7th Workshop on Numerical Methods in Applied
Science and Engineering (NMASE 08)
Vall de Núria, 09 a 11 de enero de 2008

SIMPLE ASSESSMENT OF THE NUMERICAL WAVE NUMBER IN THE FE SOLUTION OF THE HELMHOLTZ EQUATION

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Key words: Helmholtz equation, wave number, dispersion and pollution effect

Abstract. *When numerical methods are applied to the computation of stationary waves, it is observed that 'numerical waves' are dispersive for high wave numbers. The numerical wave shows a phase velocity which depends on the wave number 'k' of the Helmholtz equation. Recent works on goal-oriented error estimation techniques with respect to so-called quantities of interest or output functionals are developing. Thus, taken into account such aspects, the main purpose of this paper is a posteriori error estimation through of a assessment of the numerical wave number in finite element solution for the simulation of acoustic wave propagation problems addressed by the Helmholtz equation. A method to measure the dispersion on classical Galerkin FEM is presented. In this analysis, the phase difference between the exact and numerical solutions is researched. Fundamental results from a priori error estimation for one-dimensional are presented and issues dealing with pollution error at high wave numbers also are discussed.*

1 Introduction

A mathematical theory for estimating discretization error has become of paramount importance to the computational sciences. The main reason is that, all the computational results obtained by using mathematical models of an event involve numerical errors. The knowledge of such errors permits to asses the reliability of the computation, as well as to be a basis for adaptive control of the numerical process.

More recently, techniques for computing local estimates and estimates of errors in "quantities of interest" began to appear, were presented by Oden and Prudhomme (2001, 2002) and Becker and Rannacher (2001). Mathematically, such quantities are presented as functionals on the solutions of boundary- and initial-value problems. These estimates provide the basis of the so-called "goal-oriented adaptivity" where in adaptive meshing procedures are devised to control error in quantities such as point values of the solution, average of the solution over a sub-domain, or average gradients, but could also be defined as any linear or nonlinear functionals of the solution. A variety of applications of these ideas have appeared since 2000 (e.g. Oden and Vemaganti, 2000; Vemaganti and Oden, 2001; Romkes and Oden, 2004; Pardo et al., 2005).

The simulation of the acoustic wave propagation, addressed by the Helmholtz equation is today a field of intense development, because the acoustic performance of a product is required either by some legal rules or by sales argument. Numerical methods have been developed in order to compute approximate solutions for coupled (vibro-acoustics) or uncoupled problems, on finite or infinite domains. The most popular method is the standard Galerkin Finite Element Method (FEM).

Today it is well known that the standard FEM presents the pollution effect when is applied, to compute solutions of the Helmholtz equation for high wave numbers. That effect consists mainly on dispersion phenomena, i.e. the numerical wave length is longer than the exact one. The pollution effect is related to the loss of stability of the Helmholtz operator at large wave numbers. Thus, the pollution error is related to the *phase lag* between the computed and the exact wave, growing with the wave number, arises from a numerical pollution related to the dispersive character of the discrete medium. The pollution effect has been studied in more detail in Ihlenburg and Babuska (1995a) and Deraemaeker et al. (1999).

In this paper we present results which allow to identify and distinguish discretization error and pollution error. We see that pollution effects usually degrade the quality of a posteriori error estimators for high wave numbers and that, in estimation of error in quantities of interest, the wave number can be defined as a possible linear quantity of interest.

2 Modeling acoustic wave propagation

Although wave propagation phenomena are manifested in a broad range of applications, we delimit the present worked to acoustic wave propagation problems.

2.1 Strong form

Consider the domain Ω with boundary Γ, and let c be the speed of sound inside this medium. The equation of wave propagation derived from the fundamental equations of continuum mechanics is

$$\Delta P' = \frac{1}{c^2}\frac{\partial^2 P'}{\partial t^2} \tag{1}$$

where P' denotes the field of acoustic pressure. If the phenomenon is assumed steady harmonic, P' can be described as:

$$P' = ue^{iwt} \tag{2}$$

where w is the angular frequency and u the spatial distribution of the acoustic complex pressure inside Ω. Substituting (2) into (1), we have to solve the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad (3)$$

The equation (3), with appropriate boundary conditions is the strong form of the problem, where $k := w/c$ denotes the wave number and is defined as the ratio between the angular frequency and the speed of sound.

The physical pressure is the real part of u, and the gradient of pressure is linked to the velocity ($\mathbf{v}$) through the equation of motion which can be written as

$$i\rho w\mathbf{v} + \nabla u = 0 \quad (4)$$

where ρ the specific mass of the fluid.

For interior problems, three sets of boundary conditions are possible:

- *Dirichlet* (the acoustic pressure is prescribed)

$$u = \bar{u} \quad \text{at} \quad \Gamma_D \quad (5)$$

- *Neumann* (the normal component of the velocity is prescribed)

$$\frac{\partial u}{\partial \mathbf{n}} = -i\rho w\bar{v_n} \quad \text{or} \quad v_n = \bar{v_n} \quad \text{at} \quad \Gamma_N \quad (6)$$

- *Robin*

$$\frac{\partial u}{\partial \mathbf{n}} = -i\rho w A_n u \quad \text{or} \quad v_n = A_n u \quad \text{at} \quad \Gamma_R \quad (7)$$

where v_n is the normal component of the velocity for the excitation by the vibrating panels, $\mathbf{n}$ is the exterior unit normal vector, and A_n is the admittance coefficient modelling the structural damping. Neumann boundary conditions correspond to vibrating panels while Robin boundary conditions correspond to absorbant panels.

