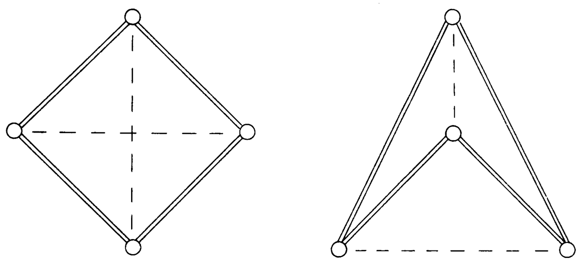

Figure 5

The reader may be surprised to learn that such structures do actually exist and are not difficult to describe. Consider two collections A and B of points in space. A bar and joint structure is then created by connecting every element of A to every element of B by a bar (as shown in **Figure 4** where set A has four elements and set B three elements). By examining the stresses of such structures, Bolker and I (Bolker 1979) show that if both A and B consist of five or more points then the resulting bar and joint structure is generically rigid in three-dimensional Euclidean space.

For strut and cable structures, the plot thickens. Now we find that a graph may fail to have a generic classification. For example, the strut and cable structures shown in **Figure 5** both arise from the same graph (where cables are indicated by dashed lines and struts by double lines). The structure shown in (a) is rigid in the plane, while that shown in (b) is a mechanism. However, neither structure is carefully contrived; both exhibit the typical behavior of strut and cable structures arising from the given graph for which the positions of the nodes are close to the positions shown in (a) and (b). Nevertheless, the phenomenon exhibited here is understood and can be explained by the distribution of signs (indicating tensions or compressions) in the stresses of the structures. Furthermore, Whiteley and I in a recent preprint (Roth 1980) identify a dense open set of well-behaved locations of the nodes (called points in **general position**) for which the strut and cable structure is rigid if and only if it is infinitesimally rigid. Moreover, the classification of a strut and cable structure as rigid or not rigid is constant on the connected pieces (or components) of this dense open set.

On the basis of results such as these, one could argue that generic properties are reasonably well understood for structures in general. However, I have thus far avoided the last question concerning exceptions. That is, what can be said about structures obtained by locating the nodes of a graph at points which are not regular points (or not in general position)? Examples of such points, called **singular points**, are provided by the structures shown in **Figure 1(b)** and **Figure 3.** Although it is known that such points have connections with projective geometry, little seems to be known about the rigidity of structures at singular points.

3. Singular points

One particularly important class of singular points arises from bar and joint structures which are rigid but not infinitesimally rigid, such as those shown in **Figure 1(b)** and **Figure 3(b)**. In each of these examples the removal of a bar, say one of the horizontal bars at the bottom, gives a structure which is a mechanism. As each mechanism moves, this distance attains a minimum value as the mechanism passes through the given position of the nodes (as shown in **Figure 6**). A different choice of the bar to be removed may lead to a maximum value for the distance during the finite motion of the resulting mechanism; the noteworthy fact is the attainment of an extremum.

Is, as James Clerk Maxwell seems to suggest, the behavior of these structures typical of structures which are rigid but not infinitesimally rigid? If so, how can this be used to further our understanding of such structures? The former question can be affirmatively answered. Consider an independent bar and joint structure which is a mechanism. If one adds additional bars to this structure in such a way that (a) the square of the distance between each pair of nodes joined by a new bar has derivative zero as every finite motion of the original mechanism passes through the given position of the nodes and (b) every finite motion of the original mechanism is prevented by at least one of the new bars, then one obtains a bar and joint structure which is rigid but not infinitesimally rigid. Furthermore, all rigid but not infinitesimally rigid bar and joint structures arise in precisely this way.

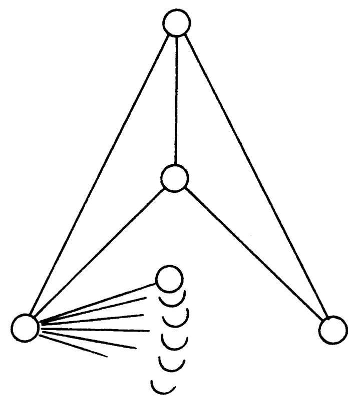

Figure 6

Next, how might we use this characterization? To do so, we need techniques for finding and classifying the critical points of functions of the form $||x-y||^2$ defined on certain subsets of Euclidean space given by the original independent mechanism. The effectiveness of this point of view in some simple examples suggests that it may be a promising approach, especially in settings where special geometric features (such as convexity) are present.

One final line of attack on the behavior of structures at singular points relies on energy methods introduced by Robert Connelly several years ago and systematically exploited in his recent preprint (Connelly 1980). One simple example in which these techniques shed some light is the collinear triangle consisting of three points a, b and c, where b is the midpoint of the segment ac. The corresponding bar and joint structure shown in **Figure 7** is rigid but not infinitesimally rigid. A nonzero stress of the structure is given by assigning the number 2 to each of the two short bars and the number -1 to the long bar. Multiplying the vector representing each bar by the scalar assigned to that bar, one obtains at each node of the structure an internal tension compression equilibrium (or **stress**). Of course, the negative of this stress (given by -2's for the short bars and 1 for the long bar) is also a nonzero stress. Interpreting positive scalars as tensions and negative scalars as compressions in the bars, it is easy to see that the original stress 2, 2, -1 gives a stable equilibrium while its negative gives an unstable equilibrium. Furthermore, if one replaces the bars with positive coefficients by cables and those with negative coefficients by struts, the original stress gives a rigid strut and cable structure while its negative gives a strut and cable structure which is a mechanism.

Some understanding of all this is gained by considering an energy function of a structure and a stress. For instance, consider the energy function of the collinear triangle and its stress 2, 2, -1 defined by

$$E(x,y,z) = 2||x-y||^2 + 2||y-z||^2 - ||x-z||^2$$

where x, y and z are arbitrary points in the plane. We find that the given position a, b, c of the nodes is a critical point of E. In fact, this critical point is a minimum of the energy function E which explains the stability of the equilibrium. Of course, the correspon-ding energy function for the same structure with the stress -2, -2, 1 is -E which attains a maximum at a, b, c, giving an unstable equilibrium. Finally,

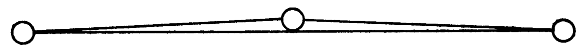

Figure 7

the behavior of the strut and cable structures obtained via the stresses exemplifies a simple general result. That is, for any rigid bar and joint structure together with a stress such that the corresponding energy function attains a local minimum at the given position of the nodes, the structure obtained by replacing bars with positive coefficients by cables, those with negative coefficients by struts, and retaining bars with zero coefficients, is also rigid.

These observations represent little more than the recognition that energy functions promise to be a fruitful tool in the study of rigidity. The various connections between rigidity, infinitesimal rigidity, stresses, the extrema of energy functions, unique embeddability and various related matters await explanation.

Bibliography

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Asimow 1978 L. Asimow and B. Roth A—M—R	**The rigidity of graphs** Trans. Amer. Math. Soc. 245 (1978), 279-289.	Rigidity, regular points, generic rigidity, rigidity of planar graphs, and convex polyhedra.
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