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SHOCK-CAPTURING WITH DISCONTINUOUS GALERKIN METHODS

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Abstract. *A shock capturing strategy for high order Discontinuous Galerkin methods for conservation laws is proposed. We present a method in the one-dimensional case based on the introduction of artificial viscosity into the original equations. With this approach the shock is capture with sharp resolution maintaining high-order accuracy. The ideas for the extension to the two-dimensional case are also set.*

1 INTRODUCTION

In the last decades high-order discontinuous Galerkin (DG) methods have been the center of many studies to deal with non-linear conservation laws and convection-dominated problems. This kind of problems may develop strong shocks and jump discontinuities in finite time even when the initial condition is smooth. Despite DG methods exhibit an inherent stability at discontinuities, it is not only sufficient to stabilize the solution in the presence of shocks and discrete solutions might suffer from spurious oscillations near the discontinuities.

The most straightforward approach consists on avoiding the presence of sharp gradients with some non-linear projection operators, namely slope limiters, developed by Cockburn and Shu in a series of papers [2, 3, 5]. When a DG solution is limited, most methods reduce the solution to first-order accuracy, and much of the advantage of high-order methods is lost. In [1] Biswas et al. tried to improve this deficiency and generalize Cockburn limiter, but it still reduces the order of the interpolation. Despite this, in the one-dimensional case slope limiting techniques in combination with h-refinement procedures seem to be the natural solution.

The adaptation of the one-dimensional slope limiting techniques to the multidimensional case raises numerous problems, among them the problem of stability. In the one-dimensional case the role of the slope limiter $\Lambda\Pi_h$ is to enforce the TVDM or the TVBM property. In the case of multi-dimensional problems neither of these desirable properties have been proven for monotone schemes on general meshes. Instead, a local maximum principle for the RKDG methods is enforced. Since now, no other class of schemes have proven a maximum principle for non-linearities on arbitrary triangulations. Other extensions of the one-dimensional case which are more straightforward are those ones which use rectangular grids and tensor product basis of the one-dimensional basis functions. Hence, it is relatively simply to apply the one-dimensional projection limiter along each of the two spatial direcctions. This approach can be found in [5, 8].

Nowadays, more sophisticated techniques, ENO and WENO schemes are proliferating in the context of DG methods, see for instance [10]. These methods guarantee high-order reconstructions but they need a wide stencil and increase too much the computational cost. They also require the use of structured grids. For general unstructured meshes shock-capturing with high-order DG methods is still an open problem.

Another classical way to avoid spurious oscillations is the introduction of artificial viscosity into the original equation. This earliest method was first proposed by VonNeumann and Richtmyer in the 50's. The main difficulty is the determination of the amount of artificial viscosity required: on one hand an excessive amount of diffusion will smear the solution too much, thus obtaining a non-realistic approximation and, on the other hand, if not enough viscosity is introduced the numerical solution will suffer from oscillations. Recently, Persson and Peraire [9] have proposed an artificial viscosity term based on the element size h and the polynomial interpolation degree p. It is used in conjuction with a discontinuity detection strategy and the obtained viscosity is typically or order h/p.

Our method is also based in the introduction of artificial viscosity in the original equation. Because of the proliferation of reconstruction techniques, the amount of artificial diffusion is computed using the main ideas of the slope limiters, but in contrast with them, the proposed method maintains the high-order of the approximation. The obtained value for the viscosity scales like $\varepsilon \sim \mathcal{O}(h^k)$, which in general is smaller than h/p, obtaining then a sharp shock profile and achieving high accuracy.

In our previous work [7] we first provide a way to compute a constant amount of artificial diffusion to introduce in each element. Here the shock detection algorithm defined in [9] is used as a discontinuity sensor. Second, we also propose a performance of this discontinuity sensor with the aim of detecting the shock more precisely also with coarse meshes. Instead of detecting the shock elementwise, it is capable to detect it with subcell resolution.

The introduction of the artificial viscosity term requires the discretization of second order derivatives with DG methods. Several methods have been proposed to deal with elliptic operators. Here the Local Discontinuous Galerkin (LDG) approach of Cockburn and Shu presented in [4] is used.

