Uniform Ant-hills in the World of Golden Isozonohedra

by Koji Miyazaki and Ichiro Takada

Structural Topology #4, 1980

Abstract

This paper has three parts.

Part 1 (by Koji Miyazaki) shows the curious world of Golden Isozonohedra, which have only one type of face, the golden diamond. Urban plans and architectural designs around the world have such periodic, sometimes perhaps essentially non-periodic, patterns.

Part 2 (by Ichiro Takada) deals with the uniform Ant-hills in which regular polyhedra are connected to each other face-to-face, leaving many finite (not infinite) tunnels. Their form suggests a nest made by ants, or linked rings of molecules.

Part 3 (by Koji Miyazaki) combines the two ideas described above. The golden isozonohedra are constructed as uniform Ant-hills born from regular dodecahedra. In other words, the curious non-periodic pattern seen in the world of golden isozonohedra can be constructed using regular dodecahedra only.

Topologie Structurale #4, 1980

Résumé

Cet article est composé de trois parties.

La première partie (par Koji Miyazaki) introduit le monde étrange des «isozonaèdres dorés» qui n'ont qu'un seul type de face, le diamant doré. Les plans urbanisés et les designs architecturaux de par le monde sont basés sur de tels modèles périodiques, ou parfois essentiellement non-périodiques.

La seconde partie (par Ichiro Takada) traite des fourmilières uniformes dans lesquelles les polyèdres réguliers sont liés les uns aux autres face à face, laissant maints tunnels finis (non-infinis). Leur forme fait penser à une fourmilière, ou aux anneaux liés des molécules.

La troisième partie (par Koji Miyazaki) associe les deux idées décrites ci-dessus. Les isozonaèdres dorés sont construits comme des fourmilières uniformes nés de dodécaèdres réguliers. En d'autres termes, le curieux modèle non-périodique que l'on voit dans le monde de l'isozonaèdre doré peut être construit en utilisant tout simplement des dodécaèdres réguliers.

1. On the Golden Isozonohedra

A rhombus whose diagonal lengths have the golden ratio, 1 : 1.618, is called the **golden diamond** (Baer 1970).

The adjacent two face angles of the golden diamond are equal to the two and only two angles under 180° which are subtended at its body-center by arbitrary pairs of vertices of the regular icosahedron. In other words, the regular icosahedron being dual to the regular dodecahedron, a golden diamond can be constructed by regular dodecahedral ball joints and sticks of equal length, as shown in **Figure 1**.

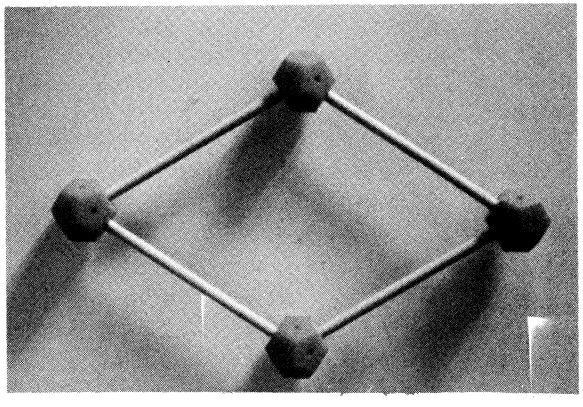


Figure 1

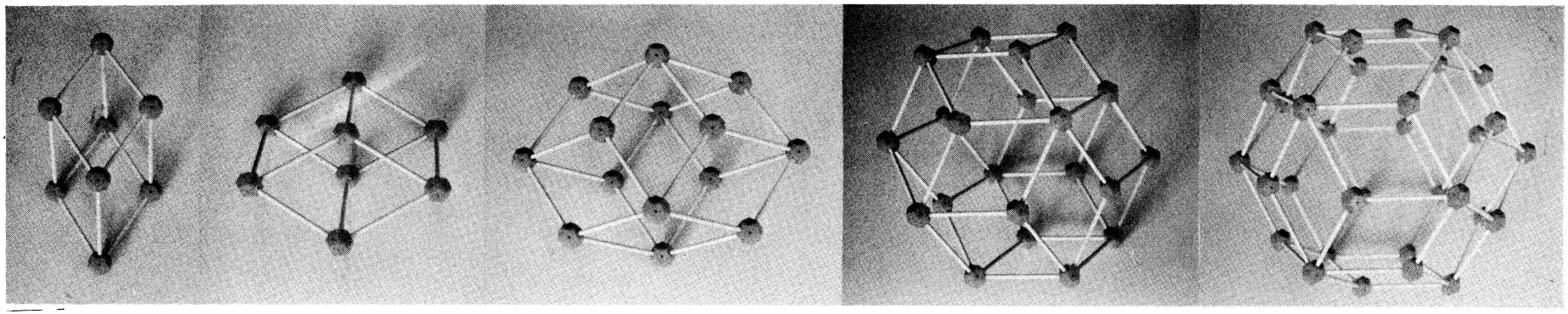


Figure 2

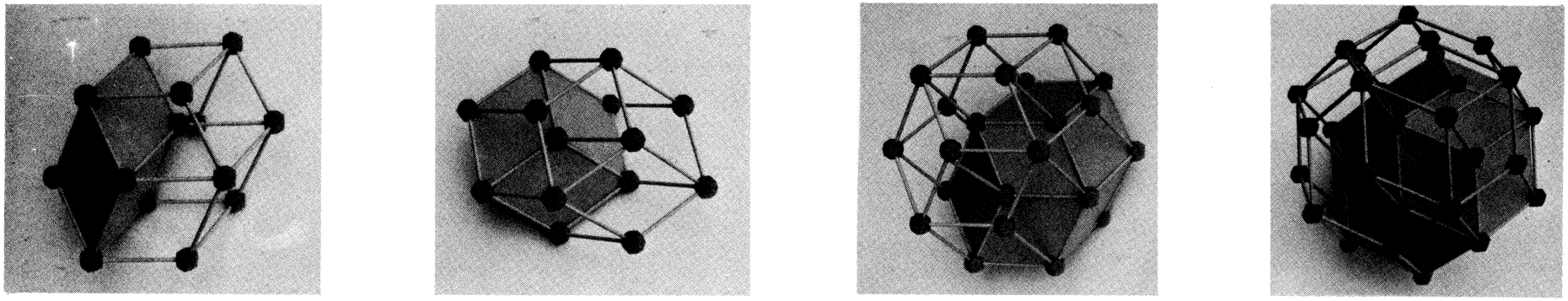


Figure 3

There are only five convex polyhedra derived from golden diamonds (**Figure 2**): two sorts of hexahedra, a dodecahedron, an icosahedron and a triacontahedron. Clearly, all can be made of the above-mentioned ball joints and sticks.

They are zonohedra, and were named the «golden isozonohedra» by Coxeter (in correspondence with the author). Among them, the dodecahedron, icosahedron and triacontahedron were discovered already by Bilinski, Federov and Kepler respectively, and the remaining two are well known as the acute and obtuse hexahedra. Thus, in the order in which they appear in **Figure 2**, they are denoted A_6, O_6, B_{12}, F_{20} and K_{30} respectively.

