Mathematical Questions concerning Zonohedral Space-Filling

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Abstract

An exposition of the basic mathematical properties of zonohedra and their diagrams, arranged so as to bring into focus the many challenging unsolved problems involved in implementing the procedures for synthesizing space-fillings which were described in the article "Polyhedral Habitat".

Note aux lecteurs francophones:
Une traduction française de cet article est disponible sur demande. Un bon de commande est inclus.

Introduction

In his article "Polyhedral Habitat", Janos Baracs introduces a technique for synthesizing zonohedral space fillings. It is an effective drafting technique, since it enables the user quickly to arrive at a single correct plane projection of a spatial framework, from which the required spatial information may then be surmised, either by intuition, or by calculation based upon a very few free choices. The technique proceeds in three stages:

A) successive **splittings** of a plane tessellation, to produce tessellations by concave clusters of convex zonagons.
B) **staggering** (relative parallel transport) of two copies of the resulting tessellation, to arrive at a projection of a space-filling.
C) **lifting** the projection into space, to arrive at a space-filling by concave clusters of convex zonohedra.
Associated with this technique, but not analyzed in as much detail in the abovementioned article, is the method of
D) beginning with a known space-filling by a convex parallelohedron, to arrive, by successive **spatial splittings**, at space-fillings using more general concave parallelohedra.

Each of these procedures raises a number of interesting and challenging problems, with which we hope to tempt some of our mathematically-minded readers. Certain lines of inquiry, primarily those closest to the domain of intended architectural application, have already been pursued during the years since "Polyhedral Habitat" was written, by Nabil Macarios, Yves Dumas, Pierre Granche, Vahe Emmian, and other members and associates of the Structural Topology research group. The problems outlined in the present article are those which arise when one begins to piece together a certain mathematical theory of space-filling, a theory designed to shed some light on space-filling polyhedra which are formed as concave clusters of zonohedra. Recent work in this direction was carried forward during meetings of the research group, mainly through the efforts of Janos Baracs, Walter Whiteley, Marc Pelletier, and the present author. As we outline the problems which we see arising, we will point out those areas where we have been able to make some progress, and will give some references to what we feel is relevant mathematical literature, chiefly to the papers of Coxeter, Grunbaum, Shephard and McMullen.

Zonohedra

A **zonohedron** is a convex polyhedron expressible as the convex sum of a finite set of line segments (see Grunbaum 1967). This construction is as follows. Given n vectors at the origin in three-dimensional space, vectors not lying in a single plane through the origin, their **convex sum** is the set of points whose position vectors may be written as linear combinations of the given vectors, using scalars chosen between 0 and 1, inclusive. The boundary of this convex sum is a convex polyhedron, the zonohedron, and the original array of vectors we call the **star** of that zonohedron. See **Figure 1**.

Given a 3-dimensional model of the star, it is easy to generate a set of parallelograms and other 2n-gons, using in turn all possible pairs of vectors, or other larger coplanar sets of vectors, in the star. These zonogons, cut from cardboard, are precisely the faces needed for a model of the zonohedron. But this is not the way we would proceed in practice, because there is really no need to construct a spatial model of the star. Say instead we have a bit more information in a plane diagram: not only a plane projection of the star, but also the intersection points of the lines of the star with the plane of projection, as in **Figure 2**. We may assume that the projection is an orthogonal projection, and that the actual centre of the star is at distance 1 above its image in the plane. Then standard constructions in descriptive geometry provide correct plane drawings of the real lengths and angles of the faces. We see this construction for a typical parallelogram face in **Figure 3**. Here the lines of the vectors 0A and 0B meet the plane of projection on points M and N respectively. The point L is the foot of the perpendicular through 0 to the line MN, the line 0K is parallel to MN, the segment 0K has length 1, and the line L0' is equal in length to LK (a compass construction). This line L0' gives the true length of the line L0, because LK is the hypotenuse of a right triangle with altitude 1, and base L0. By dropping a perpendicular to MN through the point A until it meets the line 0'M, we find the location A' of the point A as the face 0AB is rotated about the line MN until it lies in the plane of projection. Doing the same construction at B, we find B', and completing the parallelogram, we find a picture of the face 0AB with its true edge lengths and angles.

