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3D NURBS-ENHANCED FINITE ELEMENT METHOD

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Abstract. *An improvement of the classical finite element method is proposed in [1], the NURBS-Enhanced Finite Element Method (NEFEM). It is able to exactly represent the geometry by means of the usual CAD description of the boundary with Non-Uniform Rational B-Splines (NURBS). For elements not intersecting the boundary, a standard finite element interpolation and numerical integration is used. But elements intersecting the NURBS boundary need a specifically designed piecewise polynomial interpolation and numerical integration. This document presents preliminary work on the 3D extension of NEFEM.*

1 INTRODUCTION

The importance of the geometrical model in finite element (FE) simulations has recently been pointed out by several authors, see [1, 2, 3] to name a few. In some applications, such as compressible flow problems, if a Discontinuous Galerkin (DG) formulation is adopted, see [4], an important loss of accuracy is observed when a linear approximation of the boundary is used, see [2]. Bassi and Rebay [2] show that, in the presence of curved boundaries, a meaningful high-order accurate solution can only be obtained if the corresponding high-order approximation of the geometry is employed (i.e. isoparametric FE).

Maxwell equations are also very sensitive to the quality of the boundary representation. Reference [5] studies the error induced by the approximation of curvilinear geometries with isoparametric elements. The 3D Maxwell equations are solved in a sphere with isoparametric FE and with an exact mapping of the geometry. The exact mapping provides more accurate results with errors differing by an order in magnitud. Thus, in some applications, an isoparametric representation of the geometry is far from providing an optimal numerical solution for a given spatial discretization.

Non-Uniform Rational B-Splines (NURBS, see [6]) are widely used for geometry description in CAD (Computer Aided Design). This fact has motivated new numerical methodologies considering an exact representation of the computational domain with NURBS, such as the *isogeometric analysis* [3] and the NURBS-Enhanced Finite Element Method (NEFEM) [1]. The *isogeometric analysis* considers the same NURBS basis functions for both the description of the entire geometry and for the approximation of the solution. NEFEM also considers an exact representation of the domain but it differs from the *isogeometric analysis* in two main points: the geometry is given by the NURBS description of the boundary (i.e. the information usually provided by CAD), and standard FE polynomial interpolation is considered for the approximation of the solution. Thus, in the large majority of the domain —for elements not intersecting the boundary— a standard FE interpolation and numerical integration is used, preserving the computational efficiency of classical FE techniques. Specifically designed piecewise polynomial interpolation and numerical integration is only required for those FE along the NURBS boundary.

In [1] NEFEM is presented in two dimensional simulations. Poisson and Maxwell problems demonstrate the applicability of the proposed method in both a continuous and discontinuous Galerkin frameworks. When the quantities of interest are defined on curved boundaries NEFEM is at least one order of magnitude more precise than the corresponding isoparametric FE. Moreover, the exact representation of the boundary allows to mesh the domain independently of the geometric complexity of the boundary whereas classical isoparametric finite elements need h-refinement to properly capture the geometry.

This paper presents preliminary work on the 3D extension of NEFEM. Section 2 introduces the basic concepts on NURBS surfaces. In section 3 fundamentals of NEFEM are presented in three dimensions. Special attention is paid to the interpolation and numerical integration in those elements affected by the NURBS description on the boundary. Finally, in section 4 a Poisson example demonstrate the applicability of the proposed method.

2 BASIC CONCEPTS ON NURBS SURFACES

In this section NURBS curves are briefly introduced in order to present NURBS surfaces. For a more detailed presentation on NURBS see [6] .