These five are intimately related. For instance, as shown in **Figure 3**, each of the larger can be packed with some of the smaller ones, leaving no gaps;

$$B_{12} = 2 \cdot A_6 + 2 \cdot O_6$$
$$F_{20} = B_{12} + 3 \cdot A_6 + 3 \cdot O_6 = 5 \cdot A_6 + 5 \cdot O_6$$
$$K_{30} = F_{20} + 5 \cdot A_6 + 5 \cdot O_6 = 10 \cdot A_6 + 10 \cdot O_6.$$

Placed face-to-face, they can fill various portions of space. **Figure 4** shows some of examples. (1) is by A_6s only, (2) by O_6s only and (3) by B_{12}s only. **Figure 5** is an arbitrary honeycomb derived from all golden isozonohedra. Some suitable units may be set in each hollow space, until no gaps remain. It is also possible to fill all of 3-space, using quite different combinations of golden isozonohedra.

All of the above examples were constructed under the condition that adjacent isozonohedra have faces in common. But if adjacent isozonohedra are permitted to interpenetrate, the common region being one of the smaller golden isozonohedra contained in the interiors of B_{12}s, F_{20}s or K_{30}s, then more fruitful constructions can be derived. For instance, **Figure 6** shows a model of a pavilion for an exposition designed by Takanori Yamagiwa. In this case, K_{30}s were used in such a way that they share an O_6 between them. Yamagiwa and his colleagues also constructed a practical shelter for a restaurant, using F_{20}s and K_{30}s (**Figure 7**).

Next, 3-space may be filled with other new units derived from A_6s and O_6s. If an A_6 and O_6 are connected to each other across an arbitrary pair of faces, then two fist-shaped concave polyhedra (**Figure 8**) can be obtained. At the left and right hand of each picture, the front and rear views are shown. The surface of the one is the mirror image of that of the other; the former resembling the fist of the right hand is called an **R-unit** and the latter resembling that of the left hand an **L-unit**.

The colors printed in light and shade (five in all) and the Arabic numerals (from 1 to 5) of each face are assigned so that an R-unit becomes the mirror image of an L-unit and parallel faces of each unit have the same appearance. The numerals are aligned with the shorter diagonals of faces in R-units, and with the longer diagonals in L-units; moreover, the numerals on the outer faces of each unit are underlined. Consequently, all of the twenty faces of R- and L-units are distinguished from each other.

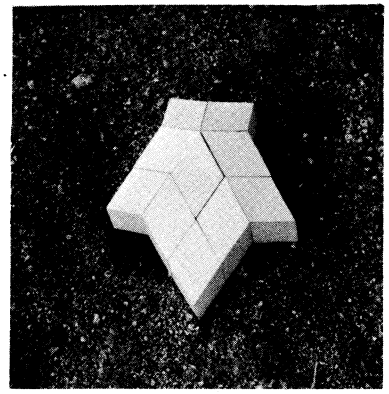
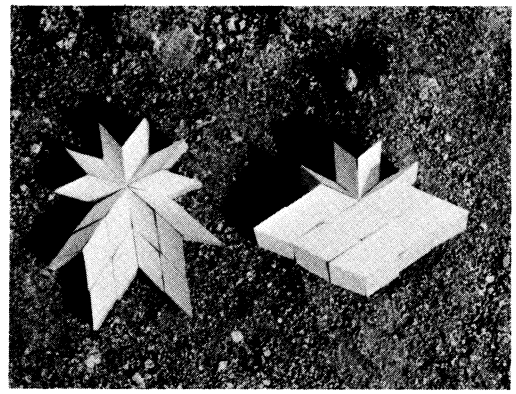
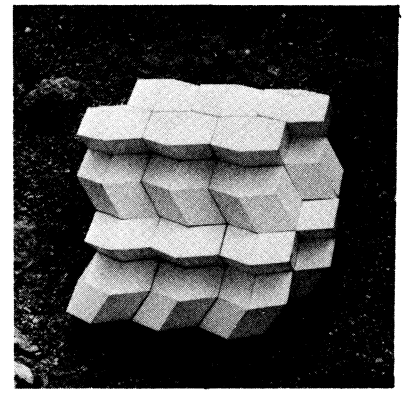


Figure 4





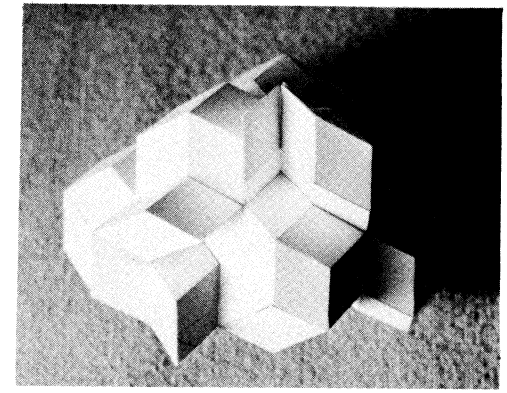


Figure 5

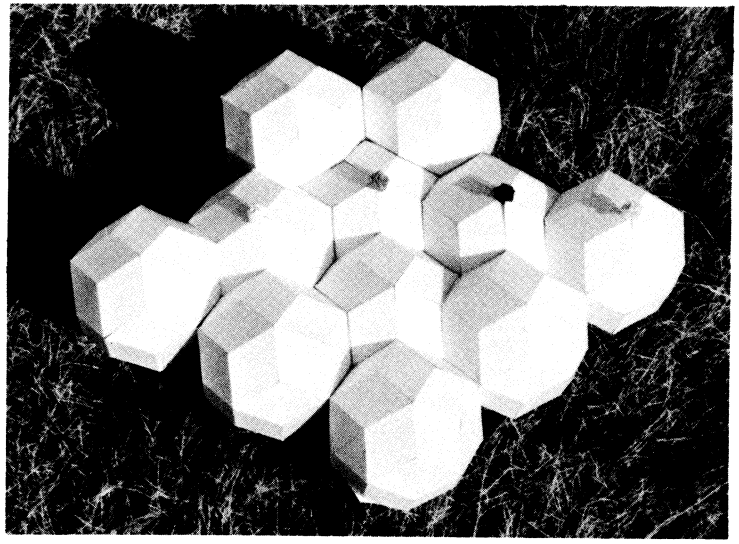


Figure 6

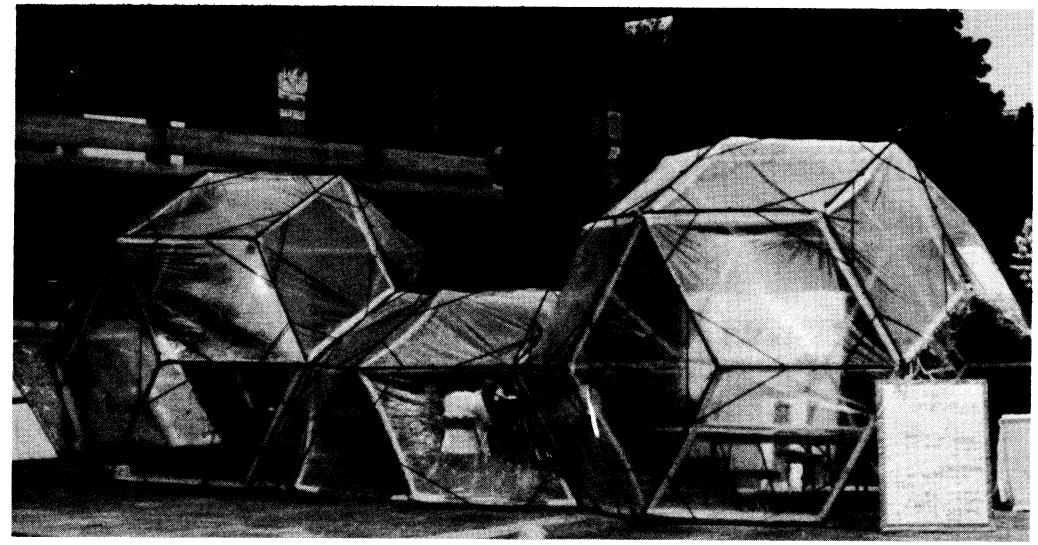


Figure 7

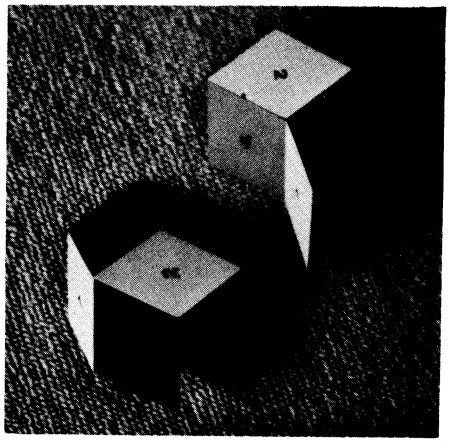


Figure 8(1)