The adaptation of the method to multidimensional unstructured problems is less straightforward than one may think, specially because x and y are coupled in the expansion of the solution. In this work we propose an extension of the one-dimensional method based on the computation of the artificial viscosity for each face of the triangle. This approach arises several difficulties, specially concerning to the direction associated to each face. Decoupling the directions in the expansion of the solution will be the focus of our research to extend the method.

The outline of the remainder paper is as follows. In section 2 a scope of the LDG method of Cockburn and Shu is presented. Then, in section 3 the necessary background is reviewed, namely, the basic ideas of the one-dimensional method. Numerical results illustrate its good behaviour. Section 4 presents an outline of the ideas and the discussion for the extension to the two-dimensional case.

2 The LDG method

Consider a scalar hyperbolic conservation law of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = 0 \tag{1}$$

The idea behind the artificial viscosity approach is to add a dissipative model term to the original equations of the form

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F}(u) = \nabla \cdot (\varepsilon \nabla u) \tag{2}$$

Here, the parameter ε models the artificial diffusion term and it is assumed to be constant for each element.

The DG discretization of equation (2) is carried out with the Local Discontinuous Galerkin approach described in [4]. In order to apply the LDG method, equation (2) is written as a system of first order hyperbolic equations by introducing the auxiliary variables σ:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{F} - \nabla \cdot \sigma = 0 \tag{3a}$$

$$\sigma - \varepsilon \nabla u = 0 \tag{3b}$$

In the original paper [4] the details of the method are explained.

3 One-dimensional shock-capturing method

In [7] a one-dimensional shock-capturing method for conservation laws is proposed. The computation of the amount of artificial viscosity to introduce at each time step within each element is performed combining the ideas of the slope limiters and the shock-capturing methods. The total amount of artificial viscosity scales like $\varepsilon \sim \mathcal{O}(h^k)$ for some $k \geq 1$, which is typically smaller than h/p, thus obtaining a sharp shock profile. In this section a brief review of the one-dimensional method is presented.

3.1 Artificial diffusion

Consider the interval $I_j =]x_j, x_{j+1}[$ of Ω. In order to simplify the method we write the solution of degree p within each element I_j in terms of a hierarchical family of orthogonal polynomials, the Legendre polynomials:

$$u(x) = \sum_{i=0}^{p} c_i P_i(x) \tag{4}$$

The strategy consists on using the weak formulation to deduce the shock-capturing term. Multiplying (2) by a test function δv, integrating over the element I_j and rearranging terms the next equation is obtained:

$$\int_{I_j} u_t(\delta v)\, dx - \int_{I_j} f(u)(\delta v)_x\, dx + \Big[\hat{F}_{\rm n}(\delta v)\Big]_{x_j}^{x_{j+1}} + \int_{I_j} \varepsilon u_x(\delta v)_x\, dx = 0 \tag{5}$$

where the approximate Riemann solver $\hat{F}_{\rm n} = \hat{F}(u){\rm n}_j$ has been introduced. Here ${\rm n}_j$ denotes the outward unit normal to node x_j. Notice that in 5 the shock capturing term is explicitly obtained.

With the aim of introducing the effect of the slope limiters into the scheme, the method is splitted in two steps: the convection-diffusion operator splits into a sum of two components, one containing the convection operator and the other with the diffusion one (see chapter 5 in [6]). The link between the slope limiters and the shock-capturing scheme can be easily established imposing that the effect of the limited function must be the same that the effect of adding some amount of artificial diffusion to the solution of the hyperbolic problem, i.e:

$$\int_{I_j} \Lambda\Pi_h(v^{n+1})(\delta v)\, dx = \int_{I_j} v^{n+1}(\delta v)\, dx + \Delta t\Big[\int_{I_j} (\delta v)_x \varepsilon v_x^{n+1}\, dx\Big] \tag{6}$$

where, $\Lambda\Pi_h(v^{n+1})$ is the reconstructed solution with the moment limiter defined in [1], which is obtained after several projections. v^{n+1} is the not reconstructed solution of the hyperbolic problem. It is assumed that viscosity goes to zero on the boundaries.