2.2 Weak form

To formulate the problem in the weak form, the spaces corresponding the kinematically admissible trial functions u and the homogeneous test functions v are introduced $U = \{u \in \mathrm{H}^1(\Omega), u|_{\Gamma_D} = \bar{u}\}$ and $V = \{v \in \mathrm{H}^1(\Omega), v|_{\Gamma_D} = 0\}$, where $\mathrm{H}^1(\Omega)$ is the Hilbert space, that is a Sobolev space of square integrable functions and first derivatives.

Applying the integral formulation, the divergence theorem and some mathematics relations, the problem can be rewritten as:

Find $u \in U$ such that

$$a(u, v) = l(v), \quad \forall v \in V \quad (8)$$

where

$$a(u,v) = \int_\Omega \nabla u^t \nabla \tilde{v} d\Omega - \int_\Omega k^2 u \tilde{v} d\Omega + \int_{\Gamma_R} i\rho u \tilde{v} A_n d\Gamma$$

$$l(v) = -\int_{\Gamma_N} i\rho w \bar{v_n} \tilde{v} d\Gamma$$

where $\tilde{\cdot}$ denotes the complex conjugate. Note that, the bilinear form $a(.,.)$ is symmetric, but not Hermitian.

2.3 Finite Element Method

Spaces of finite dimension $U_H \subset U$ and $V_H \subset V$, can be defined in order to approach the spaces of infinite dimension U and V, respectively. The superscript H denotes that these spaces are relative to the mesh discretization of Ω, where H is so the size of the characteristic element. Using these spaces of finite dimension, the classic method of Galerkin consists in:

Find $u_H \in U_H$ such that for all $v_H \in V_H$

$$a(u_H, v_H) = l(v_H) \tag{9}$$

When a standard Galerkin procedure is used, and the test functions is restricted to the discrete subspace corresponding to

$$u_H = \sum_{j=1}^{n} N_j u_j = \mathbf{N}\mathbf{u}_H \tag{10}$$

where $u_j \in \mathbb{C}$, for $j = 1, 2, ..., n$ are the nodal values the pressure field, and N_j are the shape functions.

Therefore, the discrete acoustic problem resulting in a linear system of equations

$$(\mathbf{K}_H + i\rho w \mathbf{C}_H - k^2 \mathbf{M}_H)\mathbf{u}_H = -i\rho w \mathbf{f}_H \tag{11}$$

where $\mathbf{K}_H$ is the stiffness matrix:

$$\mathbf{K}_H = \int_\Omega (\nabla \mathbf{N})^t (\nabla \mathbf{N}) d\Omega$$

$\mathbf{C}_H$ is the damping matrix, modeling Robin boundary conditions:

$$\mathbf{C}_H = \int_{\Gamma_R} A_n \mathbf{N}^t \mathbf{N} d\Gamma$$

and $\mathbf{M}_H$ is the mass matrix:

$$\mathbf{M}_H = \int_\Omega \mathbf{N}^t \mathbf{N} d\Omega$$

The right-hand side contains the prescribed normal velocities of the Neumann boundary condition

$$\mathbf{f}_H = \int_{\Gamma_N} \mathbf{N}^t \bar{v_n} d\Gamma$$

3 Dispersion and Pollution effect

We will see in this Section that the pollution effect is directly related to the dispersion, thus we develop here a tool in order to measure the dispersion and so are able to measure the pollution on Galerkin FEM. Moreover, we explain the concepts of pollution, dispersion and their effect on the error estimation explaining for a one-dimensional problem mainly.

When solving the Helmholtz equation with the classical Galerkin FEM, it is well known that the accuracy of the numerical solution deteriorates with increasing non-dimensional wave number k (Ihlenburg and Babaska, 1997). This effect is the so-called 'pollution effect'. The main effect of the 'pollution' is that the wave number of the FEM solution is different from the wave number of the exact solution, and this is what is called 'dispersion' (Ihlenburg and Babuska, 1995a).

In one dimension, the 'pollution' effect has been largely studied (Babuska and Sauter, 2000; Ihlenburg and Babuska, 1995a and 1995b). This leads to uneconomical meshes and non-accurate error estimation for high wave numbers and gives the motivation to look for a pollution-free method. In one dimension, different approaches have been proposed and lead to solutions that do not suffer the pollution effect (Babuska et al., 1995; Babuska and Melenk, 1997). In two and three dimensions however, it has been proved (Babuska and Sauter, 2000) that the pollution cannot be avoided. Many attempts have been made in order to reduce the pollution effect by adapting the methods developed in one dimension.

3.1 A priori estimate of the dispersion error

In case one-dimensional, for uniform meshes, the dispersion error can be predicted (Ihlenburg and Babuska, 1995a). Consider the patch of elements surrounding node in a one-dimensional mesh, as it is shown in Figure 1.

