> *"En Italia, en cincuenta años de dominio de los Borgia no hubo sino horror, guerras y matanzas, pero surgieron Miguel Ángel, Leonardo da Vinci y el Renacimiento. En Suiza, por el contrario, tuvieron quinientos años de amor, democracia y paz. ¿Y cual fue el resultado? El reloj de cuco".*

Siempre me irritó este alegato antidemocrático del personaje, ingenioso, pero profundamente injusto e inexacto. Hay más cosas que el reloj de cuco. En Suiza se refugiaron en 1583 los protestantes Bernoulli, huyendo de la persecución religiosa que la familia sufría en Anvers (en la católica Flandes), para no ser víctimas de una especie de noche de San Bartolomé que se preparaba. La familia Bernoulli acabó recalando en Basilea. Jacques Bernoulli (1654-1705) enseñó matemáticas a Paul Euler, padre de Leonhard, que a su vez fue alumno de Jean Bernoulli (1667-1748). De este modo, la más tolerante y democrática Suiza puede enorgullecerse, no sólo de su gran industria relojera (relojes de cuco incluidos), sino de haber sido la cuna de algunas de las mentes matemáticas más ilustres de la humanidad, entre las cuales destaca la de Leonhard Euler.





# EULER TRANSGRESSING LIMITS: THE INFINITE AND MUSIC THEORY

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## 1.- Introduction.

The Irish satirist Jonathan Swift once said:

> *"Elephants are drawn always smaller than life, but a flea always larger."* (FELLMANN, 2007: XIII).

Whoever would like to speak about Euler has to solve exactly this problem:

> *"How to do justice to this universal, richly detailed and inexhaustible mathematician?"* (SIMMONS, 2007: 168).

The leading idea of my considerations is Euler's creativity. It seems to me that it can be characterized by the transgression or removal of limits especially when he dealt with the infinite. Sometimes his audacity led him astray. In 1727 he submitted his *Dissertatio physica de sono* (*Physical dissertation on sound*). In the appendix he raised the following problem:

> *"What would happen if a stone dropped into a straight tunnel drilled to the centre of the earth and onward to the other side of the planet?"*

According to Euler it reaches infinite velocity at the centre and immediately returns to the point from which it had fallen. Only in his *Mechanica* did Euler justify this false solution saying:

> *"This seems to differ from the truth … However that may be, here we have to trust more in the calculation than in our judgement and confess*

*that we do not understand the jump at all if it is made from the infinite to the finite*[1]*".*

Euler's result was the consequence of his mathematical modelling of the situation (ROBINS, 1739: 12).

In this paper I would like to consider the four examples:

1. Mathematical rigour
2. Zeta-function for s = 2
3. Divergent series
4. Music theory

## 2.- Four examples.

### 2.1.- Mathematical rigour.

In the 18th century the differential calculus was criticized for lack of rigour and for suspect conclusions. Euler was well aware of this and tried to overcome these difficulties in his *Institutiones calculi differentialis* (*Elements of instruction of the differential calculus*) published in Berlin in 1755.

In his opinion the reason for these difficulties had to be sought in the false conception of differentials. Many authors conceived of them as infinitely small quantities that did not completely vanish but retained a certain magnitude though smaller than any assignable quantity:

> *"Thus they have been rightly reproached with neglecting geometrical rigour and with conclusions thus drawn […] being deservedly suspect*[2]*".*

These infinitely small quantities might be conceived of as arbitrarily small, Euler continued, yet, this error can eventually result in an enormous error because not only single quantities but also many and even innumerable quantities have to be disregarded at the same time.

1 "Hoc quidem veritati minus videtur consentaneum… Quicquid autem sit, hic calculo potius quam nostro iudicio est fidendum atque statuendum, nos saltum, si fit ex infinito in finitum, penitus non comprehendere." (EULER, 1736: 88).

2 "His igitur iure est obiectum rigorem geometricum negligi et conclusiones inde deductas […] merito esse suspectas." (EULER, 1755: 6).