The next value of artificial viscosity is obtained:

$$\varepsilon_{I_j}^n = \frac{\displaystyle\int_{I_j} \Big(\Lambda\Pi_h(v^{n+1}) - v^{n+1}\Big)(\delta v)\, dx}{\displaystyle\Delta t\Big[\int_{I_j} (\delta v)_x v_x^{n+1}\, dx\Big]} \tag{7}$$

Remark 1 *Notice that although of being constant within each element, the artificial diffusion introduced is nonlinear, depending on the solution u at each time step:* $\varepsilon^n = \varepsilon(u^n)$.

Because of the adaptive procedure of the limitation algorithm, an amount of viscosities is obtained: $\{\varepsilon_i\}_{i=1,\ldots,p}$, each one corresponding to the limited moments in the approximate solution (4). The following simplification is assumed: having all them computed (which is not a hard task due to the orthogonality and hierarchy of Legendre polynomials), the maximum of all the viscosities is chosen and it is introduced in the weak form of (3b). Then, the method is fully defined.

This choice corresponds to the last of the viscosities computed, which is the associated one to the lowest-order degree of freedom and also it is related to the i^{th} derivative of the solution (for details see [1, 7]):

$$\varepsilon_{I_j} = \max_{\{i|\tilde{c}_i \neq c_i\}} \{\varepsilon_i\} = \varepsilon_{\min\{i|\tilde{c}_i \neq c_i\}} \tag{8}$$

With the choice (8) we only need to compute a single value ε at each time step, thus reducing the computational cost of the procedure and also simplifying the algorithm. As it is shown in the numerical tests it is enough to maintain the solution stable and avoid oscillations.

The introduced diffusion is typically small: it scales like h^k, where k depends on the last limited coefficient. In the worst of cases $k = 1$ which equals the effect of slope limiters. However, numerical tests show that this case is rarely achieved and a sharp shock profile, is obtained.

3.2 Shock detection

The artificial viscosity is introduced with the aim of avoiding oscillations near discontinuities, therefore in smooth regions it is unnecessary. In order to identify “troubled cells”, namely those elements where discontinuities or sharp gradients are present, the discontinuity detector proposed in [9] for each element is used:

$$S_{I_j} = \frac{\displaystyle\int_{I_j} |u - \hat{u}|^2 \, dx}{\displaystyle\int_{I_j} |u|^2 \, dx} \tag{9}$$

where $\hat{u}$ represents the approximation of order $p-1$ and u the approximation of order p. The discontinuity detector is sustented on the rate of decay of the expansion coefficients in the polynomial approximation (4).

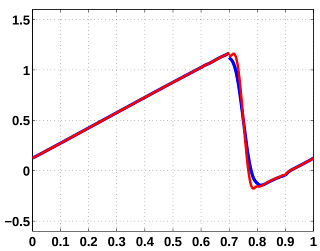
Despite numerical tests confirm that S_{I_j} is a remarkably reliable indicator, the problem remains when coarse meshes are used: for a sharp shock the element by element detecting procedure doesn't provide enough information and smooth parts of the solution can be identified as discontinuities. In these cases the introduction of constant artificial viscosity within each element can flatten the solution too much. The default and obvious solution is the local refinement of these elements.

In order to avoid adaptivity procedures and obtain shock detection with subcell resolution, a performance of the discontinuity sensor based on the introduction of an auxiliar submesh is proposed. With this new approach the shock can be detected within a small region of the element, typically of size h/p, thus allowing us to introduce some form of localized viscosity within each element and to use coarser meshes.

3.3 Numerical tests

3.3.1 Burgers equation

The inviscid Burgers' equation with initial condition $u(x) = 1/2+sin(2\pi x)$ and periodic boundary conditions on the interval $[0, 1]$ is tested. In figure (1) results at time $T = 0.5$ are shown and compared with those ones obtained introducing a diffusion of order h/p, as Persson and Peraire in [9] suggest. With our method the shock is accurately detected and the amount of required viscosity is at least one order smaller.



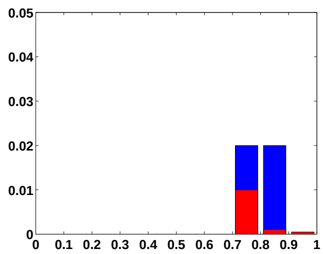


Figure 1: Solution of the Burgers equation adding artificial diffusion: in blue diffusion of order (h/p), and in red diffusion of order (h^k).