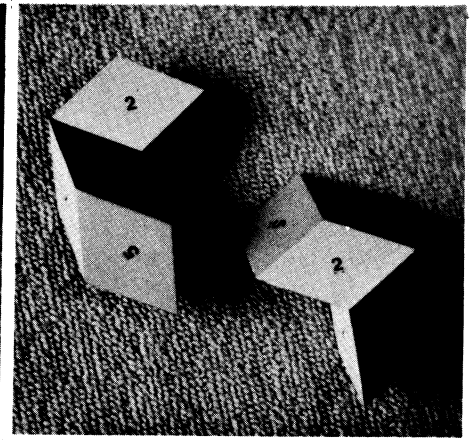


Figure 8(2)

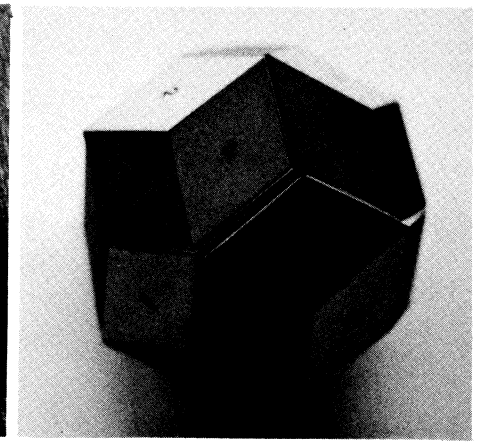


Figure 9(1)

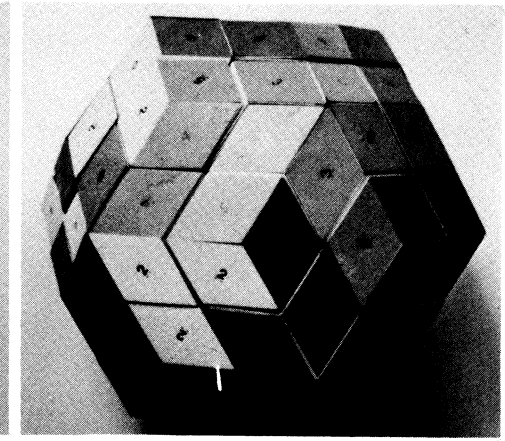


Figure 9(2)

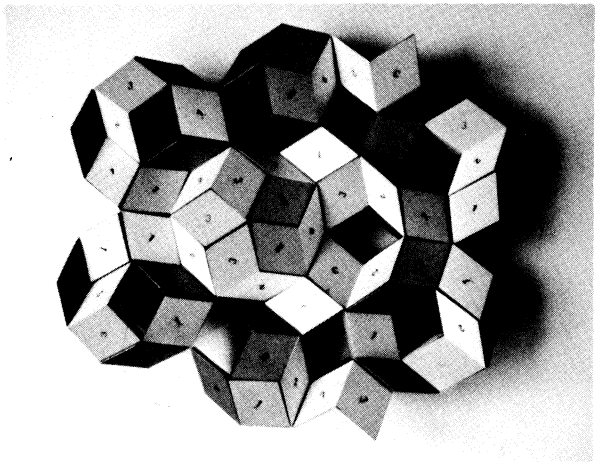


Figure 10(1)

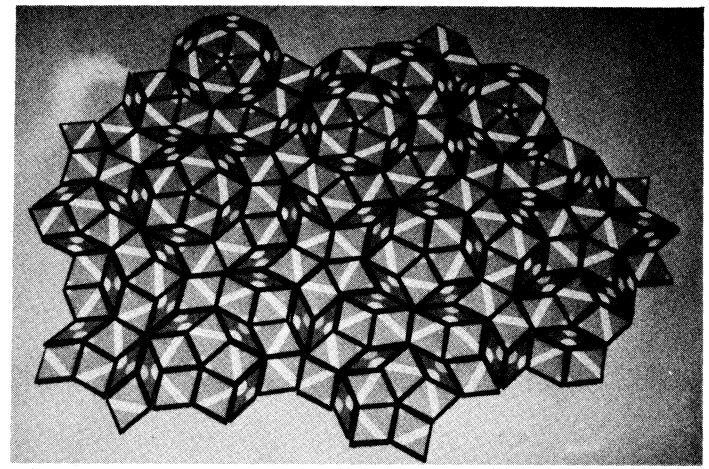


Figure 10(2)

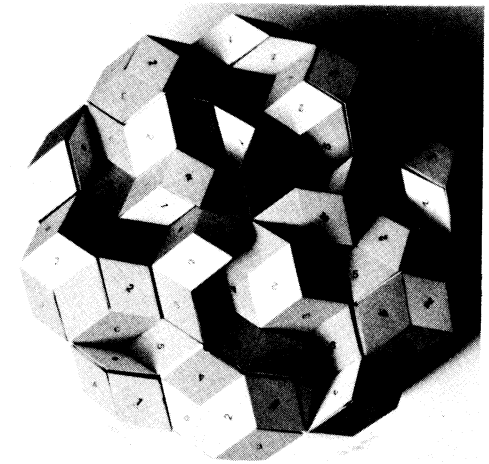
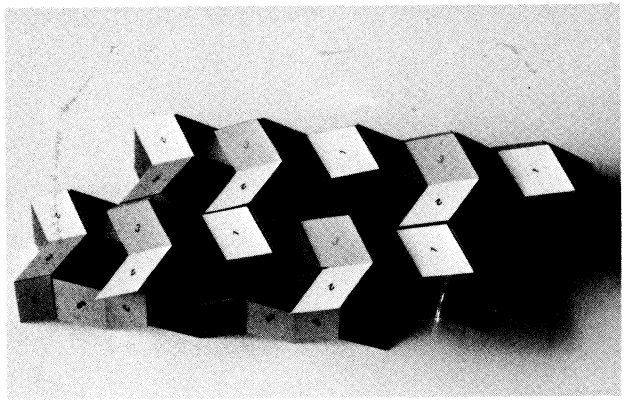
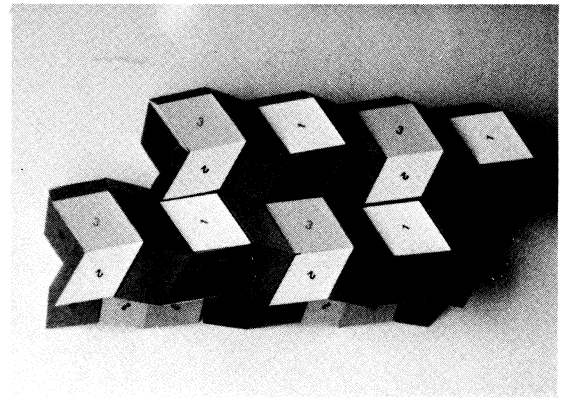