For Euler there was only one solution of this problem:

> *"Hence they are called differentials which are also called infinitely small because they are deprived of a quantity. Thus they are to be interpreted by their nature in such a way that they are considered to be nothing at all or equal to zero*[3]*".*

The whole third chapter is dedicated to "the infinite and infinitely small quantities" (*De infinitis atque infinite parvis*). Euler's definitions of these two types of quantities are of the greatest interest:

> *"Hence a quantity that is so large that it is larger than any finite quantity, can not be not infinite. … by this sign a quantity is denoted that is larger than any finite or assignable quantity*[4]*".*

Euler's infinite quantities are by definition actually infinite quantities. But what about "infinitely small quantities"?

> *"But there cannot be any doubt that every quantity can be diminished to such a degree, that it completely vanishes and comes to nothing. Yet, an infinitely small quantity is nothing else but a vanishing quantity and will be therefore in fact equal to zero. This definition of infinitely small quantities is in agreement with the one according to which they are called smaller than every assignable quantity. For if a quantity should be so small that it is smaller than every assignable quantity, it will be impossible that it is not equal to zero. Because if it would not be equal to zero, a quantity could be assigned to be equal to it this would be contrary to the hypothesis. Thus to somebody asking what is an infinitely small quantity in mathematics we will answer that it is in fact equal to zero*[5]*".*

3 "Vocantur itaque differentialia, quae, cum quantitate destituantur, infinite parve quoque dicuntur, quae igitur sua natura ita sunt interpretanda, ut omnino nulla seu nihilo aequalia reputentur." (EULER, 1755: 7).

4 "Hinc, quae quantitas tanta est, ut omni quantitate finita sit maior, ea non infinita esse nequit. … quo denotatur quantitas omni quantitate finita seu assignabili maior." (EULER, 1755: 69).

5 "Nullum autem est dubium, quin omnis quantitas eousque diminui queat, quoad penitus evanescat atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens ideoque revera erit = 0. Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabile minora; si enim quantitas tam fuerit parva, ut omni

Euler's remark is right and inevitably so: the inevitable conclusion depends on his definition "smaller than any *assignable* quantity." We might describe this situation as follows:

For all infinitely small quantities $i$ and for all assignable quantities $aq$

$$i < aq \Rightarrow i = 0$$

It is worth mentioning that originally Gottfried Wilhelm Leibniz, the intellectual forerunner of Euler, had tested the same definition in the spring of 1673 (LEIBNIZ, 2008: preface). Yet, exactly owing to the inevitable conclusion he replaced this definition by a far better one:

For every given quantity $gq > 0$ there is an infinite $i(gq) > 0 \Rightarrow i(gq) < gq$

An infinitely small quantity is a variable quantity: its value depends on the given quantity. This notion of infinitely small quantity can be translated without any difficulty into the ε-δ-language of Weierstrass's analysis of the $19^{th}$ century.

Euler, however, was forced to calculate with zeros of different values.

From $2 \cdot 0 = 1 \cdot 0$

he deduced $2:1 = 0:0$

In order to get a correct proportionality, he had to assume that the left zero was a bit larger than the right zero (EULER, 1755: 70). Obviously a consistent theory of real numbers could not be founded in such a way.

In his writings on mechanics and astronomy he needed infinitely small quantities that were unequal to zero. It does not change this result that one might interpret Euler's treatment of infinitely small quantities using modern methods of non-standard analysis.

Yet, Euler was convinced that he had refuted the critics of the differential calculus. He personally stated:

> *"Hence that objection by which the analysis of the infinite is accused of neglecting geometrical rigour collapses by itself because nothing is rejected beyond that which is really nothing. And for that reason one can justly claim that in this more elevated science, the geometrical rigour found in the books of the Ancients, is equally diligently observed*[6]*".*

quantitate assignabili sit minor, ea certe non poterit non esse nulla; namque nisi esset = 0, quantitas assignari posset ipsi aequalis, quod est contra hypothesin. Quaerenti ergo, quid sit quantitas infinite parva in mathesi, respondemus eam esse revera = 0."