The lengths of vectors in a star can be changed freely without affecting the topology of its zonohedron, so the lengths can all be set equal to one without loss of generality. The number of distinct faces needed for the construction of a zonohedron can be reduced by assuring that there are relatively few different angles between pairs of vectors in the star. The vectors might be taken as pointing outward toward the vertices of a regular polyhedron, for example.

Given a plane through the origin, a plane containing none of the vectors of the star, we obtain a partition of the vectors into two subsets: those on one side of the plane, those on the other. By adding the vectors in these two subsets, we obtain the position vectors of a pair of opposite vertices of the zonohedron. The converse is also true: at any vertex of the zonohedron, find a plane which touches the zonohedron only at that vertex. The plane parallel to this plane, and passing through the origin, produces the required partition of the vectors.

Every edge of the zonohedron, considered as a free vector, is equal to one of the vectors in the star. If we fix our attention on all the edges parallel to any one vector, we find they form a belt, or **zone** around the zonohedron. The zonohedron is centrally-symmetric, about a point whose position vector is exactly half the sum of all the vectors in the star. This symmetry is visible in plane projection, by virtue of the fact that the upper cap and lower cap are obtained from one another by central reflection **(Figure 4)**. This will be important when we come to study staggerings and lifting, below. We have also used **Figure 4** to illustrate how hexagonal faces are formed when we take three vectors coplanar in the star, here bdf and ace.

(Coxeter 1962) simplifies the development of zonohedra one step further, by studying the figure obtained by intersecting the lines of the vectors in the star with a plane not through the origin, such as in **Figure 2**, above. He calls this plane figure the **(primal) projective diagram** of the zonohedron.

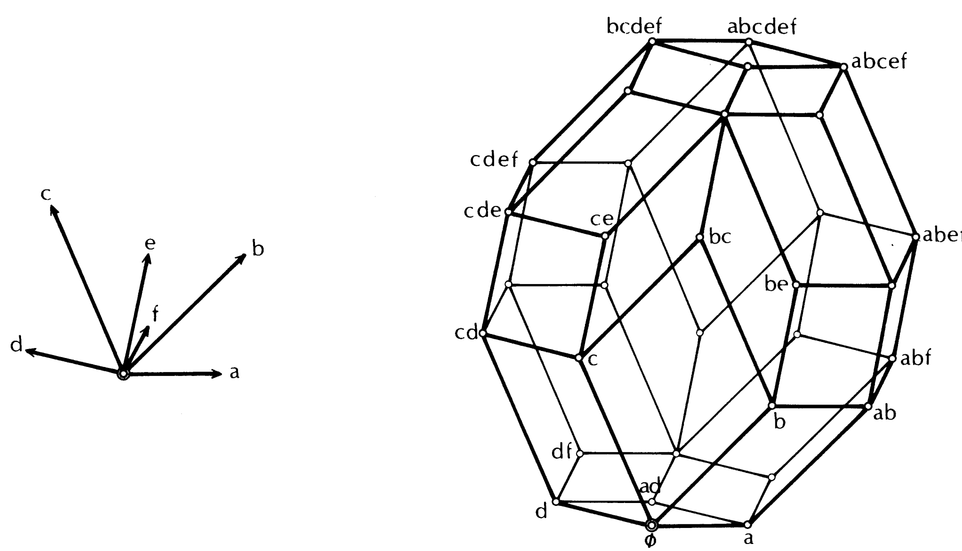

Figure 1. Each vertex of the zonohedron is reached by adding some (indicated, in a few cases) subset of the set of vectors in its star.