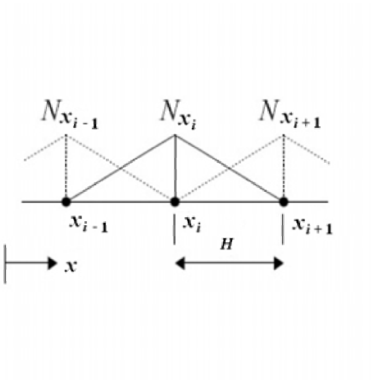

Figure 1: *Element patch in uniform FEM mesh for dispersion analysis.*

Modified exact problem

Let the modified problem defined by

$$\Delta u_m + k_H^2 u_m \quad = \quad 0 \tag{12}$$

and in simple case one-dimensional the numerical solution of this equation is represented by propagation wave

$$u_m(x) \quad = \quad e^{ik_H x} \tag{13}$$

with the numerical wave number k_H.

By applying standard Galerkin FEM approach, the discrete system of equations is obtained to the Helmholtz problem

$$(\mathbf{K}_H + i\rho w \mathbf{C}_H - k^2 \mathbf{M}_H)\mathbf{u}_H \quad = \quad \hat{\mathbf{f}}_H \tag{14}$$

where $\hat{\mathbf{f}}_H$ results of the boundary conditions.

The goal of this analysis is to find k_H such that $\mathbf{u}_H \approx u_m$ solution of modified problem (equation 12). Thus, if linear elements with basis functions N_{i-1}, N_i and N_{i+1} are used, the one-dimensional homogeneous Helmholtz equation can be written in algebraic form for node i as

$$Ru_{i-1} + 2Su_i + Ru_{i-1} \quad = \quad 0 \tag{15}$$

with

$$R \ := \ -1 - \frac{1}{6}(kH)^2$$
$$S \ := \ 1 - \frac{1}{3}(kH)^2$$

where u_i represent the nodal pressure at position $x_i = iH$.

The substitution of the solution (13) in the differential equation (15) leads to:

$$\left[Re^{-ik_H H} + 2S + Re^{ik_H H}\right] e^{ik_H x_i} \quad = \quad 0 \tag{16}$$

thus, it is derived that

$$\cos(k_H H) \quad = \quad -\frac{S}{R} = \frac{1 - \frac{1}{3}(kH)^2}{1 + \frac{1}{6}(kH)^2} \tag{17}$$

Applying Taylor expansion and some further mathematical derivations result in the following explicit dispersion relation

$$k_H H \quad \approx \quad kH - \frac{1}{24}(kH)^3 + O((kH)^5) \tag{18}$$

Thus, the phase difference between exact and numerical solutions can be predicted as

$$k - k_H \approx \frac{k^3 H^2}{24} \tag{19}$$

and therefore, the dispersion error can be defined as

$$E^{pri} = \frac{k^3 H^2}{24} \approx k - k_H \tag{20}$$

3.2 Relation between the dispersion, pollution, wave number and mesh size

The dispersion error produces pollution effect that modify the standard a priori estimates in energy norm and the dispersion increases with the wave number k and the mesh size H.

In order to illustrate the fact that the pollution also increases with these factors, we will see an error estimate given by Ihlenburg and Babuska (1995a)

$$\frac{|u - u_H|_1}{|u|_1} \leq C_1\theta + C_2 k\theta^2 \tag{21}$$

being $\theta = (kH/p)^p$ (p-order of polynomial, H-mesh size) the scale of the finite element mesh and the constants C_1 and C_2 independent of k and H.

Thus, for particular case of $p = 1$

$$\frac{|u - u_H|_1}{|u|_1} \leq C_1 kH + C_2 k^3 H^2 \tag{22}$$

The first term in equation represents the local error and the second term represents the pollution error. In order to keep the local error constant, it is sufficient to keep

kH constant. That is the so-called 'rule of the thumb' which corresponds to taking a certain number of elements per wavelength. As we can see, this is not sufficient to keep the pollution error under control. The pollution error increases with k. To obtain the accurate finite element solution, the second term has to be controlled. This leads to the necessity to use fine meshes for high wave numbers. For well-refined meshes, the exact error is close to the interpolation error, which means that there is not much pollution. As the mesh becomes less and less refined, we see the influence of the pollution error on the accuracy of the finite element solution.

3.3 Measure of the Dispersion - a posteriori error

We have seen that the error on the FE solution for the Helmholtz equation is made of two parts, the local error and the pollution error which is directly related to the dispersion. Because the pollution effect leads to uneconomical meshes for high wave numbers, there have been many attempts to find new numerical methods which control the pollution. In this section, we develop a tool in order to measure the dispersion for FEM as a measure of the pollution.

In this work, in this examples of application the problem of Helmholtz is solved for several meshes. The more refinate of all the mesh is defined as reference mesh. Thus, we have the FE solution u_H such that

$$[\mathbf{K}_H + i\rho w\mathbf{C}_H - k^2\mathbf{M}_H]\mathbf{u}_H = \hat{\mathbf{f}}_H \tag{23}$$

and for reference mesh

$$[\mathbf{K}_h + i\rho w\mathbf{C}_h - k^2\mathbf{M}_h]\mathbf{u}_h = \hat{\mathbf{f}}_h \tag{24}$$

thus, the error

$$\mathbf{e} = \mathbf{u}_h - [\mathbf{u}_H]_h \tag{25}$$

where $[\mathbf{u}_H]_h$ is solution in the coarse mesh evaluate in the reference mesh.