6 "Quare illa obiectio, qua analysis infinitorum rigorem geometricum negligere arguitur, sponte cadit cum nil aliud reiiciatur, nisi quod revera sit nihil. Ac propterea iure affirmare licet in





**2.2.- Zeta-function for s=2.**

Euler applied the principle that Niels Henrik Abel formulated for the first time (ENGEL/SCHLESINGER, 1913: XI):

> *"To conceive of problems so that they can be solved".*

A seemingly unsolvable problem was the summation of the reciprocal square numbers. James Bernoulli recognized the convergence of the series because the series of the reciprocal triangular numbers is its convergent majorant series. Yet, complaining about the difficulty of the problem he said in 1689:

> *"If somebody should find and tell us what was too much for our efforts up to now, we will be most grateful to him."* (STÄCKEL, 1925: 160).

Euler found the value $\frac{\pi^2}{6}$ in autumn 1735 and communicated it to his friend Daniel Bernoulli (STÄCKEL, 1925: 102; HOFMANN, 1959: 185).

Algebra was, as Condorcet said in his eulogy on Euler, a "science très-bornée", a very limited science (CONDORCET, 1786: 290). Euler transgressed these limits. A power series is a polynomial continued up to infinity. One might calculate with it as with polynomials. The analysis of the transcendental functions is nothing but a natural expansion of algebra. The decomposition of polynomials into factors can be applied to transcendental functions that are equations of infinite degree (EULER, 1748: chapter 10). Hence one gets (POLYA, 1954: 41-46; SIMMONS, 2007: 267-269):

a)
$$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots = 0$$

with the infinitely many roots $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

b) Division by x leads to:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \pm \cdots = 0$$

hac sublimiori scientia rigorem geometricum summum, qui in veterum libris deprehenditur, aeque diligenter observari." (EULER, 1755: 71).

This "infinite equation" has the roots $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

c) The decomposition into factors of a polynomial of even degree 2n with the 2n distinct roots $\pm a_1, \pm a_2, \ldots, \pm a_n$ $(a_i \neq 0)$ reads:

$$b_0 - b_1 x^2 + b_2 x^4 + \cdots (-1)^n b_n x^{2n} = b_0 \left(1 - \frac{x^2}{a_1^2}\right)\left(1 - \frac{x^2}{a_2^2}\right)\cdots\left(1 - \frac{x^2}{a_n^2}\right)$$

According to the Girard-Newton-Leibniz formulas involving the coefficients and roots of algebraic equations the following relation holds:

$$b_1 = b_0 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right)$$

d) An analogy leads from the finite to the infinite equation:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

Now $b_0 = 1, b_1 = \frac{1}{3!}$,

hence $\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots$ or $\frac{\pi^2}{3!} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

At this time Euler still wrote p instead of π (EULER, 1740). His method enabled him to calculate the sum of the series $1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots$ if *n* is even. His second publication on this subject already appeared in Berlin (EULER, 1743).

In 1743, he wrote;

> *"In any case […] the method was new and completely unusual in such a problem because it was based on the solution of an infinite equation all of whose roots, infinite in number, had to be known*[7]*".*

[7] "Methodus […] utique erat nova et in eiusmodi instituto plane non usitata; nitebatur enim in resolutione aequationis infinitae, cuius omnes radices, quarum numerus erat infinitus, nosse oportebat."





For a while, he believed that there is no way other than through the solution of an *infinite equation*. Yet, he found such other methods. In the Berlin treatise published in 1743 he introduced the letter π for half of the circumference of a circle of radius 1.

a) Euler applied a rule to a case for which it was not established. Instead of an algebraic equation he took a non-algebraic equation: a conclusion by analogy. Yet, there were reasons why he believed in his result.
b) The numerical accord with the directly calculated sum of a finite number of terms.
c) The deduction of the Leibniz series for $\frac{\pi}{4}$.