Figure 11(1)





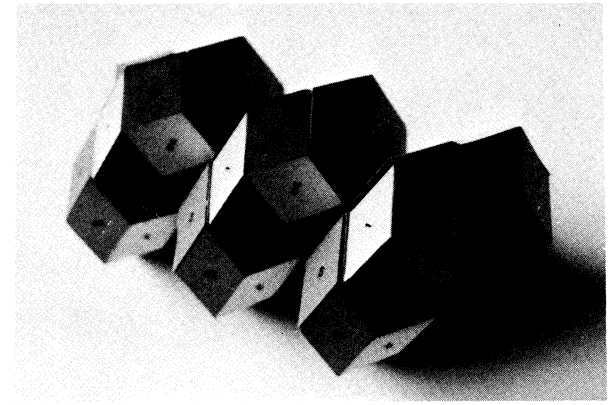
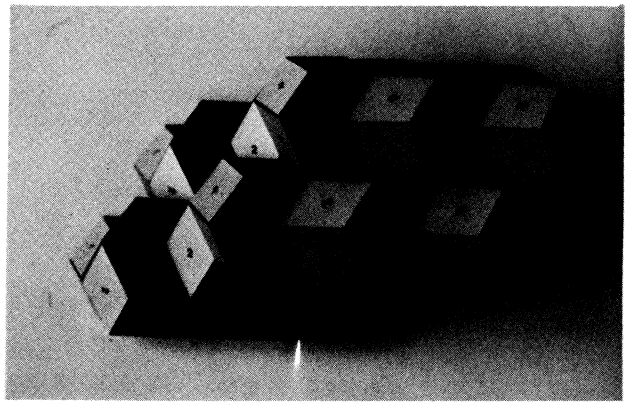
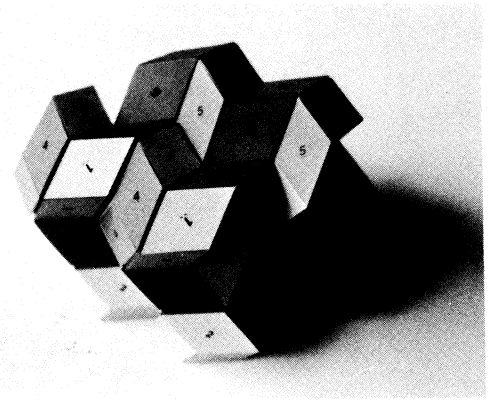


Figure 11(2)



Figures 12



Under such conditions, if R- and L-units are assembled with faces in common, without any restriction, then a B_{12}, F_{20} and K_{30} of frequencies one or two can be constructed (**Figure 9**).

A spatial tessellation, one layer of which appears as **Figure 10(1)**, covers Penrose's essential non-periodic tessellation shown in **Figure 10(2)**, as introduced in *Scientific American*, March 1977. There are some holes, but into them we may insert parts having the shape of A_6 in R- or L-units, so that this spatial tessellation develops in multiple layers.

Figure 11 (1) shows a whirlpool and (**2**) a tornado, each of which continues indefinitely without a gap.

All of the above-mentioned examples lead to non-periodic honeycombs, but the non-periodicity is not essential, for some typical periodic honeycombs (such as **Figure 12**) are also possible.

Figure 13 shows two of the curious periodic honeycombs. In both, non-periodic layers stack periodically along a trunk shown at the top of the picture.

To forbid each periodic honeycomb, we may impose restrictions on the method of assembly:

(1) Two faces having the same colors (having the same numerals, whether or not underlined) must not be joined to each other.

(2) Within a single unit, faces numbered 1 and $\underline{4}$, $\underline{1}$ and 4, must not be joined to each other.

(3) In different units, faces numbered 2 and $\underline{3}$, $\underline{2}$ and 3 must not be joined to each other.

Even so, a troublesome honeycomb appears (**Figure 14**) In this case, no parallel translation, central symmetry or mirror image is observed. It may be an essential non-periodic honeycomb.

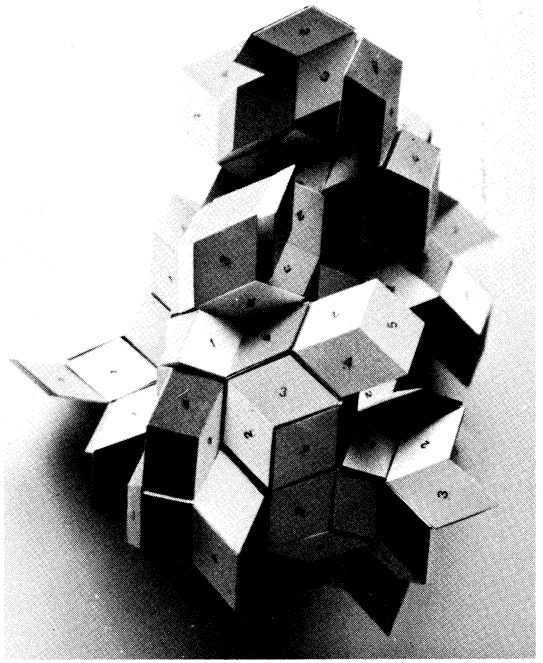


Figure 14

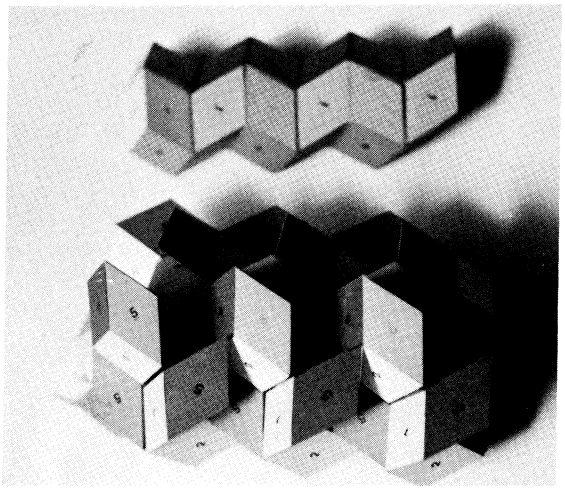


Figure 13(1)

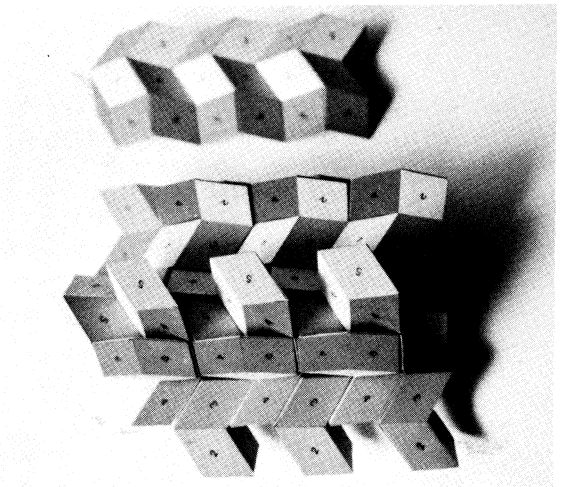


Figure 13(2)

2. On Uniform Antcombs

The aim of this section is to introduce some interesting stackings of polyhedra which, by outward appearance, are closed polyhedra having many tunnels (in the topological sense, having numerous genera).