For compute the global error in norm L^2 ($\| \mathbf{e} \|_{L^2}$) is necessary compute $\mathbf{e}$ that requires global computation of $\mathbf{u}_h$. However, not requires for compute dispersion error in k.

3.3.1 Global version

The main objective is the assessment of the wave number k as quantity of interest in order to measure the dispersion.

Modified reference problem

Analogous applied procedure a priori, the goal of this analysis is to find k_H that better accommodates $\mathbf{u}_H$ such that $\mathbf{u}_H \approx \mathbf{u}_m$ is the best solution of the modified reference problem defined as:

$$[\mathbf{K}_h + i\rho w\mathbf{C}_h - k_H^2\mathbf{M}_h]\mathbf{u}_m = \hat{\mathbf{f}}_h \tag{26}$$

rewriting

$$[\mathbf{L}_h - \alpha \mathbf{M}_h]\mathbf{u}_m = \hat{\mathbf{f}}_h \tag{27}$$

with $\mathbf{L}_h = \mathbf{K}_h + i\rho w \mathbf{C}_h$, $\hat{\mathbf{f}}_h = -i\rho w \mathbf{f}_h$ and $\alpha = k_H^2$.

Here we are not in the simple case and not has previous knowledge of $\mathbf{u}_m$. Thus, replace $\mathbf{u}_H$ (in fact $[\mathbf{u}_H]_h$) FE solution in modified reference problem and compute a residual defined as

$$\mathbf{r}(\alpha) \;=\; \hat{\mathbf{f}}_h - (\mathbf{L}_h - \alpha \mathbf{M}_h)[\mathbf{u}_H]_h \tag{28}$$

where $\mathbf{r}$ depends of $\alpha = k_H^2$.

Now, denoting

$$\mathbf{a} \;=\; \hat{\mathbf{f}}_h - \mathbf{L}_h[\mathbf{u}_H]_h \tag{29}$$

$$\mathbf{b} \;=\; \mathbf{M}_h[\mathbf{u}_H]_h \tag{30}$$

the residual transforms into

$$\mathbf{r}(\alpha) \;=\; \mathbf{a} + \alpha \mathbf{b} \tag{31}$$

The goal is to find α^{eq} to α physical, such that $||\mathbf{r}||$ is minimal, that is,

Find α^{eq} such that

$$\alpha^{eq} := \arg(\min_{\alpha}(\tilde{\mathbf{r}}\mathbf{r})) = -\frac{\texttt{real}(c_1)}{c_2} + i\frac{\texttt{imag}(c_1)}{c_2} \tag{32}$$

where $c_1 = \tilde{\mathbf{a}}\mathbf{b}$ and $c_2 = \tilde{\mathbf{b}}\mathbf{b}$

Therefore,

$$k_H^{eq} = \sqrt{\alpha^{eq}} \tag{33}$$

Thus, finally we can define a measure of the dispersion as a error estimate in k by

$$E^{pos} \;= k - k_H^{eq} \tag{34}$$

Remarks:

- explicit computation, requires assembling of matrices $\mathbf{K}_h, \mathbf{M}_h, \mathbf{C}_h$ and $\tilde{\mathbf{f}}_h$, but no solving reference problem h.
- possible localization (local version, next section)

3.4 Local version

Now, we intent to evaluate the equivalent wave number in local *patches* of elements (Figure 2).

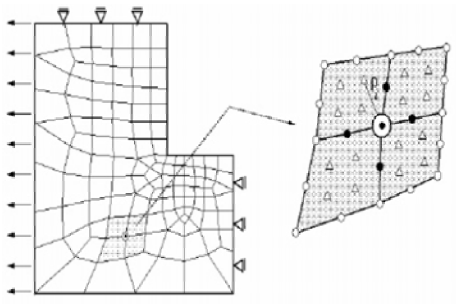

Figure 2: *Example of detached patch.*

The geometric support of the elements for a given mesh are open subdomains denoted by $\Omega_n, n = 1, ..., n_{el}$, where $\bar{\Omega} = \bigcup_n \bar{\Omega}_n$. Let $\mathrm{p}^j, j = 1, ..., n_{np}$ denoted as the nodes of the elements in the computational mesh of FE and N^j the corresponding linear shape functions. The support of N^j is denoted by τ^j and it is called of *patch* associated with node p^j, that is, $\tau^j = \mathrm{supp} N^j$.

For each node p^j of the coarse mesh is generated a patch of elements, that is one finite mesh $\Omega_n^j, n = 1, ..., n_{el}^j$ over Ω, where Ω_n^j is refined the same number of times that the global problem was refined. Finishing the refine process, obtains one patch reference mesh and also the coarse solution interpolated in the patch reference mesh. Analogous global problem, assembling the matrices and vectors to solve the Helmholtz problem in the new local domain (reference patch), but not is necessary to compute the solution. Then, the residual $\mathbf{r}^j(\cdot)$ correspondent to patch j is defined as following

$$\mathbf{r}^j(\alpha) \quad = \quad \hat{\mathbf{f}}_h^j - (\mathbf{L}_h^j - \alpha^j \mathbf{M}_h^j)[\mathbf{u}_H]_h^j \tag{35}$$

where $\mathbf{r}^j$ depends of $\alpha = k_H^j$.