**2.3.- Divergent series.**

In 1754/55 Euler wrote his paper *De seriebus divergentibus* (On divergent series) that appeared in 1760 (EULER, 1760).

At first he defined the notions of convergent or divergent series.

Def. 1: Series are called convergent if their terms form a strictly monotone null sequence.

The series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ might serve as an example. The series consists of *termini continue decrescentes*, of *continuously decreasing terms*. This is not the modern notion of convergence because the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

would be convergent according to Euler's notion of convergence while it is divergent according to the modern understanding. That is, the sum is not finite: the sequence of its partial sums exceeds every finite value. Euler only states a necessary condition for convergence.

Def. 2: Series are called divergent if their terms remain finite or increase up to infinity.

Euler says: *termini infinitesimi non in nihilum abeunt* (the infinite terms do not shrink to nothing).

According to the modern understanding Euler states only a sufficient condition for divergence. It is not a necessary condition because the harmonic series also diverges in the modern sense of the notion.

In other words: Euler's and the modern notions of convergence and divergence differ. Euler defines them by means of the size of the terms. Today one defines them by means of the limit of the sequence of partial sums.

We have to accept: Euler used a different notion of convergence from what we use today. In 1913, ENGEL and SCHLESINGER (1913: XII) justly mentioned that an estimation of the formal process was lacking in order to reach a prescribed degree of exactness. Even if Euler stated it differently: he did not achieve the rigour of the Greeks that is of Archimedes (EULER, 1755: 71). We will return to this issue.

The starting point was the alternating series 1-1+1-1+1…

On January 10, 1714 Leibniz communicated the convergence criterion that was later named after him to Johann Bernoulli, later Euler's teacher:

> *"Alternating series are convergent if the sequence of the terms of the series form a strictly monotone null sequence* (LEIBNIZ, 1756: 926)."

There was a connection between his communication and a letter to Christian Wolff dating from 1713 (LEIBNIZ, 1856). He knew that $1-x+x^2\mp\cdots=\frac{1}{1+x}$ for every $x$ such that $-1<x<1$. He substituted $x=1$ formally, getting $1-1+1-1\cdots=\frac{1}{2}$.

He believed the result to be correct though there was a quarrel about it between Guido Grandi,Wolff, Annibale Marchetti, Nikolaus I and Daniel Bernoulli, and Christian Goldbach. Leibniz was interested in a justification. He found it in the game of dice: if two values have the same probability their mathematical expectation is their arithmetical mean, in our case $\frac{0+1}{2}=\frac{1}{2}$.

One could call this solution a first idea of Cesàro summability (VARADARAJAN, 2006: 126). Grandi had proposed an analogy with arbitration in the case of two heirs but Leibniz had rejected this proposal.

Euler resumed Leibniz's letter to Wolff (EULER, 1760: 589). However, he dealt immediately with Leibniz's alternating series in the general context of his *divergent* series according to the definition explained above. He began with the expansions into power series which Leibniz had already considered:

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4\mp\cdots$$





or
$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$$

or
$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+5x^4+\cdots$$

If one formally substitutes the values 1, 2, 3 etc. for $x$, without taking account of the radius of convergence in the modern sense, one gets the equations (EULER, 1755: 81; EULER, 1760: 591)

$$\frac{1}{2}=1-1+1-1+1-1\cdots \quad (1)$$
$$\frac{1}{3}=1-2+4-8+16-32\cdots \quad (2)$$
$$\frac{1}{4}=1-3+9-27+81-243\cdots \quad (3)$$

or
$$\frac{1}{0}=1+1+1+1+1+\cdots \quad (4)$$
$$-1=1+2+4+8+\cdots \quad (5)$$
$$-\frac{1}{2}=1+3+9+27+\cdots \quad (6)$$

or
$$\frac{1}{0}=1+2+3+4+\cdots \quad (7)$$
$$1=1+4+12+32+\cdots \quad (8)$$

All of these series are divergent according to Euler's definition. His aim was to retain the utility of such divergent series. Thus the question had to be answered: Can these equations be justified?