A qth-degree NURBS curve is a piecewise rational function defined in parametric form as

$$\boldsymbol{C}(\lambda) = \frac{\displaystyle\sum_{i=0}^{q} \nu_i \, \boldsymbol{B}_i \, C_{i,q}(\lambda)}{\displaystyle\sum_{i=0}^{q} \nu_i \, C_{i,q}(\lambda)} \qquad 0 \leq \lambda \leq 1, \tag{1}$$

where $\{\boldsymbol{B}_i\}$ are the coordinates of the *control points* (forming the *control polygon*), $\{\nu_i\}$ are the control weights, and the $\{C_{i,q}(\lambda)\}$ are the normalized B-spline basis functions of degree q, which are defined recursively as:

$$C_{i,0}(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [\lambda_i, \lambda_{i+1}[, \\ 0 & \text{elsewhere,} \end{cases}$$
$$C_{i,k}(\lambda) = \frac{\lambda - \lambda_i}{\lambda_{i+k} - \lambda_i} C_{i,k-1}(\lambda) + \frac{\lambda_{i+k+1} - \lambda}{\lambda_{i+k+1} - \lambda_{i+1}} C_{i+1,k-1}(\lambda),$$

for $k = 1 \dots q$, where λ_i, for $i = 0, \dots, m$, are the *knots* or *breakpoints*, which are assumed ordered $0 \leq \lambda_i \leq \lambda_{i+1} \leq 1$.

A NURBS surface of degree q in λ direction and degree l in κ direction is a piecewise rational function defined in parametric form as

$$\boldsymbol{S}(\lambda, \kappa) = \frac{\displaystyle\sum_{i=0}^{q} \sum_{j=0}^{l} \nu_{ij} \, \boldsymbol{B}_{ij} \, C_{i,q}(\lambda) \, C_{j,l}(\kappa)}{\displaystyle\sum_{i=0}^{q} \sum_{j=0}^{l} \nu_{ij} \, C_{i,q}(\lambda) \, C_{j,l}(\kappa)} \qquad 0 \leq \lambda, \kappa \leq 1, \tag{2}$$

where $\{\boldsymbol{B}_{ij}\}$ are the coordinates of the *control points* (forming the *control net*), $\{\nu_{ij}\}$ are the control weights, and $\{C_{i,q}(\lambda)\}$ and $\{C_{i,l}(\kappa)\}$ are the normalized B-spline basis functions of degree q and l respectively, defined on the so-called *knot vectors*

$$\Lambda^q = \{\underbrace{0, \dots, 0}_{q+1}, \lambda_{q+1}, \dots, \lambda_{m^q - q - 1}, \underbrace{1, \dots, 1}_{q+1}\},$$

$$\Lambda^l = \{\underbrace{0, \dots, 0}_{l+1}, \kappa_{l+1}, \dots, \kappa_{m^l - l - 1}, \underbrace{1, \dots, 1}_{l+1}\}.$$

Note that each 2D B-spline basis function is a cartesian product of two 1D B-spline basis functions. Figure 1 shows two 2D B-spline basis functions associated with knot vectors

$$\Lambda^q = \{0, 0, 0, 0, 0.4, 1, 1, 1, 1\},$$
$$\Lambda^l = \{0, 0, 0, 0.2, 0.6, 0.6, 1, 1, 1\}.$$

This figure also represents 1D B-spline basis functions associated to Λ^q and Λ^l knot vectors.

Note that NURBS curves change their definition at breakpoints. Then, NURBS surfaces change their definition at *knot lines*, that is, at (λ_i, κ) for $i = 1, \dots, m^q$ and (λ, κ_j) for $j = 1, \dots, m^l$. An example of a NURBS surface is represented in Figure 2 with the corresponding control net. Knot lines are represented on the NURBS surface in order to stress the discontinuous nature of the parametrization.