Analogous the global version, solving the minimization problem we obtain the equivalent wave number $k_H^{eq^j} = \sqrt{\alpha^{eq^j}}$, where α^{eq^j} is defined as

$$\alpha_{eq^j} := \arg(\min_{\alpha}(\tilde{\mathbf{r}^j}\mathbf{r}^j)) = -\frac{\mathtt{real}(c_1)}{c_2} + i\frac{\mathtt{imag}(c_1)}{c_2} \tag{36}$$

where c_1 and c_2 is defined over the patch as global problem.

4 Examples of application

Numerical tests are performed on two problems. The first example is the problem one-dimensional of a rectangular tube. In this case, we solve the problem in uniform meshes to compare the a priori estimate (equation 20) with estimates a posteriori of the dispersion error.

The second example is a real-life problem, where the noise inside of an car is analyzed, in order to show that is really efficient measure the dispersion effect.

In the presented examples, we observe mainly related aspects with the numeric error in the solutions of FEM, such as: dispersion, asymptotic behavior, convergence, etc.

The examples are filled with air ($\rho_a = 1.225[Kg/m_3]$) and sound speed ($c = 340[m/s]$).

4.1 Band

The model problem is a tube of length $L_x = 1[m]$, such that

$$\begin{cases} \frac{d^2u}{dx^2} + k^2 u & = & 0 & \in \Omega(0 \leq x \leq 1) \\ u(0) & = & & 1 \\ \frac{du}{dx}(1) & = & & iku(1) \end{cases} \tag{37}$$

where Dirichlet boundary conditions are prescribed on one end of the tube, while Robin conditions are submitted to the other end.

This problem has analytical solution (equation 13). Thus, for each mesh we can measure a posteriori the numerical wave number. Let u =real($u(x)$) $= \cos(kx)$ and u_H, the analytical and numerical solution, respectively, such that

$$u(x_f - \Delta\lambda) \quad = \quad u_H(x_f) \tag{38}$$

with

$$u_H(x_f) \quad = \quad \frac{\sum_i^m u_{H_i}(x_f)}{m} \approx \beta \tag{39}$$

where u_{H_i} are all the coarse solutions evaluated in the final point x_f and $\Delta\lambda$ is the difference between the numerical and the analytical wavelength.

Assuming

$$\cos(k(x_f - \Delta\lambda)) \quad = \quad \beta \tag{40}$$

then

$$\Delta\lambda \quad = \quad -\frac{arcos\beta}{k} + x_f \tag{41}$$

On the other hand, we have

$$\Delta\lambda \quad = \quad n\lambda - n\lambda_H \tag{42}$$

where $\lambda := 2\pi/k$ is the wavelength and n is period number.

Consequently,

$$k_H = \frac{k}{(1 - \frac{\Delta\lambda k}{2\pi n})} \tag{43}$$

Therefore, is possible to measure a posteriori the dispersion as

$$E^{meas} = k - k_H \tag{44}$$

where the numerical wave number is given by equation (43).

4.1.1 Numeric Results

The problem is solved in uniform meshes of triangular elements. Some results of solutions, error and convergence are shown in different frequencies (340Hz, 680 Hz and 1020 Hz).

1. Dispersion

For frequency $f = 340$ Hz, $k = 2\pi$ the wavelength is $\lambda = 2\pi/k = 1$. Then by 'rule of the thumb' $(kh \leq 1), h \leq 0.1591$ captures the wavelength and the presented error is predominantly due to interpolation. We start with a mesh $h = 0.125$ and can be clearly that same satisfied the 'rule of the thumb' the solution presents dispersion. However, for frequency $f = 680$ Hz ($k = 4\pi$, $\lambda = 0.5$), from $h \leq 0.0795$ satisfy the 'rule of thumb'. But, as we start with a mesh $h = 0.125$ notice that the dispersion in the solution coarse mesh is greater (Figure 3).

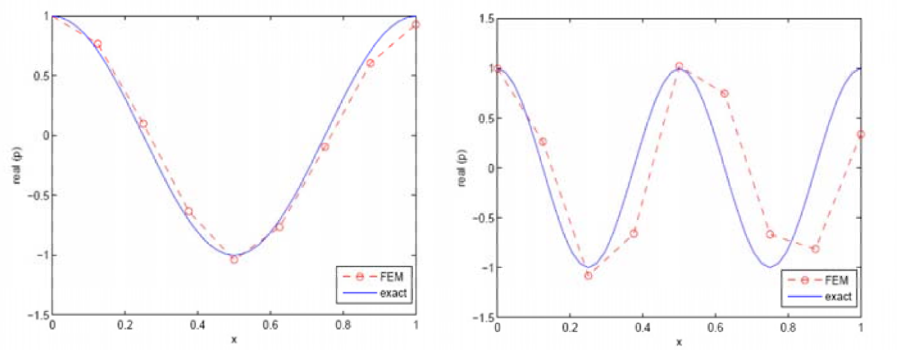

Figure 3: *FEM solution* vs. *exact wave for* $h = 0.125$, $f = 340$ *Hz and* $f = 680$ *Hz.*

2. Error Analysis

Error in norm L^2

The total error is calculated in norm L^2 as the difference between the solution in the coarse meshes and the solution in the refined reference mesh. The Figure 4 show the error convergence for the three frequencies.