a) In any case $\frac{1}{2}$ and the alternating series $1-1+1-1\pm\cdots$ are *equivalent quantities* (Euler, 1760: 593). The question was whether $\frac{1}{2}$ could be called the sum of this series.

b) $\frac{1}{3}$ is not the sum of the alternating series $1-2+4-8\pm\cdots$ in the sense of a stepwise summation because the results removed more and more from $\frac{1}{3}$ if one calculated the partial sums stepwise.

c) Euler said about the result of the equations (5), (6):

> *"It can be much less reconciled with the usual ideas*[8]*".*

His audacious proposal read:

> *"In the meantime, however, it seems to be in accordance with truth if we say that the same quantities that are smaller than zero, can be valued as larger than infinite at the same time*[9]*".*

Euler does not explain the nature of such quantities that can be *larger than infinite*. For him, all difficulties can be reduced to the notion of sum. It must not be understood operationally (EULER, 1755: 82; 1760: 593):

> *"Thus we should say that the sum of any infinite series is the finite expression from the expansion of which that series originates*[10]*".*

This definition seems to occur for the first time in a letter to Christian Goldbach dating from August 7, 1745 (FABER, 1935: XII). Euler mainly used three justifications for his new definition of a sum:

a) The principle of permanence. The new definition coincides with the old one in the case of convergent series.

b) His transformation in the case of alternating series. The result coincides with the result of his transformation method:

Let $s = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots$ He calculated the first, second etc. differences of the successive terms of the series:

$$\Delta a = a_2 - a_1, a_3 - a_2, a_4 - a_3, \cdots$$

$$\Delta^2 a = a_3 - 2a_2 + a_1, a_4 - 2a_3 + a_2, \cdots$$

$$s = \frac{1}{2}a_1 - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a \mp \cdots$$

$$\Delta a = 0,0,0,\cdots, a_1 = 1$$ (GRATTAN-GUINNESS, 1970: 68-70)

8 "Multo minus cum solitis ideis conciliari potest".

9 "Interim tamen veritati consentanum videtur, si dicamus easdem quantitates, quae sint nihilo minores, simul infinito maiores censeri posse". (EULER, 1760: 592).

10 "Dicemus ergo seriei cuiusque infinitae summam esse expressionem finitam, ex cuius evolutione illa series nascatur."





Thus we get s = $\frac{1}{2}$

In the same way the method leads to s = $\frac{1}{3}$ for equation (2) s = $\frac{1}{4}$ for equation (3) (EULER, 1755: 224)

By the new definition Euler is able to show that divergent series are useful. He comments:

> *"By means of this definition we shall be able to uphold the utility of divergent series and to protect them from all injustice*[11]*".*

The question remains whether Euler's definition is well defined. To that end he would have to demonstrate that there is only exactly one *finite expression* whose expansion leads to the series concerned. Faber gave a counterexample (FABER, 1935: XIII).

For us the following four aspects are important:

a) Euler consciously left aside *habitual ideas* as in the case of the Basel problem. Such behaviour is characteristic of creativity. If one only considers power series, it is a matter of Abel's summation method: $\sum a_n = f(1), f(x) = a_0 + a_1 x + \ldots$ is the generating function (VARADARAJAN, 2006: 130).

b) An analogical consideration leads to a rule-stretching that is justified by a concept-stretching.

c) The results deduced by analogy are not doubted but a justification was looked for. Euler asked: How do I define? He did not ask: What is?

d) The two authors differed only in the methods of justification. Leibniz used an analogical consideration; he demonstrated a similarity between infinite series and probability theory. Euler chose the method of interpreting, of defining anew old notions. This was the method resumed by the English mathematicians in the circle of John Peacock, Robert Woodhouse, and Augustus de Morgan at the beginning of the 19th century. They used the principle of permanence of formal power series. Indeed, Euler's definition derives a certain mathematical meaning from the notion of analytical continuation (FABER,1935: XIII).

