SOLUCIONS ALS PROBLEMES PROPOSATS AL VOLUM 23 N. 2

PROBLEMA N. 76

1. De $\Pi_n = \dfrac{\lambda_0\,\lambda_1\,\cdots\,\lambda_{n-1}}{\mu_1\,\mu_2\,\cdots\,\mu_n}\,\Pi_0$, al ser $\lambda_n = \lambda$, $\mu_n = \mu$, $\forall n \;\Rightarrow$

$$\Pi_n = \Psi^n\,\Pi_0,\; \psi = \frac{\lambda}{\mu}$$

$$1 = \sum_{n=0}^{\infty} \Pi_n =^{1} \frac{1}{1-\psi}\,\Pi_0 \;\Rightarrow\; \Pi_0 = 1-\psi \;\Rightarrow\; \boxed{\Pi_n = \psi^n(1-\psi)\;\forall n}$$

2. La función generatriz de probabilidades resulta:

$$G(s) = \sum_{n_0}^{\infty} s^n\,\psi^n(1-\psi) = (1-\psi)\sum_{n=0}^{\infty}(s\,\psi)^n = (1-\psi)(1-s\,\psi)^{-1}$$

El número medio de clientes en el sistema será por tanto:

$$L = G'(s)\bigg|_{s=1} = -\psi(1-\psi)(1-s\,\psi)^{-2}\bigg|_{s=1} = \frac{\psi}{1-\psi}$$

El número medio de *estaciones desocupadas* será: $\emptyset = 1\times\Pi_0 = 1-\psi$

$$L = L_q + s - \emptyset \;\Rightarrow$$

El número medio de clientes esperando: $L_q = \dfrac{\psi}{1-\psi} - 1 + (1-\psi) = \dfrac{\psi^2}{1-\psi}$

3. $P(X \leqslant n) = \displaystyle\sum_{i=0}^{n} \psi^n(1-\psi) = (1-\psi)\,\frac{1-\psi^{n+1}}{1-\psi} = 1-\psi^{n+1}$

La probabilidad de espera: $P(X > s=1) = 1-\Pi_0 = 1-(1-\psi) = \psi$.

El tiempo medio de espera: $T_q = \dfrac{L_q}{\lambda} = \dfrac{\psi}{\mu-\lambda}$

[1] La existencia de régimen permanente supone que $\psi < 1$.

4. Como $\psi < 1 \Rightarrow \Pi_0$ es la mayor probabilidad $\equiv n = 0$ es el estado más probable.

5. $\Pi'_n(\psi) = -\psi^n + (1-\psi)n\,\psi^{n-1}$;

   $\Pi'(\psi) = 0 \Rightarrow \psi = (1-\psi)n \Rightarrow 1/n = 1/\psi - 1 \Rightarrow$

$$1/n + 1 = \frac{1+n}{n} = 1/\psi \Rightarrow \boxed{\psi = \frac{n}{n+1}} \qquad \begin{array}{l} n=1,\ \psi = 1/2 = 3/6 \\ n=2,\ \psi = 2/3 = 4/6 > 3/6 \end{array}$$

$$\max_{\psi} \Pi_n(\psi) = \left(\frac{n}{n+1}\right)^n \overbrace{\left(1 - \frac{n}{n+1}\right)}^{1/n+1} = \frac{n^n}{(n+1)^{n+1}} = \begin{cases} 1/4 & n=1 \\ 4/9 & n=2 \end{cases}$$

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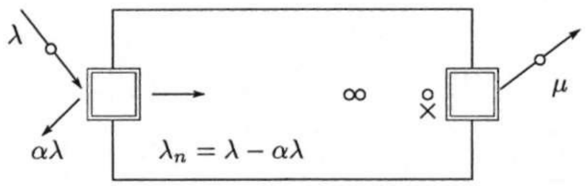
PROBLEMA N. 77

a) Las tasas de *entrada* son ahora $\lambda_0 = \lambda$; $\lambda_n = \lambda(1-\alpha), n \geqslant 1$.



1. $\Pi_1 = \frac{\lambda}{\mu}\Pi_0 = \psi\,\Pi_0$, $\Pi_2 = \frac{\lambda\lambda(1-\alpha)}{\mu\mu}\Pi_0 = \psi^2(1-\alpha)\,\Pi_0, \ldots, \Pi_n =$ $= \psi(1-\alpha)\,\Pi_{n-1} = \psi^n(1-\alpha)^{n-1}\,\Pi_0$ **Nota**: $\psi(1-\alpha) < 1$.