3.3.2 A system of conservation laws: Euler equations

We now consider the system of Euler equations. The selected case is the Sod's shock problem (see for instance chapter 4 in [6]). The diffusion is computed for each of the characteristic variables with the described method. Then it is mapped back into the conserved variables space by multiplication with the matrix of right eigenvectors, thus obtaining a full tensor viscosity matrix which also couples all the variables. Entropy has been used as a sensing variable. Results at time $T = 0.2$ are shown in figure 2. The different discontinuities and also the rarefaction waves are well-captured. The introduced viscosity avoids significant overshoots, and we obtain a sharp but smooth profile of the solution for each variable.

4 The extension to 2D case

A great deal of effort has been oriented to the construction of numerical schemes for the multidimensional case which might be able to eliminate non-physical oscillations without

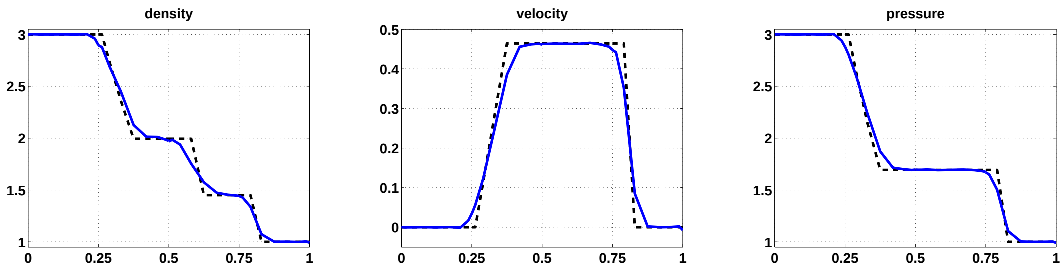


Figure 2: Computed Euler solutions profiles of the outputs of interest with non-constant artificial viscosity matrix using a mesh of 4 elements and an approximation of degree 8.

adding excessive numerical viscosity. In this section we first show an illustrative case using the actual slope limiting techniques Then we introduce the main ideas to extend our one-dimensional artificial viscosity method for triangular meshes.

4.1 Actual techniques: an illustrating example

An extension of the generalized slope limiter $\Lambda\Pi_h$ of Cockburn and Shu has been made by the same authors in [3, 5] to the case of triangular meshes for linear approximations. Unfortunately, it also becomes a second order method and requires regularity restrictions on the computational mesh.

Consider the two-dimensional version of Burger's equation with periodic boundary conditions:

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^2}{2}\right)_y = 0 \quad \text{in } [-1,1]\times[-1,1] \tag{10a}$$

$$u(x,y,0) = \frac{1}{4} + \frac{1}{2} sin(\pi(x+y)) \tag{10b}$$

In figure 3 we reproduce the numerical results of [3] using a mesh of 100 triangular elements. Spurious oscillations are avoided but the limited solution loses accuracy. As the method is based on linear approximations, it gives rise to a lack of information for high-order DG methods. This fact is much more noticeable when increasing the interpolation degree.

4.2 Extension of the one-dimensional artificial viscosity method

The objective is to compute one single viscosity ε_j in each face of the triangle Γ_j, for $j = 1, 2, 3$ using the one-dimensional method.

For this purpose it is necessary to choose a direction associated to every face and reduce to one-dimensional approximations the approximation in the triangle

$$u(x,y) = \sum_{i=1}^{N(p)} N_i(x,y)u_i \tag{11}$$

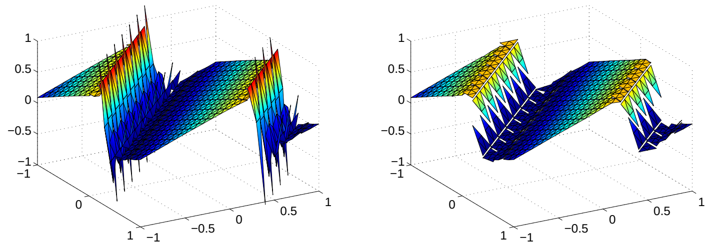


Figure 3: Approximation of degree 2 at time $T = 0.6$. On the left without limiting and on the right with Cockburn limiter. In the elements where reconstruction is necessary the approximation is reduced to a linear one.

where $N(p)$ is the number of degrees of freedom. Then, the method described in section 3 is applied to each of the one-dimensional approximations.