11 "Ope huius definitionis utilitatem serierum divergentium tueri atque omnibus iniuriis vindicare poterimus." (EULER, 1755: 82).

## 2.4.- Music theory.

Since ancient times music theory tried to explain which numbers can be used to get consonant intervals, and tried to find a criterion in order to separate consonances from dissonances. In a famous letter dating from 1712 to Euler's later friend and colleague Christian Goldbach, Gottfried Wilhelm Leibniz had written:

> *"In music we do not count beyond five like those peoples that did not go beyond three in arithmetic… If we were possessed of a little more subtlety, we could proceed to the prime number seven*[12]*".*

Euler knew this letter that was first published in 1734 and referred to it several times. He chose a completely new, purely mathematical number-theoretical approach in order to solve this problem of delimitation. In his *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae* (Essay of a music theory that is clearly explained on the basis of the most certain principles of harmony), he explained his method:

> *"But because, on the one hand, it is difficult to define the boundaries of consonances and dissonances, and indeed, on the other hand, because this distinction is less in accordance with our way of operating … we will assign the name of consonance to all sounds that consist of several simple tones sounding at the same time*[13]*".*

In other words Euler eliminated the traditional distinction between consonances and dissonances. To that end he first defined a degree of agreeableness $d(n)$ for every natural number

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \quad \text{(EULER, 1739: 232)}$$

[12] "Nos in Musica non numeramus ultra quinque, similes illis populis, qui etiam in Arithmetica non ultra ternarium progrediebantur … Si paulo plus nobis subtilitatis daretur, possemus procedere ad numerum primitivum 7." (LEIBNIZ, 1734, 240).

[13] "At quia partim difficile est consonantiarum et dissonantiarum limites definire, partim vero haec distinctio cum nostro tractandi modo minus congruit … omnibus sonitibus, qui ex pluribus sonis simplicibus simul sonantibus constant, consonantiae nomen tribuemus." (EULER, 1739: 246).





$$d(n) = a_1(p_1 - 1) + \cdots + a_m(p_m - 1) + 1,\ G(1) := 1$$

If $n = p$ is a prime number, $d(n) = 1(p-1)+1 = p$, that is equality holds.

It might happen that $d(p) = d(m)$ for $n \neq m$ that is the function is not injective.

Example: $n = 12 = 2^2 \cdot 3^1$ with $d(n) = 2(2-1)+1(3-1)+1 = 5$

What Euler still needed was an evaluation of an arbitrary interval. The degree of consonance of an interval or chord should be the degree of agreeableness of the least common multiple (*lcm*) of the numbers entering the ratios:

$$d(a:b) = d(lcm(a,b))$$

Two examples might illustrate Euler's definition. The major third 4:5: *lcm*(4,5) = 20, d(20) = 2(2-1)+1(5-1)+1 = 7

The major chord 4:5:6: *lcm*(4,5,6) = 60, d(60) = 9

Intervals with smaller degrees are more agreeable and easier to use. Yet, Euler's method led to two problems:

a) The notion of consonance could not be inserted into the classification of degrees (VOGEL, 1960: L). The same degree 8 was attributed to different chords like the whole tone 9:8, the minor third 6:5, the minor sixth 8:5.
   While Hugo Riemann and Carl Stumpf criticized Euler's theory, Felix Auerbach and Hermann von Helmholtz judged it more positively (VOGEL, 1960: LI).
b) The use of the least common multiple admitted reductions or extensions of the chords by omitting or inserting certain tones. This is not permitted in musical practice.

The chords *ceg* sharp or 16:20:25 and *cg* sharp or 16:25 have the same degree of consonance 13 because *lcm*(16, 20, 25)=*lcm*(16, 25).The same is true of the chords *ch*, *ceh*, *cgh*, *cegh* or 8:15, 8:10:15, 8:12:15, 8:10:12:15. All of them have the common degree 10.The chord *ghdf* or 36:45:54:64 and the whole scale of *C* major *gahcdef* or 36:40:45:48:54:60:64 have the same degree 17.