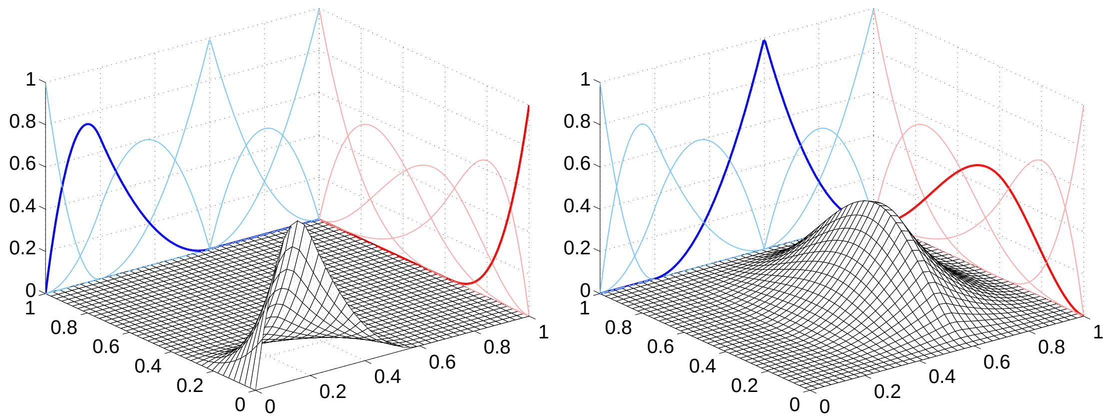


Figure 1: Two examples of 2D B-splines basis functions. Complete 1D B-spline basis are also represented: in red for knot vector Λ^q and, in blue for knot vector Λ^l.

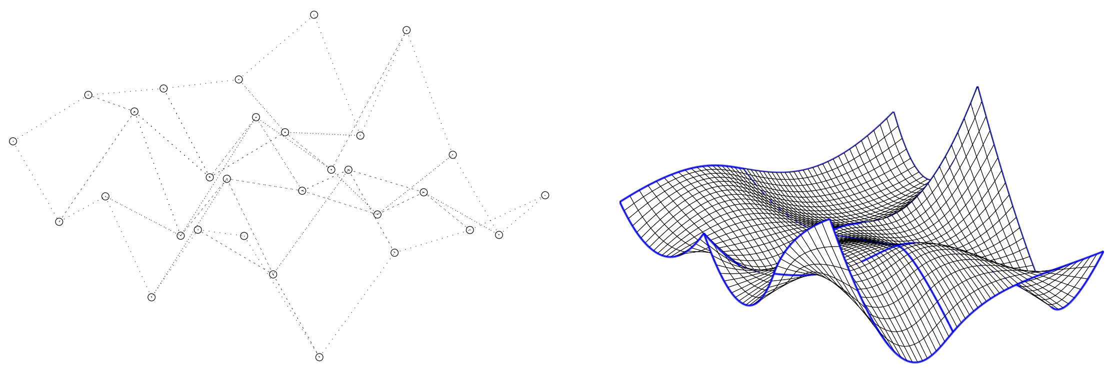


Figure 2: Control net (left) and NURBS surface with knot lines (right)

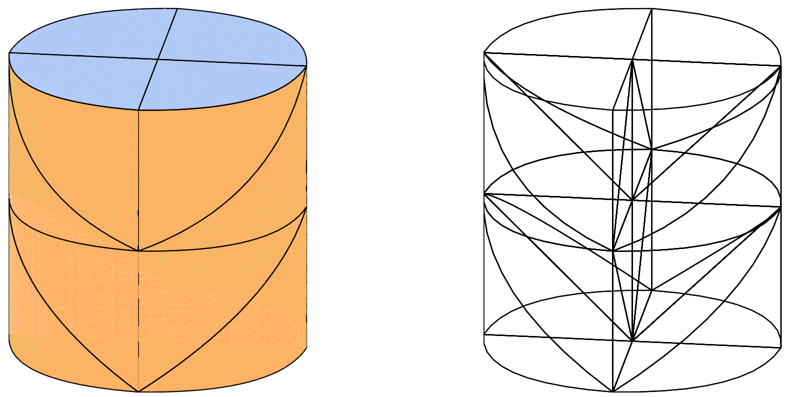
3 NURBS-ENHANCED FINITE ELEMENT METHOD (NEFEM)

Consider a physical domain $\Omega \subset \mathbb{R}^3$ whose boundary, $\partial\Omega$, or a portion of it is defined by NURBS surfaces. Every NURBS is assumed to be parametrized by

$$\boldsymbol{S} : [0,1]^2 \longrightarrow \boldsymbol{S}([0,1]^2) \subseteq \partial\Omega \subset \mathbb{R}^3.$$

A regular partition of the domain $\overline{\Omega} = \bigcup_e \overline{\Omega}_e$ in tetrahedrons is assumed such that every element Ω_e has at most one face on the NURBS boundary. Figure 3 shows a domain with part of the boundary described by a NURBS surface corresponding to a cylinder, and a valid tetrahedral mesh for NEFEM.