First of all, if three regular octahedra be piled up as in **Figure 15**, then the first and third octahedra are parallel translates of one another, along the line containing the body-centers of these three octahedra. The reason is that all faces of a regular octahedron, taken in pairs, are situated in point symmetry about its body-center, and the composite of two such symmetries is a translation.

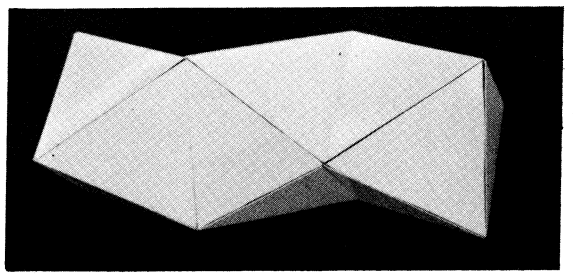


Figure 15

By applying this property of the regular octahedra, a «ring» composed of eight regular octahedra can be made as shown in **Figure 16**. Outwardly, it is a torus-shaped polyhedron, but its inner space has many separating walls.

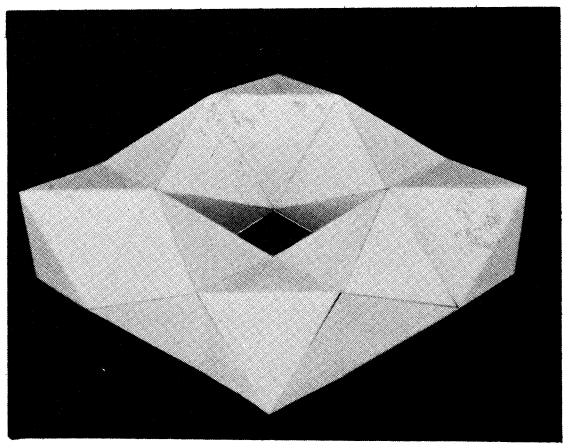


Figure 16

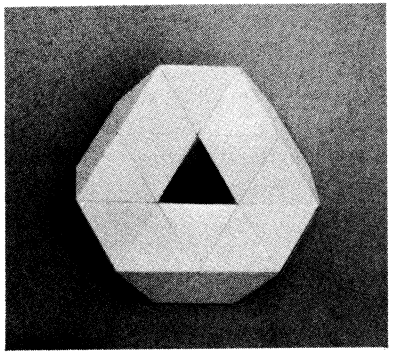


Figure 17

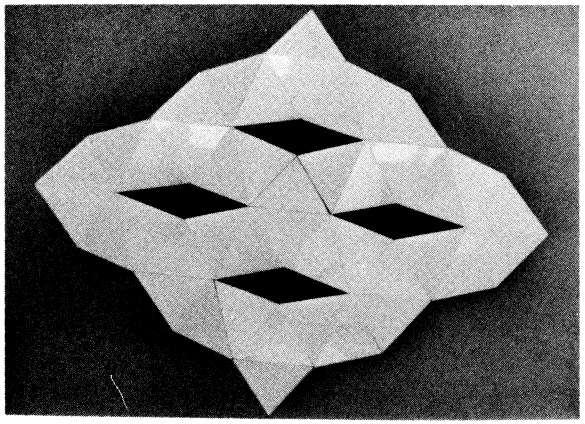


Figure 18

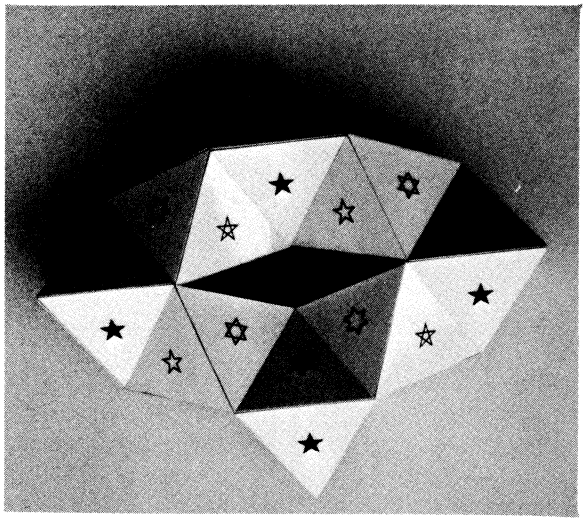


Figure 19

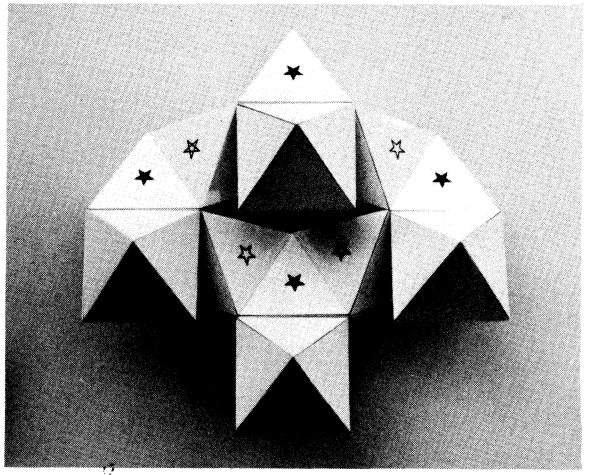


Figure 20(1)

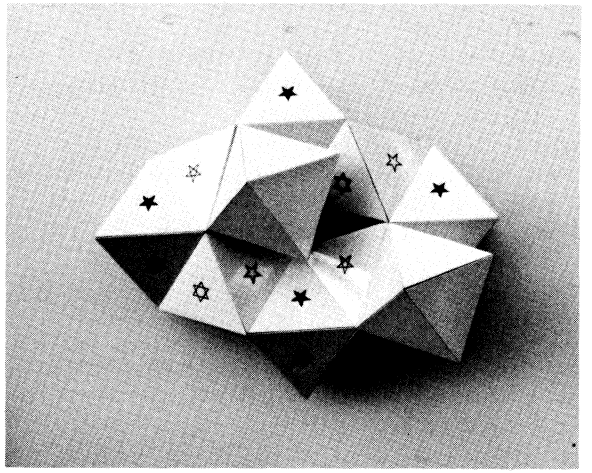


Figure 20(2)

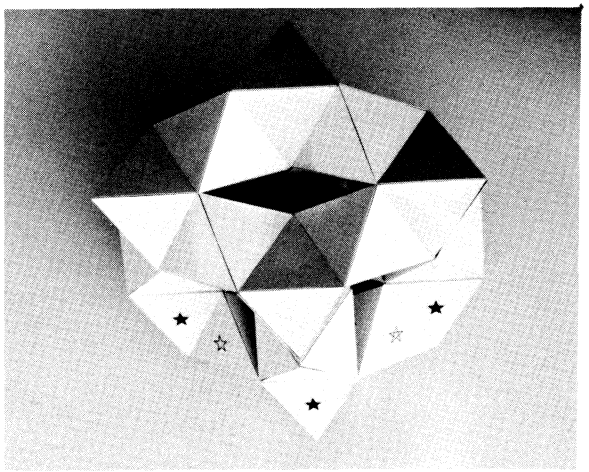


Figure 21(1)