$$1 = \Pi_0 + \sum_{n=1}^{\infty}\Pi_n = \Pi_0 + \frac{\psi\,\Pi_0}{1-\psi(1-\alpha)} = \frac{1+\alpha\,\psi}{1-\psi(1-\alpha)}\Pi_0 \Rightarrow$$

$$\Pi_0 = \frac{1-(1-\alpha)\psi}{1+\alpha\,\psi} \Rightarrow$$

$$\boxed{\Pi_n = \psi^n(1-\alpha)^{n-1}\,\Pi_0 = \psi^n(1-\alpha)^{n-1}\,\frac{1-(1-\alpha)\psi}{1+\alpha\,\psi}\ \forall\, n > 0} \underset{\alpha=0}{=} \psi^n(1-\psi)$$

2. $G(s) = \Pi_0 + \sum_{n=1}^{\infty} s^n\left(\psi^n(1-\alpha)^{n-1}\right)\Pi_0 = \Pi_0\left(1 + s\psi\sum_{n=1}^{\infty}(s\psi(1-\alpha))^{n-1}\right) =$ $\Pi_0\left(1 + s\psi\frac{1}{1-s\psi(1-\alpha)}\right)$

$$G'(s) = \Pi_0\left(\psi\frac{1-s\psi(1-\alpha)+s(\psi(1-\alpha))}{(1-s\psi(1-\alpha))^2}\right) \Rightarrow$$

$$G'(1) = \frac{1(1-\alpha)\psi}{1+\alpha\psi}\left(\psi\frac{1}{(1-\psi(1-\alpha))^2}\right) \Rightarrow$$

$$L = \frac{\psi}{(1+\alpha\psi)(1-\psi(1-\alpha))} = \begin{cases} \dfrac{\psi}{1-\psi} & \alpha = 0 \\ \dfrac{\psi}{1+\psi} & \alpha = 1 \end{cases}$$

$\emptyset = \Pi_0$

$$L_q = L - s + \emptyset = L - (1 - \Pi_0) = \frac{\psi}{(1+\alpha\psi)(1-\psi(1-\alpha))} - \frac{\psi}{1+\alpha\psi} =$$

$$\frac{\psi}{1+\alpha\psi}\left(\frac{\psi(1-\alpha)}{(1-\psi(1-\alpha))}\right) \underset{\alpha=0}{=} \frac{\psi^2}{1-\psi}$$

3. $P(X > s = 1) = 1 - \Pi_0 = 1 - \dfrac{1-(1-\alpha)\psi}{1+\alpha\psi} = \dfrac{\psi}{1+\alpha\psi} \underset{\alpha=0}{=} \psi$

$$\overline{\lambda} = \lambda\,\Pi_0 + \sum_{n=1}^{\infty} \lambda(1-\alpha)\Pi_n = \lambda\left(\Pi_0 + (1-\alpha)\sum_{n=1}^{\infty}\Pi_n\right) =$$

$$= \lambda\,(\Pi_0 + (1-\alpha)(1-\Pi_0)) = \lambda\,(1 - \alpha(1-\Pi_0)) =$$

$$= \lambda\left(1 - \alpha\frac{\psi}{1+\alpha\,\psi}\right) = \frac{\lambda}{1+\alpha\,\psi}$$

$$T_q = \frac{L_q}{\overline{\lambda}} = \frac{1}{\lambda}\frac{\psi^2(1-\alpha)}{(1-\psi(1-\alpha))} \underset{\alpha=0}{=} \frac{\lambda}{\mu-\lambda}$$

b. Si $\alpha = 1$ el rechazo a entrar cuando la unidad está ocupada es total:

$$\lambda_n = \begin{cases} \lambda & n = 0 \\ 0 & n \geqslant 1 \end{cases} \Rightarrow \Pi_1 = \frac{\lambda}{\mu}\Pi_0,\ \Pi_n = \frac{0}{\mu}\Pi_0 = 0\ \forall\, n > 1 \Rightarrow$$

$$\Psi\Pi_0 + \Pi_0 = 1 \Rightarrow \boxed{\Pi_0 = \frac{1}{1+\psi},\ \Pi_1 = \frac{\psi}{1+\psi}} \quad \boxed{\emptyset = 1\,\Pi_0 = \frac{1}{1+\psi}}$$

$$\boxed{L = 0\,\Pi_0 + 1\,\Pi_1 = \Pi_1 = \frac{\psi}{1+\psi}} = \underset{0}{L_q} + \underset{1}{s} - \underset{\frac{1}{1+\psi}}{\emptyset}$$

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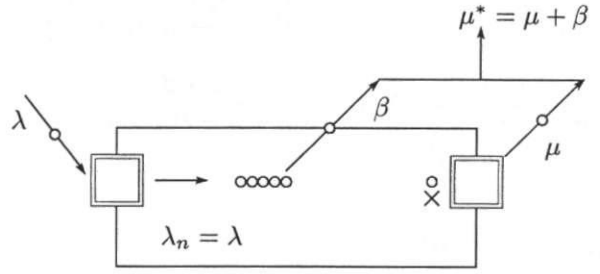
PROBLEMA N. 78