As usual in FE mesh generation codes, it is assumed that every curved boundary face belongs to a unique NURBS, $\Upsilon_e^F \subseteq \boldsymbol{S}([0,1]^2)$. That is, one element face can not be defined by portions of two (or more) different NURBS surfaces. But on the contrary, it is important to note that knot lines, which characterize the piecewise nature of NURBS, are independent of the mesh discretization. Thus, NURBS parametrization can change



(a) (b)

Figure 3: (a) Physical domain with part of the boundary defined by a NURBS surface and (b) NEFEM tetrahedral mesh

its definition inside one face, that is breakpoints may belong to element faces and do not need to coincide with FE nodes. This is another major advantage with respect to the isogeometric analysis [3].

Every element of a 3D NEFEM tetrahedral mesh has 0, 1 or 3 edges on the NURBS boundary. An element with no edges defined by NURBS has planar faces and is defined and treated as a standard FE or DG element. Therefore, in the vast majority of the domain, interpolation and numerical integration are standard. Specifical numerical strategies for the interpolation and the numerical integration are needed only for those elements affected by the NURBS description of the boundary.

An element with one edge on the NURBS boundary has two curved faces defined from the NURBS edge and an interior tetrahedral vertex, see Figure 4 (b). Finally, for an element with three edges on the NURBS, all his faces are curved: a boundary face defined by NURBS is denoted by Υ_e^F, and a curved interior face defined from a NURBS edge is denoted by Υ_e^E, see Figure 4 (a).

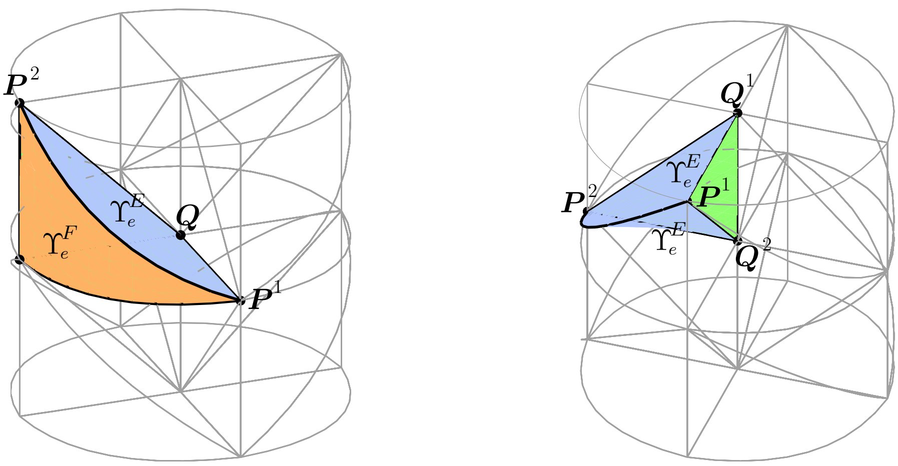
3.1 FE polynomial basis

For each element Ω_e, a tetrahedral T_e is defined using its vertices. A linear mapping $\mathbf{\Psi} : I \longrightarrow T_e$ is used, which goes from the reference tetrahedral I to the tetrahedral T_e. The inverse of this linear transformation maps the tetrahedral T_e into the reference tetrahedral I and, more important, also maps the actual element Ω_e, which is in the physical domain, into a curved element in local coordinates, namely

$$I_e := \mathbf{\Psi}^{-1}(\Omega_e).$$

I_e is called the *local curved element* for the actual element Ω_e.

Note that the reference tetrahedral I is the same for all elements Ω_e. However, the local curved element I_e depends on the trimmed NURBS defining the curved face Υ_e^F of



(a) (b)

Figure 4: (a) An element with a face on the NURBS boundary and (b) an element with an edge on the NURBS boundary. Curved faces on the NURBS boundary are represented in orange, curved faces defined from an edge on the NURBS boundary in blue and planar faces in green

Ω_e, and therefore it is different for every element Ω_e intersecting the NURBS boundary.