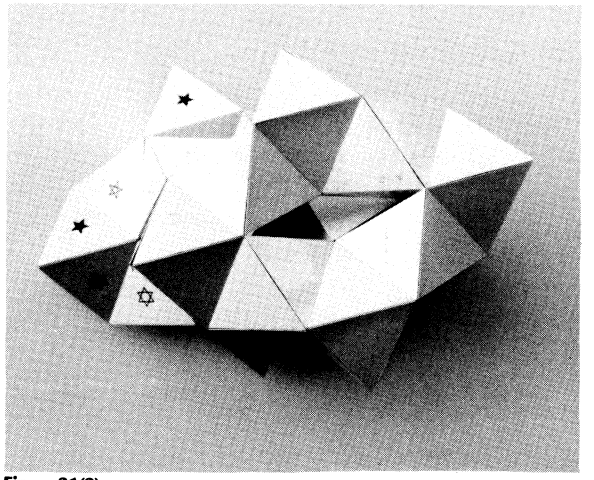


Figure 21(2)

This ring was discovered in an attempt to find the torus-shaped polyhedron which has the least number of faces, all faces being regular triangles. Perhaps, the outer surface of the above ring, having 48 faces, is minimal in this respect. There is another ring whose surface is composed of 48 regular triangles (**Figure 17**). This is derived from the octet-truss, but some of its neighboring triangles lie in a single plane. In this section, we use only the former example as «unit ring», as it has many curious features.

Now, starting from this unit ring, closed polyhedra having many tunnels can be obtained. **Figure 18** shows an example spread in one layer, having four tunnels. This is made from parallel translates of a unit ring.

As a next step, labels (different types of stars) are attached to each face of the unit ring shown in **Figure 19**, in such a way that mutually parallel faces have the same stars. By using these labels as a guide, polyhedra as in **Figures 20-22** can be obtained.

First, in the spatial extension, there are two essentially different types of fundamental construction (**Figure 20**): (1) four regular octahedra may be joined to the faces having the same four black stars in **Figure 19**, or (2) two may be joined at white stars. From this procedure, four or two oblique «legs» are obtained.

Next, if another unit ring be put on these legs, two solids such as **Figure 21** appear. **(1)** is derived from **Figure 20(1)**, and **(2)** from **Figure 20(2)**. Starting with

the construction in **Figure 21(1)**, one octahedron can be placed in the interior space (**Figure 21(3)**). These solids are named the **fundamental solids**.

The same procedure can be continued indefinitely. Two examples are shown in **Figure 22**. **(1)** is derived from **Figure 21(1)**, and **(2)** from **Figure 21(2)**. (If the construction such as **Figure 21(3)** be continued indefinitely, then 12 regular triangles will meet at each vertex.)

These polyhedra are named the **uniform ant-hills**. More precisely, «ant-hill» means the stacking face-to-face of simple polyhedra (the polyhedra being topologically equivalent to the surface of the sphere) as «unit polyhedra», so that the resulting outer surface is a closed polyhedron having finitely many (not 0) tunnels. Let G be the number of tunnels. If $G = 0$, we have the honeycomb, if $0<G<\infty$, the ant-hill, and if $G=\infty$, the three dimensional polyhedron of Wells (Wells 1977) or the infinite polyhedra of Wachmann (Wachmann 1974) having, however, many separating walls. Of all the faces which cluster around any one edge of a unit polyhedron, two and only two faces are not common to two different unit polyhedra. So the configurations in **Figures 23** and **24** do not appear in ant-hills. In the first instance, a part of the octet-truss as a honeycomb, there is an edge common to two tetrahedra and two octahedra, so all faces around that edge are common to two unit polyhedra. And in the latter instance, there is an edge common to two octahedra, so four (not two) faces meeting that edge do not belong to two different unit polyhedra.

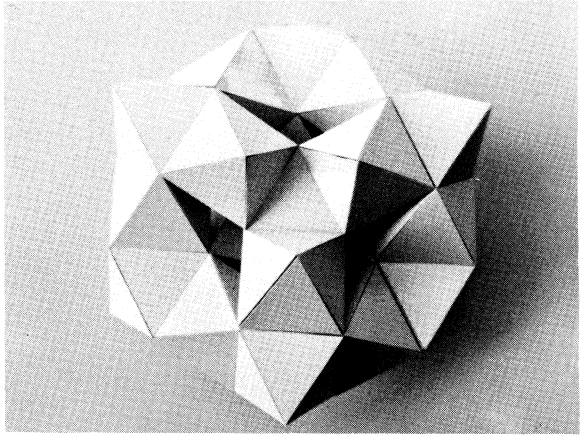


Figure 21(3)

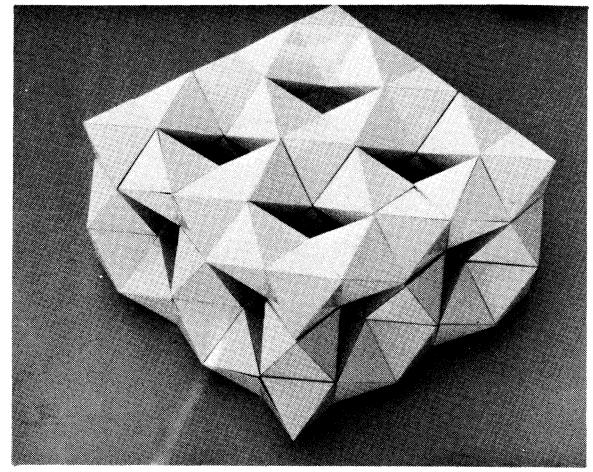


Figure 22(1)

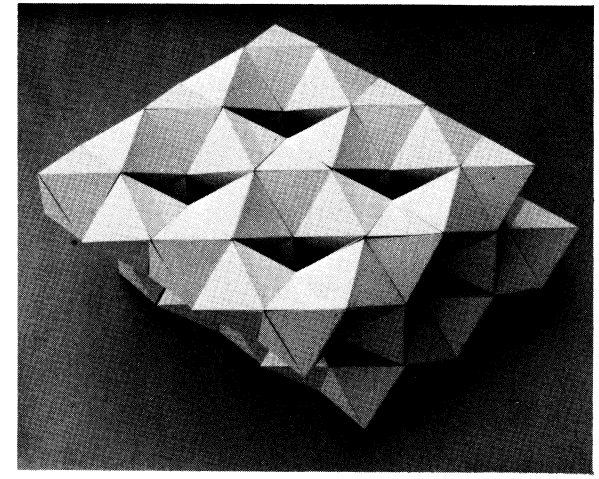


Figure 22(2)

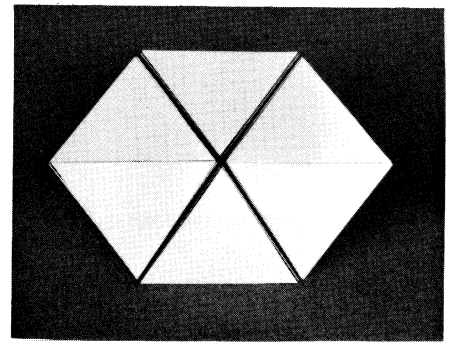


Figure 23

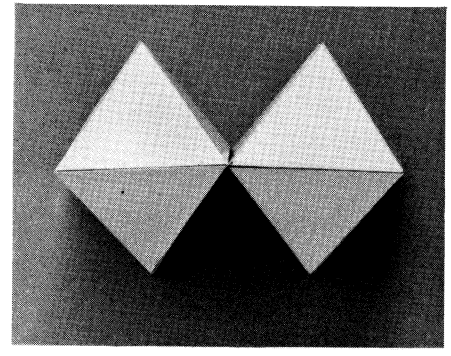


Figure 24

An ant-hill constructed of only congruent regular polyhedra as unit polyhedra is called a **uniform ant-hill**. In the uniform ant-hills, the unit ring such as **Figure 16** and the fundamental solids such as **Figure 21** are the most important elements, since any uniform ant-hill is constructed from a combination of them and their mirror images.