The error satisfy the order of convergence $p+1$ for FEM in norm L^2, keeping the asymptotic behavior. However, it can be noticed that for high frequencies (i.e, the order of convergence is 1.8 for 1020 Hz) there is a pre-asymptotic behavior for the coarse meshes, what is consequence of the pollution increment.

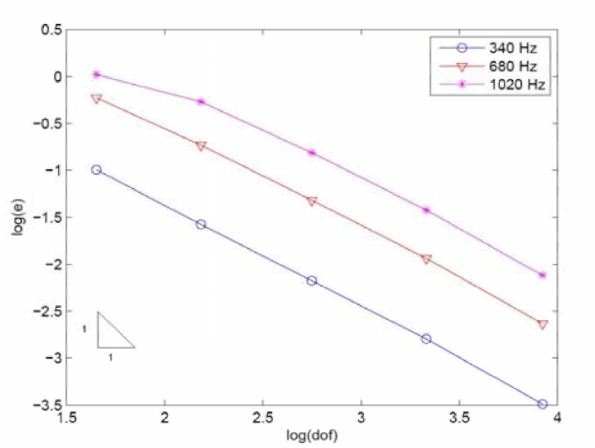

Figure 4: *Global error convergence in norm* L^2.

Error in k

The analysis of the error through of the wave number k, that show a phase shift between the exact and the finite element waves. This analysis can be estimate a priori (equation 20), "measured" (equation 44) and a posteriori as proposed in Section 3.4 (equation 34). The results obtained these estimates are:

i) $f = 340$ Hz P=1

dof	h [m]	E^{pri}	E^{meas}	E^{pos}
45	0.1250	2.5702e-02	-2.7116e-02	4.1963e-02 + 4.1055e-03i
153	0.0625	6.4255e-03	-6.7046e-03	2.1494e-02 + 1.0171e-03i
561	0.0312	1.6063e-03	-1.6697e-03	1.0131e-02 + 2.2696e-04i
2145	0.0156	4.0159e-04	-4.1751e-04	4.1996e-03 + 4.1296e-05i
8385	0.0078	1.0039e-04	-1.0456e-04	1.1877e-03 + 3.9608e-08i

ii) $f = 680$ Hz P=2

dof	h [m]	E^{pri}	E^{meas}	E^{pos}
45	0.1250	1.0280e-01	-9.6394e-02	4.8792e-02 + 7.0552e-03i
153	0.0625	2.5702e-02	-2.6239e-02	2.2637e-02 + 1.7330e-03i
561	0.0312	6.4255e-03	-6.5309e-03	1.0973e-02 + 4.2350e-04i
2145	0.0156	1.6063e-03	-1.6305e-03	4.6218e-03 + 7.9988e-05i
8385	0.0078	4.0159e-04	-4.0765e-04	1.3355e-04 - 7.8596e-08i

iii) $f = 1020$ Hz P=3

dof	h [m]	E^{pri}	E^{meas}	E^{pos}
45	0.1250	2.3131e-01	-1.3426e-01	3.4790e-02 + 5.2772e-03i
153	0.0625	5.7829e-02	-5.5414e-02	2.5150e-02 + 2.5900e-03i
561	0.0312	1.4457e-02	-1.4535e-02	1.2291e-02 + 6.3000e-04i
2145	0.0156	3.6143e-03	-3.6299e-03	5.3120e-03 + 1.1983e-04i
8385	0.0078	9.0358e-04	-9.0704e-04	1.5806e-03 - 8.2880e-08i

The Figure 5 show the convergence of the error assessment in k a priori, measure and a posteriori for the frequencies $f = 340$ Hz, $f = 680$ Hz and $f = 1020$ Hz.

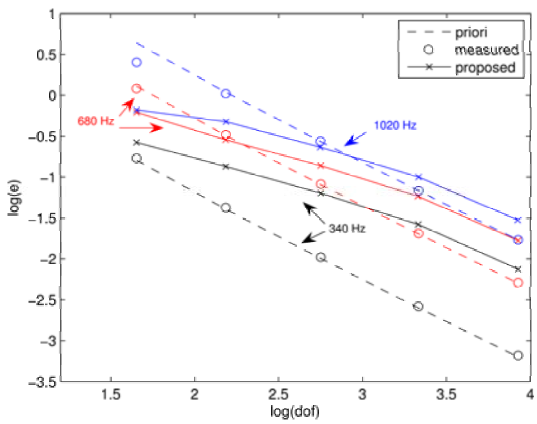

Figure 5: *Convergence estimates of the error relative a priori, measured and proposed for the three frequencies: 340 Hz (black), 680 Hz (red) and 1020 Hz (blue).*

Note that the results of a posteriori estimate error does not correspond with the values of a priori and "measured" estimates error.

4.2 Car cavity

We consider now a real-life problem, a practical example, of the analysis of noise inside of an car (Figure 6). In this case the solution is no more plane wave.