1. $\Pi_n = \dfrac{\lambda^n}{\mu(\mu+\beta)^{n-1}} \Pi_0 \; \forall \, n > 1$

$$\left(1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu(\mu+\beta)^{n-1}}\right) \Pi_0 = 1 \Rightarrow \Pi_0 = \frac{\mu(\mu+\beta-\lambda)}{\mu^2 + (\lambda+\mu)\beta}$$

$$\Pi_0^{-1} = 1 + \frac{\lambda}{\mu} \frac{1}{1 - \dfrac{\lambda}{\mu+\beta}} = 1 + \frac{\lambda}{\mu} \frac{\mu+\beta}{\mu+\beta-\lambda} = \frac{\mu^2 + (\lambda+\mu)\beta}{\mu(\mu+\beta-\lambda)}$$

$$G(s) = \Pi_0 + \sum_{n=1}^{\infty} s^n \frac{\lambda^n}{\mu(\mu+\beta)^{n-1}} \Pi_0 = \Pi_0 \left\{1 + s\psi \sum_{n=1}^{\infty} \left(\frac{s\lambda}{\mu+\beta}\right)^{n-1}\right\} =$$

$$= \Pi_0 \left\{1 + s\psi \frac{1}{1 - \dfrac{s\lambda}{\mu+\beta}}\right\} = \Pi_0 \left\{1 + \psi(\mu+\beta) \frac{s}{\mu+\beta-s\lambda}\right\} \Rightarrow$$

$$G'(s) = \Pi_0 \left(\psi(\mu+\beta) \frac{(\mu+\beta-s\lambda) - s(-\lambda)}{(\mu+\beta-s\lambda)^2}\right)$$

2. $L = G'(1) = \dfrac{\mu(\mu+\beta-\lambda)}{\mu^2+(\lambda+\mu)\beta} \left(\dfrac{\lambda}{\mu}/(\mu+\beta) \dfrac{\mu+\beta}{(\mu+\beta-\lambda)^2}\right) =$

$$= \frac{\lambda}{\mu^2 + (\lambda+\mu)\beta} \frac{(\mu+\beta)^2}{\mu+\beta-\lambda} \underset{\beta=0}{=} \frac{\psi}{1-\psi}$$

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PROBLEMA N. 79

a) $\Pi_n = \dfrac{\lambda(\lambda(1-\alpha))^{n-1}}{\mu(\mu+\alpha)^{n-1}}\,\Pi_0 \qquad \left(1+\sum_{n=1}^{\infty}\dfrac{\lambda}{\mu}\dfrac{(\lambda(1-\alpha)^{n-1}}{(\mu+\beta)^{n-1}}\right)\Pi_0 = 1$

$$\Pi_0^{-1} = 1+\frac{\lambda}{\mu}\frac{1}{1-\dfrac{\lambda(1-\alpha)}{\mu+\beta}} = 1+\frac{\lambda}{\mu}\frac{\mu+\beta}{\mu+\beta-\lambda(1-\alpha)} =$$

$$= \frac{\mu^2+(\lambda+\mu)\beta+\alpha\lambda\mu}{\mu(\mu+\beta-(1-\alpha)\lambda)}$$

b) $G(s) = \Pi_0 + \sum_{n=1}^{\infty} s^n\left(\psi\left(\dfrac{\lambda(1-\alpha)}{\mu+\beta}\right)^{n-1}\right)\Pi_0 =$

$$= \Pi_0\left(1+s\psi\sum_{n=1}^{\infty}\left(s\frac{\lambda(1-\alpha)}{\mu+\beta}\right)^{n-1}\right) =$$

$$= \Pi_0\left(1+s\psi\frac{1}{1-s\dfrac{(1-\alpha)\lambda}{\mu+\beta}}\right) = \Pi_0\left(1+s\psi\frac{\mu+\beta}{\mu+\beta-s\lambda(1-\alpha)}\right)$$

$$G'(s) = \Pi_0\left(\psi(\mu+\beta)\frac{\mu+\beta-s\lambda(1-\alpha)+s\lambda(1-\alpha)}{(\mu+\beta-s\lambda(1-\alpha))^2}\right)$$

$$L = G'(1) = \frac{\mu(\mu+\beta-(1-\alpha)\lambda)}{\mu^2+(\lambda+\mu)\beta+\alpha\lambda\mu}\left(\frac{\lambda}{\mu}(\mu+\beta)\frac{\mu+\beta}{(\mu+\beta-\lambda(1-\alpha))^2}\right) =$$

$$= \frac{\lambda}{\mu^2+(\lambda+\mu)\beta+\alpha\lambda\mu}\left(\frac{(\mu+\beta)^2}{(\mu+\beta-\lambda(1-\alpha))}\right) =$$

$$= \begin{cases} \dfrac{\lambda}{\mu^2+(\lambda+\mu)\beta}\left(\dfrac{(\mu-\beta)^2}{\mu+\beta-\lambda}\right), & \alpha = 0 \\ \dfrac{\psi}{(1+\alpha\psi)(1-\psi(1-\alpha))}, & \beta = 0 \end{cases}$$