In contrast with the case of rectangular grids, in which each face of the element is associated with a cartesian direction, with triangular meshes it is necessary to define another system of reference. The barycentric coordinates seem to be the more natural choice for the triangle. In figure 4 an schematic plot of the barycentric coordinates is showed.

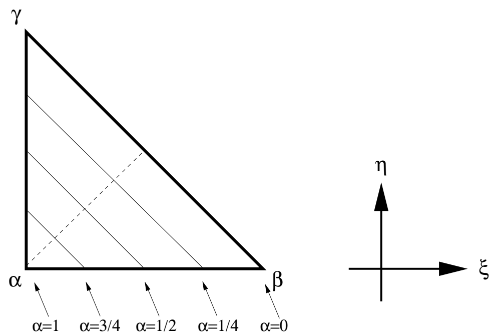


Figure 4: Barycentric coordinates in the reference triangle.

The following expression relates cartesian coordinates (ξ, η) with the barycentric ones (α, β, γ):

$$\alpha = 1 - \xi - \eta \qquad \beta = \xi \qquad \gamma = \eta \tag{12}$$

satisfying the relation $\alpha+\beta+\gamma = 1$. A point P of the triangle is defined by the coordinates $P = P(\alpha, \beta, \gamma)$.

With the barycentric coordinates the most straightforward choice for the direction associated to each face is the following: for the opposite face of coordinate α choose the direction determined by vector joining vertex α with the midpoint of the opposite face, namely m_α (see figure 4). Analogously for the opposite faces of coordinates β and γ.

Remark 2 *Notice that for the reference triangle the direction $\overrightarrow{\alpha m_\alpha}$ coincides with the α-direction of its neighboring element. For general unstructured meshes this won't be the case and we suspect that other directions might be used in order to apply the one-dimensional model.*

The one-dimensional approximation associated to the coordinate α, namely $u_\alpha(\alpha)$, is given by

$$u_\alpha(\alpha) = u\Big(\frac{1-\alpha}{2}, \frac{1-\alpha}{2}\Big) \tag{13}$$

Analogously the approximation associated directions to β and γ are given by:

$$u_\beta(\beta) = u\Big(\beta, \frac{1-\beta}{2}\Big) \qquad u_\gamma(\gamma) = u\Big(\frac{1-\gamma}{2}, \gamma\Big) \tag{14}$$

Now, expressions (13) and (14) are one-dimensional polynomials of degree p, which can be expaned in terms of Legendre basis:

$$u_\alpha(\alpha) = \sum_{i=0}^{p} a_i P_i(\alpha) \qquad u_\beta(\beta) = \sum_{i=0}^{p} b_i P_i(\beta) \qquad u_\gamma(\gamma) = \sum_{i=0}^{p} d_i P_i(\gamma) \tag{15}$$

where a_i, b_i, d_i are linear combinations of the degrees of freedom of approximation (11). The one-dimensional method described in [7] can be applied to equations (15) to obtain ε_j, for $j = 1, 2, 3$.

Due to the fact that the computed viscosities ε_j are of oder $h^k \quad (k \geq 1)$ we expect that this extension will preserve the high-oder oaccuracy.

Remark 3 *The computation of expressions (15) is not an easy task. The expression of the coefficients a_i, b_i, d_i depends on the polynomial degree p of the approximation (11) in the triangle and they must precomputed for each particular case. We believe that this will be one the hardest tasks in the extension of the method.*

5 CONCLUSIONS

We have presented a practical approach to shock-capturing with high-order DG methods which maintains high-order accuracy not only in smooth regions but also in shock regions. The method was developed in one space dimension and the ideas to extend it to two dimensions for unstructured triangular meshes are set here. We expect to obtain some preliminar results in posterior works.

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