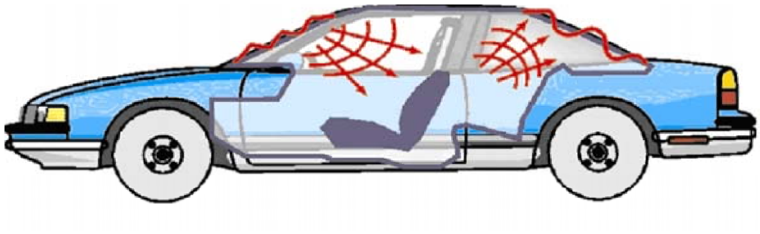

Figure 6: *Two-dimensional section of a car.*

The problem (Figure 7) is a 2D-section in the bodywork of a car. The air inside the cabin is excited by the vibrations due to the engine through the front panel (Neumann boundary conditions). The roof is covered with an absorbent material (Robin boundary conditions).

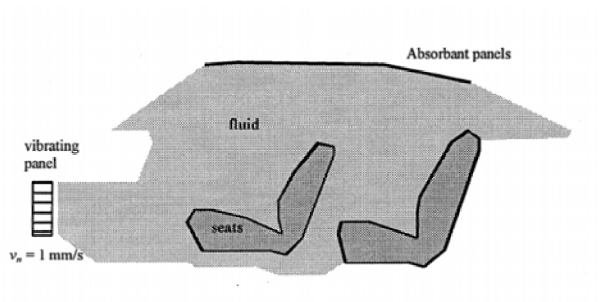

Figure 7: *Two-dimensional section of a car.*

The cavity has the characteristic dimensions $L_x \approx 2.7[m]$ and $L_y \approx 1.1[m]$. The main part of domain boundary is rigid ($\overline{v_n} = 0[m/s]$). An impedance boundary condition is imposed at the roof with $\overline{Z} = 2000[Pa.s/m]$, which introduces damping in the acoustic system. A unit normal velocity distribution ($\overline{v_n} = 1[m/s]$) at the fire wall excits the 2D car cavity.

4.2.1 Numerical Results

We consider a FEM discretization of linear elements from of 238 nodes (coarse mesh) and is used as reference the FEM solution on a highly refined mesh (193616 nodes).

We study the acoustic response inside the car in the frequency of 215, 525 and 955 Hz. Results of the distribution of the acoustic pressure inside the car is presented in the Figure 8, where can easily observe the wavefronts, especially in the region above the seat.

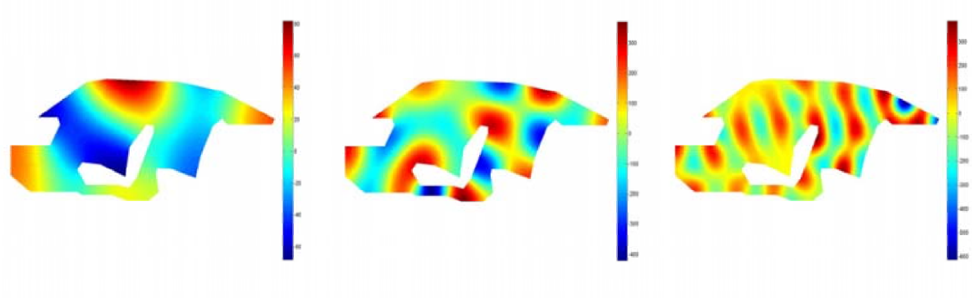

Figure 8: *Distribution of the real part of the acoustic pressure for the frequencies 215 Hz, 525 Hz and 955 Hz, respectively: solution FEM mesh (12404 nodes).*

1. Dispersion

We also presents results computed along the straight line defined in Figure 9, with goal of shown the dispersion effect for the numerical wave number as well as for the amplitude of the wave.

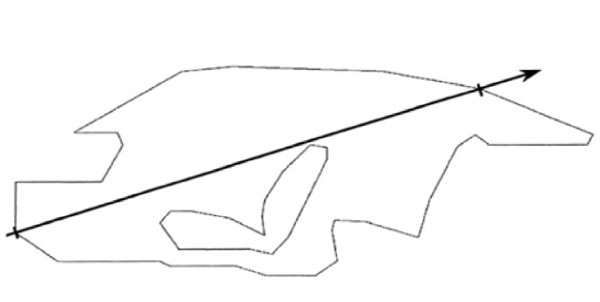

Figure 9: *Definition of the line.*

The solution FEM wave exhibits an important phase lag, when compared to a reference solution. This is confirmed in the cut view of Figure 10, where we can also see that the amplitude of the FEM wave.

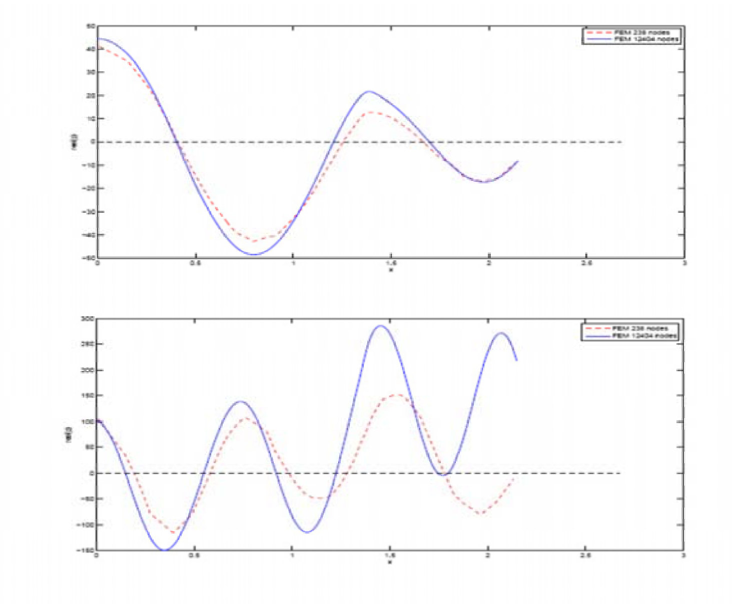

Figure 10: *FEM solution of the acoustic pressure at line for frequencies of 215 and 525 Hz.*