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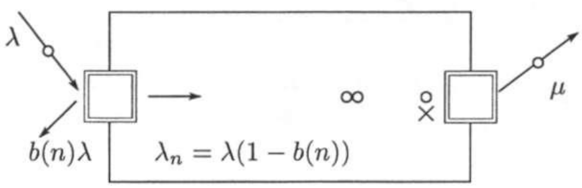
PROBLEMA N. 80

Las tasas de *entrada* son ahora: $\lambda_0 = \lambda$; $\lambda_n = \lambda(1-\alpha).n \geqslant 1$.



$$\text{a)}\ \lambda_n = \left(1 - \frac{n}{1+n}\right)\lambda = \frac{1}{1+n}\lambda \Rightarrow \Pi_n = \frac{\frac{1}{1}\lambda\frac{1}{2}\lambda\cdots\frac{1}{n}\lambda}{\mu\,\mu\cdots\mu}\Pi_0 =$$

$$= \frac{\psi^n}{n!}\Pi_0 \Rightarrow \Pi_0 = e^{-\psi} \Rightarrow \Pi_n = \frac{\psi^n}{n!}e^{-\psi} \equiv X \in \mathcal{P}(\psi)$$

b) $L = E(X) = \psi \quad L_q = L - s + \emptyset = L - (1 - \Pi_0) = \psi - 1 + e^{-\psi}$

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PROBLEMA N. 81

a) $\Pi_1 = 1\,\Pi_0,\ \Pi_2 = 1(\Pi_1 = \Pi_0),\ \ldots,\ \Pi_n = \Pi_0,\ \forall\, n \leqslant K;$

$$1 = \sum_{n=0}^{K} \Pi_0 \ \Rightarrow \ \boxed{\Pi_n = \frac{1}{1+K}, \quad n \leqslant K}$$

b) $L = \sum_{n=0}^{K} n\,\Pi_0 = \dfrac{1}{1+K}\dfrac{K(K+1)}{2} = \dfrac{K}{2}$

c) $\emptyset = 1 \times \Pi_0 = \dfrac{1}{1+K}$

d) $L_q = L - s + \emptyset = \dfrac{K}{2} - 1 + \dfrac{1}{1+K} = \dfrac{K(1+K) - 2(1+K) + 2}{2(1+K)} \overset{2}{=} \dfrac{K(K-1)}{2(1+K)}$

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[2] $L_q = \sum_{n=2}^{K} (n-1)\Pi_0 = \dfrac{1}{1+K}\sum_{n=2}^{K} (n-1) = \dfrac{1}{1+K}\sum_{n=1}^{K-1} n = \dfrac{1}{1+K}\dfrac{(K-1)K}{2}.$

PROBLEMES PROPOSATS

PROBLEMA N. 82

Let $x_1, \ldots, x_n$ independently distributed with $x_i \sim N_p(\mu_i, \Omega)$, where Ω is nonsingular. It is assumed that $n > p$ and that the matriz $M' = (\mu_1, \ldots, \mu_n)$ is of rank 1. Define

$$S = \sum_{i=1}^{n} x_i x_i'$$

Find $E(S^{-1})$ as a second order approximation to the exact solution of Steerneman (1997, 1999), given by

$$E(S^{-1}) = \frac{1}{n-(p+1)} \Omega^{-1}.$$

(The author invite readers to propose a solution).

References

Steerneman, A.G.M. (1997). «Problem 331», *Statistica. Neerlandika*, 51, 381.

Steerneman, A.G.M. (1999). «Solution 331», *Statistica. Neerlandika*, 53, 252-254.

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PROBLEMA N. 83

El estimador de regresión lineal multivariante de la media de una población finita de tamaño N, $\overline{Y} = (1/N)\sum_{i=1}^{N} y_i$, usando el principio de mínimos cuadrados en base a n observaciones de la variable de interés y, siendo x_j ($j = 1, 2, \ldots, k$) la j-ésima variable auxiliar, es:

$$\widehat{\overline{Y}} = \overline{y} + \sum_{j=1}^{k} b_j \left(\overline{X}_j - \overline{x}_j\right),$$

donde $\overline{y}$ es la media muestral de la variable de interés, b_j el coeficiente de regresión j-ésimo, y $\overline{x}_j$ y $\overline{X}_j$ son la media muestral y poblacional respectivamente para la j-ésima variable auxiliar. Demostrar que si los coeficientes de regresión están acotados, el estimador $\widehat{\overline{Y}}$ es consistente.

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