Yet, not only dissonant chords are included by the calculation with the least common multiple, but also inversions or repetitions of tones are not taken into account.

The value of the system consists in two main assumptions of this method of calculating the degree of consonance:

a) The graduation of octaves: one octave increases the degree by 1.
$d(2:1) = 2, d(4:1) = 2(2-1)+1 = 3$

b) The incorporation of prime numbers.

No prime number is excluded any longer, not even seven. Euler defended the natural chord of seventh 7:4 in order to justify the great consonance of the dominant seventh chord. For he explained the pleasure in music by the enjoyment of solving (mathematical) riddles (EULER, 1768-1772: III, 11, 23f. [letter 8]). By referring to the cited Leibnizian letter Euler jokingly remarked in 1766:

> *"All these reasons oblige us to recognize that one has to have recourse to the prime number 7 in order to explain the success of these chords … and that we shall say for that reason with the late Mr. Leibnitz that music has now learnt to count up to seven*[14]*."*

## 3.- Epilogue.

Euler's belief in the solvability in principle of mathematical problems reminds us of Hilbert's optimism. If it is necessary problems have to be formulated in such a way that they become solvable. If he was not able to solve a problem such as the algorithmic solution of the algebraic equation of higher than fourth degree (THIELE, 1982: 103), or the three body-problem of celestial mechanics, he looked for the reasons in the insufficient state of the mathematical discipline concerned.

To put it with Eduard Fueter's words dating from 1941: *"For where mathematical reason did not suffice, for Euler began the kingdom of God* (FELLMANN, 2007: 172)".

## 4.- Bibliography.

Euler's works are cited according to their reprint in the *Opera omnia* (EO), for example EO III, 12 = Series III, vol. 12. The translations have been worked out by the author.

[14] "Toutes ces raisons nous obligent à reconnoître qu'il faut recourir au nombre premier 7 pour expliquer le succès de ces accords … et partant nous pourrons dire avec feu M. de Leibniz que la musique a maintenant appris à compter jusqu'à sept. "(EULER, 1766, 525).






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# THE INTEGRAL AS AN ANTI-DIFFERENTIAL. AN ASPECT OF EULER'S ATTEMPT TO TRANSFORM THE CALCULUS INTO AN ALGEBRAIC CALCULUS

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## 1.- Introduction.

The integration of differential formulas was one of the main fields of Euler's activity. He wrote many papers on the subject. In his *Opera omnia*, about 54 articles concern the integration of functions and about 42 regard the integration of differential equations. There are also about 33 papers on elliptic integrals (the theory of elliptic integrals was part of the integral calculus) and three volumes of the massive *Institutionum calculi integralis*, published between 1668 and 1770. Moreover, Euler dealt with integration in many other papers which, even though they were devoted to different subjects, involved differential equations (especially papers that regard geometric or mechanical problems).

In this paper, I will dwell upon an important aspect of Euler's work on integration: the notion of integration as anti-differentiation. I will show that this notion requires examination within the context of Euler's strategy that aimed at transforming integral calculus into an exclusively algebraic theory[1] and that it produced several problems, the most important of which concerned the existence of the anti-differential and the nature of the functions involved in the operation of integration. I will also consider the role of general and particular integrals in Euler's theory and stress that the importance attributed to indefinite integration and general integrals was linked to the conception of analysis as the science that investigated mathematical objects in

[1] On the calculus in the 18th century, see FRASER, Craig (1989) "The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century", *Archive for History of Exact Sciences*, 39, 317-335; FERRARO, Giovanni (2007a) "The foundational aspects of Gauss's work on the hypergeometric, factorial and digamma functions", *Archive for History of Exact Sciences*, 61, 457-518, in particular pp. 459-479, and FERRARO, Giovanni (2008), *The rise and development of the theory of series up to the early 1820s*, New York, Spinger, in particular Chapter 18.