In order to work with standard FE polynomial approximations, Lagrange polynomials (that is, standard nodal interpolation) can be considered. In fact, they can be defined on the curved tetrahedral, I_e, in the reference domain or equivalently, in the actual element in the physical domain, Ω_e. The use of a linear transformation from the local (reference) coordinates $\boldsymbol{\xi} = (\xi, \eta)^T$ in I_e to the cartesian coordinates $\boldsymbol{x} = (x, y)^T$ in Ω_e, ensures that a complete polynomial interpolation of degree m in $\boldsymbol{\xi}$ leads to a polynomial interpolation with the same degree in $\boldsymbol{x}$. Thus, consistency and accuracy of the approximation is ensured even for elements Ω_e far from being a straight-sided element.

In order to make the computation of Lagrange polynomials, more systematic, for any degree and for any distribution of nodes, the implementation proposed in [7] is adopted, see [1] for more details.

3.2 Numerical integration

The weak form that must be solved requires both integrations along element faces and in the element interiors. All integrals in elements not having an edge along the NURBS boundary are computed using standard procedures. Elements Ω_e with one or three edges on the NURBS boundary require special attention. Two cases must be considered for both surface integrals (usually related to the implementation of natural boundary conditions or to flux evaluation along the face in a DG context) or volume integrals (standard integrals in the element Ω_e). As discussed in the previous section, since NEFEM uses polynomials to approximate the solution, the difficulties in numerical integration are only restricted to the definition of a proper numerical quadrature in a curved element, Ω_e, and a curved face.

3.2.1 Surface integrals

Two different integrals are studied separately: integrals on a curved face defined by NURBS, Υ_e^F, and integrals on a curved face defined by an edge on the NURBS boundary, Υ_e^E, see Figure 4 (a) and (b).

A surface integral to be computed on a curved boundary face Υ_e^F defined by a trimmed NURBS, can be written as

$$\int_{\Upsilon_e^F} f \, dx \, dy \, dz = \int_{\Lambda_e} f\big(\boldsymbol{S}(\lambda, \kappa)\big) \, \| J_{\boldsymbol{S}}(\lambda, \kappa) \| \, d\lambda \, d\kappa,$$

where f is a generic function (usually polynomial), Λ_e is a triangle (or quadrilateral) in the parametric space of the NURBS such that $\boldsymbol{S}(\Lambda_e) = \Upsilon_e^F$, and $\| J_{\boldsymbol{S}}(\lambda, \kappa) \|$ denotes the norm of the differential of the NURBS parametrization $\boldsymbol{S}$ (which, in general, is not a polynomial).

Evaluation of the previous integral requires to take into account the piecewise nature of the NURBS definition, because several changes of NURBS parametrization can occur inside Λ_e. A delaunay triangulation of Λ_e is considered to define a numerical quadrature on Υ_e^F. The triangulation is constructed from Λ_e vertexs, intersections of knot lines with Λ_e and intersections of knot lines inside Λ_e. Then, different numerical quadratures are considered at each *subelement* of the triangulation. In this work symmetric quadratures for triangles are considered, see [8].

On the other hand, surface integrals on interior curved faces must be computed when a DG formulation is adopted. An interior curved face Υ_e^E can be parametrized using the definition of the NURBS edge and the interior node $\boldsymbol{Q}$ of the tetrahedral element by

$$\begin{aligned} \boldsymbol{\Pi_Q} \ : [0,1]^2 &\longrightarrow \Upsilon_e^E \\ (\varrho, \sigma) &\longmapsto (1-\sigma)\boldsymbol{S}\big(\lambda(\varrho), \kappa(\varrho)\big) + \sigma \boldsymbol{Q}, \end{aligned}$$

where $\lambda(\varrho) = \lambda_1 + \varrho(\lambda_2 - \lambda_1)$ and $\kappa(\varrho) = \kappa_1 + \varrho(\kappa_2 - \kappa_1)$, with $\lambda_1, \lambda_2 \in \Lambda^q$ and $\kappa_1, \kappa_2 \in \Lambda^l$ such that $\boldsymbol{S}(\lambda_i, \kappa_i) = \boldsymbol{P}^i$, for $i = 1, 2$, i.e. $\boldsymbol{P}^{\mathbf{1}}$ and $\boldsymbol{P}^{\mathbf{2}}$ are the vertexs of the element on the NURBS boundary, see Figure 4 (b).