Uniform ant-hills can also be constructed using cubes, regular dodecahedra or regular icosahedra. But the regular tetrahedra cannot be so used, having no mutually parallel faces.

All uniform ant-hills resemble those built using the regular octahedra. That is, as a first step, eight unit polyhedra are always arranged in a rhombic shape. In the spatial extension phase using regular dodecahedra and icosahedra, there are four (not two) essentially different types of fundamental solids. Two of them, shown in **Figures 25(1) (2)**, correspond to the solid in **Figure 21(1)**, and the others, shown in **Figures 25(3),(4)** to the solid in **Figure 21(2)**. (**Figure 25** shows only the case of the regular dodecahedra.)

3. Uniform Ant-hills in the World of Golden Isozonohedra

The rhombus obtained by connecting the eight body-centers of a unit ring constructed of regular dodecahedra is exactly the golden diamond. That is, the sticks seen in **Figures 1** and **2** etc. can be replaced with regular dodecahedral ball joints as shown in **Figures 25(3)** and **(4)**.*

Therefore, all constructions using the golden isozonohedra mentioned in part 1 can also be made in the form of uniform ant-hills, using regular dodecahedra. In particular, A_6 corresponds to **Figure 25(1)** and O_6 to **Figure 25(2)**.

Figure 26 shows the construction of the five isozonohedra, while **Figure 27** shows the stages of growth from a single regular dodecahedron up to a large crystalline form having icosahedral symmetry.

The same operations can be continued indefinitely in 3-space, with the result that the entire world of golden isozonohedra can be packed with copies of the regular dodecahedron.

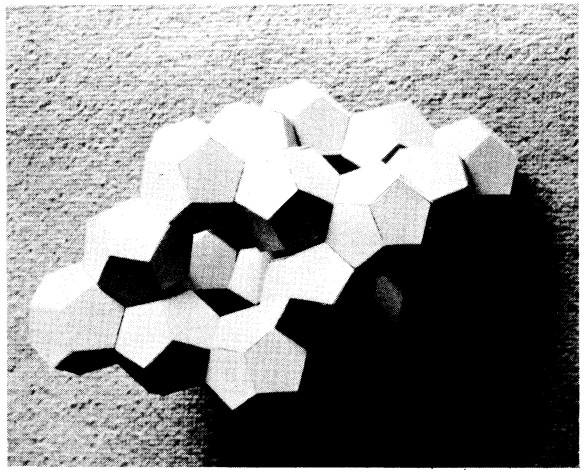


Figure 25(1)

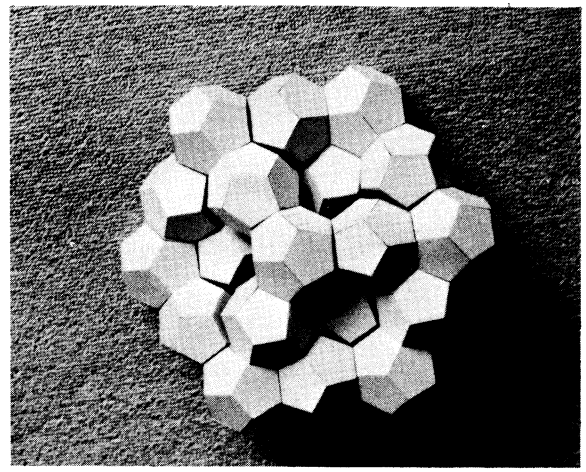


Figure 25(2)

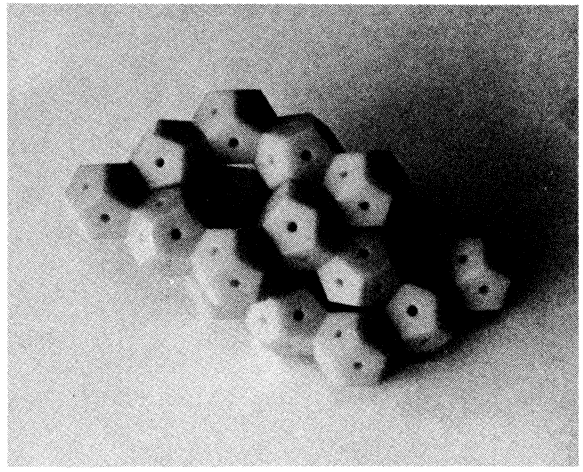


Figure 25(3)

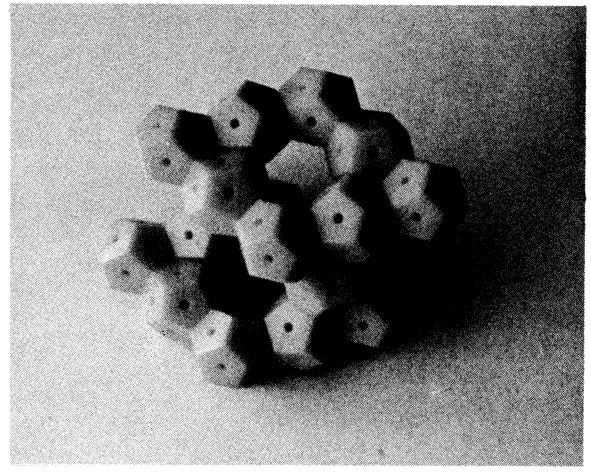


Figure 25(4)

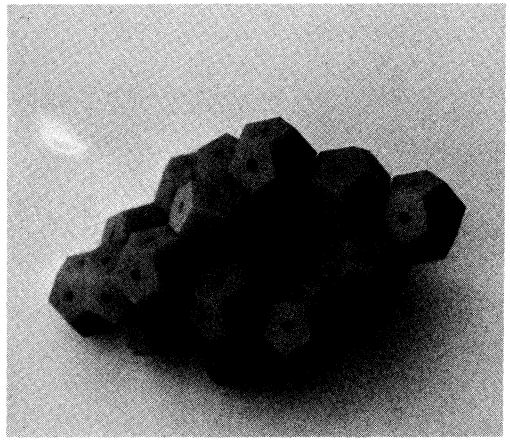


Figure 26(1)

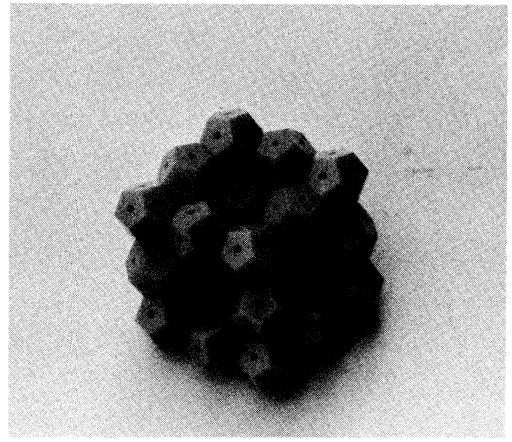


Figure 26(2)

* Note: the pairs of dodecahedra centred on adjacent edges of the basic rhombic ring do not meet, although at the acute angles of the rhombus they come close, leaving a wedge-shaped opening of only 3° 26′. (Six dodecahedra do not form a closed ring).