2. Error Analysis

Error in norm L^2

Analogous the example 1, the Figure 11 show the convergence error in the norm L^2 for three frequencies computed. The error keeps the asymptotic or pre-asymptotic behavior.

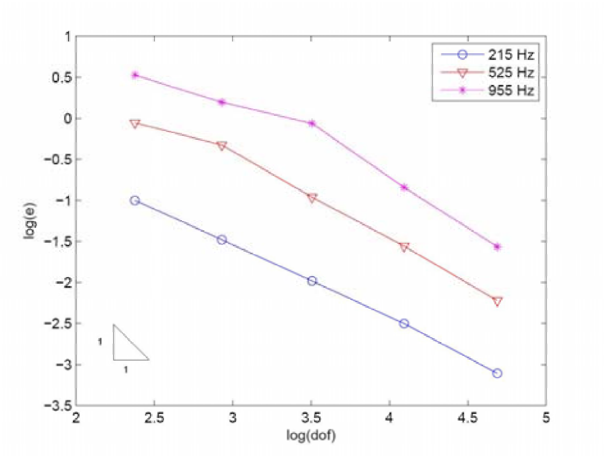

Figure 11: *Convergence of error in norm* L^2.

Error in k

Remember that analysis of the error through of the wave number k show a phase shift between the exact and FE waves. In this example cannot estimate a priori and nor estimate "measured" through of analytical solution, but we can be estimate a posteriori as proposed in the section 3.4. (equation 34). The results of relative error this estimate are:

dof	$E^{pos}(f = 215Hz)$	$E^{pos}(f = 525Hz)$	$E^{pos}(f = 955Hz)$
238	1.2268e-02 - 2.3413e-03i	2.7423e-02 - 2.4543e-03i	3.5868e-02 - 1.6004e-03i
851	9.6892e-03 - 1.7693e-03i	1.8340e-02 - 1.1183e-03i	2.9572e-02 - 3.0248e-03i
3202	5.1290e-03 - 1.0353e-03i	1.0218e-02 - 4.2717e-04i	1.4369e-02 - 7.4387e-04i
12404	2.2822e-03 - 5.2165e-04i	4.6277e-03 - 2.1885e-04i	6.5754e-03 - 2.2866e-04i
48808	7.6776e-04 - 1.9907e-04i	1.5767e-03 - 8.6208e-05i	2.1851e-03 - 7.7177e-05i

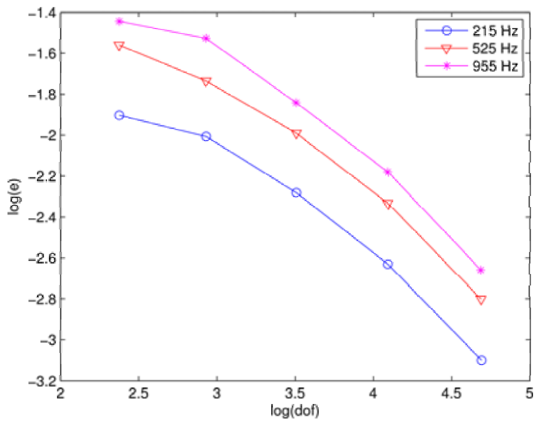

Figure 12: *Convergence estimate error proposed.*

5 Conclusion

We observed that pollution effects degrade the quality of the posteriori error estimators for high wave numbers. However, asymptotic behavior is kept in the analyzed cases, by which the order of convergence $p + 1$ of the FEM in norm L^2 is slightly lost.

In the first case analyzed, the results of a posteriori estimate error does not correspond with the values of a priori and "measured" estimates error. It can be explained by the the fact that we have assumed that our a posteriori estimator take only in account the error of dispersion, but this is not true. Our estimator actually has two contributions, one becoming from the dispersion and other from the interpolation. It mean that the two contributions must be separately dealt in order to improve the estimator.

In second case of the car cavity, where study the noise inside a car, we observed the dispersion effect especially in the region above the seat. Therefore, is important an efficient measurement of the dispersion effect.

We can conclude that FEM is applicable to real-life acoustic problems, and in this case, but leading to a error in terms of wave number and amplitude. Therefore, is necessary to measure of the dispersion and search a formulation more efficient for reduction yours effect. Moreover, that this measure and formulation are general and easily applicable to real-life problems.

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Con el apoyo de *Generalitat de Catalunya*, *Universitat Politi¿½nica de Catalunya* y *E.T.S. d'Enginyers de Camins, Canals i Ports de Barcelona*