Then, the integral on the interior curved face can be written as

$$\int_{\Upsilon_e^E} f \, dx \, dy \, dz = \int_0^1 \int_0^1 f\big(\boldsymbol{\Pi_Q}(\varrho, \sigma)\big) \, \| J_{\boldsymbol{\Pi_Q}}(\varrho, \sigma) \| \, d\varrho \, d\sigma,$$

where f is a generic function (usually polynomial), and $\| J_{\boldsymbol{\Pi_Q}}(r, s) \|$ denotes the norm of the differential of the mapping $\boldsymbol{\Pi_Q}$ (which, in general, is not a polynomial). Again, the evaluation of the previous integral requires to take into account the piecewise nature of the NURBS parametrization. Several changes of NURBS definition can occur inside the boundary edge and composite quadratures must be considered for r direction.

3.2.2 Element integrals

NEFEM also requires to compute integrals in an element Ω_e with two or four curved faces, see Figure 4. Thus, a different numerical quadrature for every curved element Ω_e is needed.

For an element with one face on the NURBS boundary, one alternative is to define a transformation from the prism $\Lambda_e \times [0,1]$ to the curved element Ω_e, namely,

$$\begin{aligned} \boldsymbol{\varphi} \ : \ & \Lambda_e \times [0,1] \longrightarrow \Omega_e \\ & (\lambda, \kappa, \vartheta) \longmapsto \boldsymbol{\varphi}(\lambda, \kappa, \vartheta) = (1-\vartheta)\boldsymbol{S}(\lambda, \kappa) + \vartheta \boldsymbol{Q}. \end{aligned} \tag{3}$$

The numerical quadrature on $\Lambda_e \times [0,1]$ is defined from a cartesian product of the quadrature on Λ_e proposed in Section 3.2.2 and a Gauss-Legendre quadrature on the ϑ direction. Then, the element integral is computed as

$$\int_{\Omega_e} f \; dx\, dy\, dz = \int_{\Lambda_e} \int_0^1 f\big(\boldsymbol{\varphi}(\lambda, \kappa, \vartheta)\big) \; |J_{\boldsymbol{\varphi}}(\lambda, \kappa, \vartheta)| \; d\lambda \; d\kappa \; d\vartheta,$$

On the other hand, an element with a curved face not on the boundary, can be parametrized using the definition of the edge on the NURBS boundary and interior nodes, $\boldsymbol{Q}^1$ and $\boldsymbol{Q}^2$, of the element Ω_e

$$\begin{aligned} \boldsymbol{\phi} \ : \ & [0,1]^3 \longrightarrow \Omega_e \\ & (\varrho, \sigma, \tau) \longmapsto \boldsymbol{\phi}(\varrho, \sigma, \tau) = (1-\tau)\boldsymbol{\Pi}_{\boldsymbol{Q}^1}(\varrho, \sigma) + \tau \boldsymbol{Q}^2. \end{aligned} \tag{4}$$

Then, integral is evaluated using Gauss-Legendre quadratures in each direction. To account for the rational definition of the NURBS edge composite quadratures are considered in the parameter ϱ. Then, the integral is computed as,

$$\int_{\Omega_e} f \; dx\, dy\, dz = \int_0^1 \int_0^1 \int_0^1 f\big(\boldsymbol{\phi}(\varrho, \sigma, \tau)\big) \; |J_{\boldsymbol{\phi}}(\varrho, \sigma, \tau)| \; d\varrho \; d\sigma \; d\tau,$$

Note that parametrizations (3) and (4) are linear in the third parameter (ϑ and τ respectively). Then, exact integration can be carried out in this direction with $p+1$ integration points.