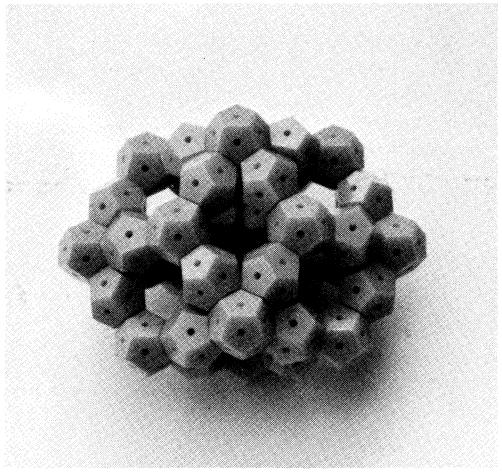


Figure 26(3)

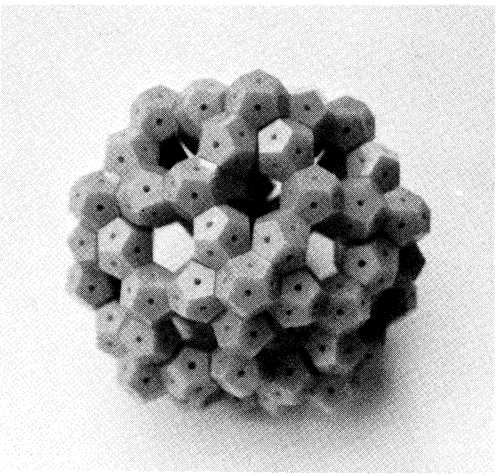


Figure 26(4)

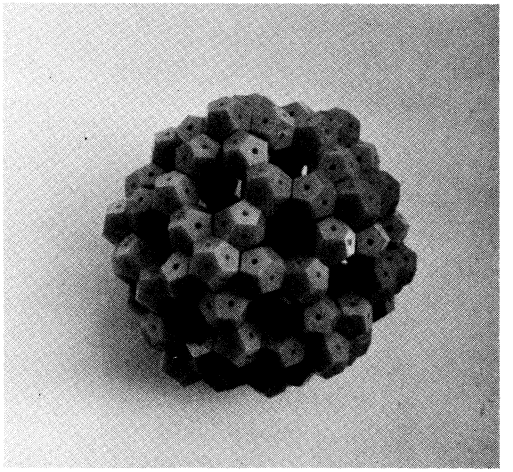


Figure 26(5)

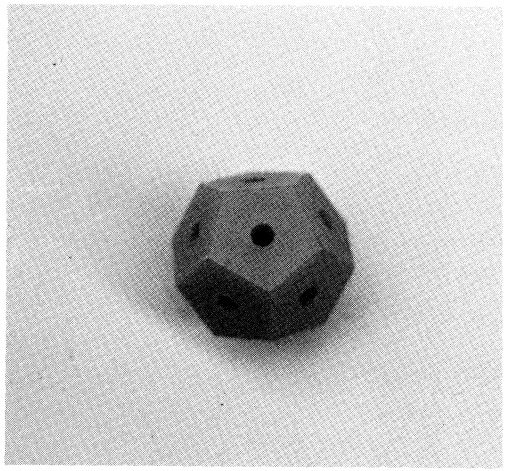


Figure 27(1)

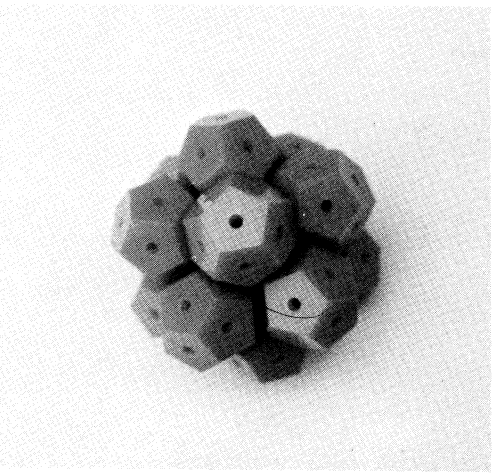


Figure 27(2)

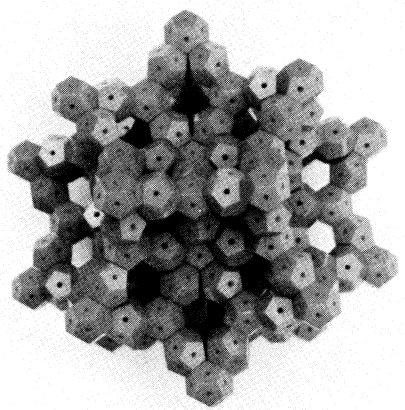


Figure 27(3)

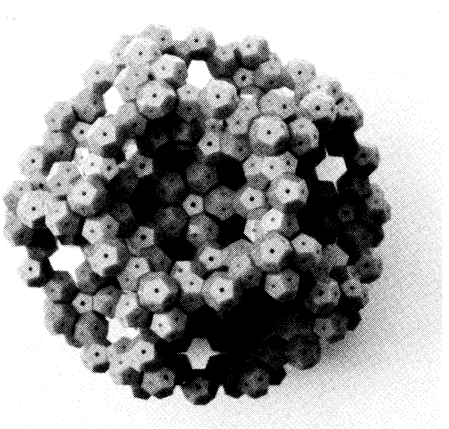
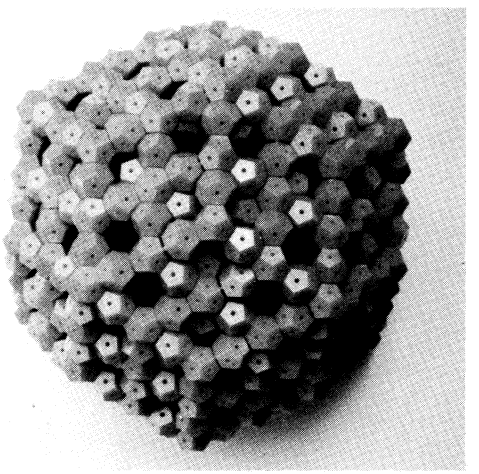


Figure 27(4)



27(5)

Bibliography

The code in the first block of each bibliographic item consists of three parts, separated by dashes. The first letter indicates whether the item is a

B ook
A rticle
P reprint, or
C ourse notes.

The middle letter(s) indicates whether the piece was intended primarily for an audience of

M athematicians
A rchitects, or
E ngineers.

The final letter(s) indicates if the piece touches on one or more of the principal themes of structural topology:

G eometry (in general),
P olyhedra,
J uxtaposition, or
R igidity.

The key words or other annotations in the third column are intended to show the relevance of the work to research in structural topology, and do not necessarily reflect its overall contents, or the intent of the author.

Baer 1970 Steve Baer B–AE–PJ	**Zome Primer** Zomeworks Corporation, Albuquerque, NM	Elements of zonohedral geometry, the golden diamond
Coxeter 1973 H. S. M. Coxeter B–M–PJ	**Regular Polytopes** Dover	Zonohedra, polytopes, honeycombs.
Coxeter 1968 H. S. M. Coxeter B–M–P	**Twelve Geometric Essays** Southern Illinois University Press	(Chapter 5) Regular sponges.
Wachman 1974 A. Wachman, M. Burt, M. Kleinman B–MA–GPJ	**Infinite Polyhedra** Technion	Infinite regular polyhedra, multi-layered polyhedra, multi-directional polyhedra.
Wells 1977 A. F. Wells B–M–PJ	**Three-dimensional Nets and Polyhedra** Wiley	Three-dimensional nets, finite and infinite polyhedra, regular three-dimensional polyhedra.