4 NUMERICAL EXAMPLE

The following model problem is solved in three dimensions

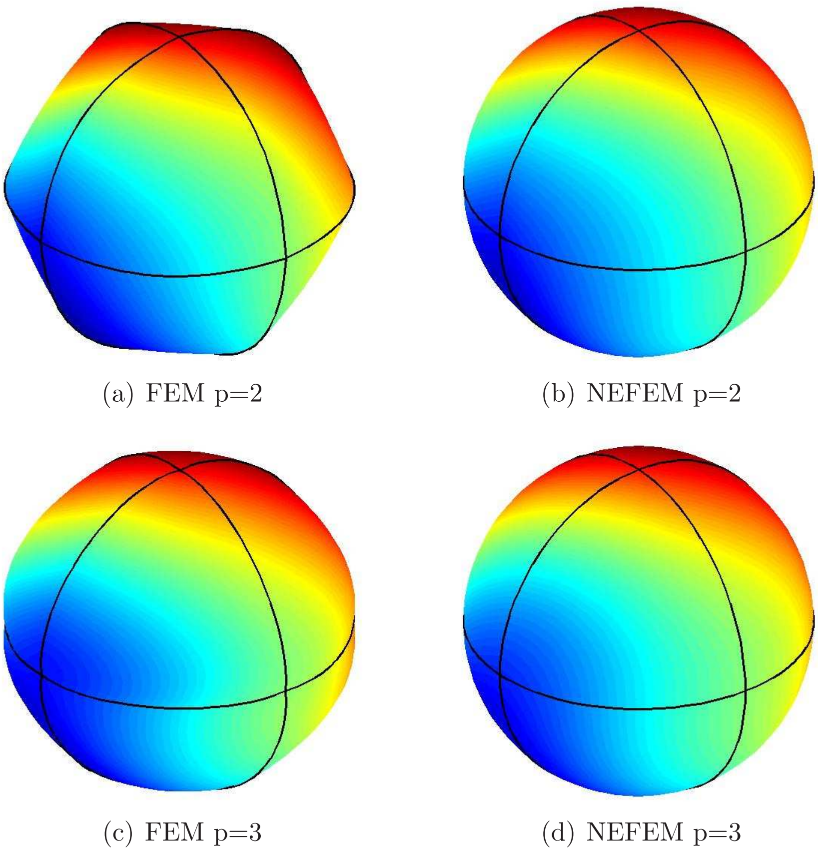
$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in} \quad \Omega \\ u & = & u_d & \text{on} \quad \Gamma_d \\ \nabla u \cdot \boldsymbol{n} & = & g_n & \text{on} \quad \Gamma_n \end{array} \right.$$

where Ω is an sphere of unit radius, $\overline{\Gamma}_d \cup \overline{\Gamma}_n = \partial\Omega$ and $\boldsymbol{n}$ is the outward unit normal vector on $\partial\Omega$. The source is given by $f(x, y, z) = x\cos(y) + y\sin(z) + z\cos(x)$, in such a way that the analytical solution of the problem is known and smooth,

$$u(x, y, z) = x\cos(y) + y\sin(z) + z\cos(x).$$

Dirichlet boundary conditions, corresponding to the analytical solution, and Neumann boundary conditions, also corresponding to the analytical normal flux, are imposed on $\Gamma_d = \partial\Omega \cap \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 0\}$ and $\Gamma_n = \partial\Omega \cap \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ respectively.

FEM and NEFEM solutions in a mesh with only eight elements are represented in Figure 5 for quadratic and cubic approximation. It is important to note that not only the solution is captured with lower accuracy with isoparametric FE but also geometric errors are observable.



(a) FEM p=2 (b) NEFEM p=2

(c) FEM p=3 (d) NEFEM p=3

Figure 5: Surface plot of FEM and NEFEM solutions

5 CONCLUDING REMARKS

The 3D extension of NEFEM is studied in this work. Although the extension is conceptually easy, special attention must be paid to geometrical aspects. Specifically designed polynomial interpolation and numerical integration is needed for elements affected by the NURBS description of the boundary. Several strategies are presented to perform the numerical integration on curved elements.

Finally, a Poisson example is solved to demonstrate the applicability of the proposed method in a continuous Galerkin framework.

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