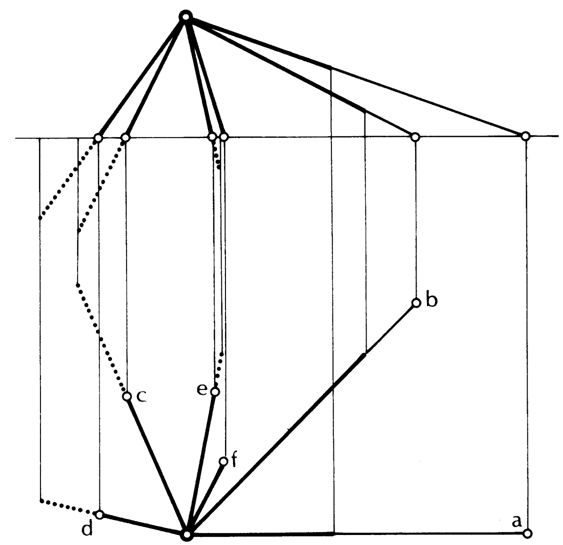

Figure 2. Intersection of all lines of a spatial star with a plane produces a (primal) projective diagram for its zonohedron.

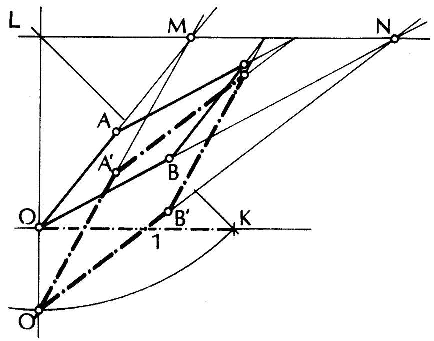

Figure 3. From a diagram like Figure 2, it is easy to construct the true dimensions of each face by "rabattement", rotating the face down into the plane of projection, using its intersection with the plane of projection as hinge for the rotation.

Planes through the origin, in the above procedure for locating pairs of opposite vertices, become lines in the plane of the projective diagram, lines which separate the points into two complementary subsets **(Figure 5).**

The topological properties of any given zonohedron are easily discernable from its projective diagram. For instance, adjacent vertices of the zonohedron come from bipartitions which differ only by the reassignment of one point to the other half, and the direction of the edge between those two vertices is that of the vector belonging to that point in the projective diagram. By this same line of reasoning, taking any given separating line as the "line at infinity", with respect to which we develop a notion of "convexity", the convex hull of the set of points of the diagram is a convex polygon, whose vertices (which are always points of the diagram) give exactly the edges incident at the two vertices corresponding to that bipartition of the points. For instance, using **Figure 5**, we know that from the vertex df we can reach adjacent vertices by adding the vector c, adding a, or subtracting f.

If we contract to zero along itself any one vector of a star, we will contract to zero the width of one zone of the zonohedron. What remains of the zone will be a **meridian** path, which traverses a sequence of edges and faces of the contracted zonohedron, using each remaining vector or its negative once (either by going along an edge in that direction, or diagonally across a face involving that zone), to reach the vertex opposite to the starting point, then continuing on the opposite route (opposite in the sense of central symmetry) back around to the starting point **(Figure 6)**. The converse is not quite true. Say we define a meridian path on a zonohedron (or on a plane projection of a zonohedron) by the above property, namely that the path uses each vector once, then returns to the starting point along the same sequence of opposite vectors. There are meridian paths which do not correspond to possible expansions of new zones on (convex) zonohedra. This difficulty can occur at several different levels. A plane drawing obtained after several successive expansions along meridian paths, even though its faces are convex and it has a convex perimeter, need not be the projection of a zonohedron in space. It may for instance be the case that, although the drawing is topologically equivalent to a zonohedral skeleton, it may not be possible to view that zonohedron with the given outline as its contour. Even worse, the drawing need not even be topologically equivalent to the skeleton of a (convex) zonohedron. **Figure 7A** with 9 zones is such a faulty drawing: it is not even topologically a zonohedron. By contracting any one zone in **Figure 7A**, and moving the resulting meridian to the perimeter, we obtain a subdivision into parallelograms of the zonogon with eight pairs of sides, a subdivision which cannot be the top cap of a (convex) zonohedron **(Figure 7C)**. By putting the word "convex" in parentheses we wish merely to underline that strict convexity is part of the definition of zonohedra. There are, as it is easy to show, many **concave** polyhedra having **Figures 7** as their parallel projections, concave polyhedra which resemble zonohedra in most respects.

These facts derive from a dual projective diagram also invented by Coxeter, and from the theory of arrangements of lines and pseudo-lines (Grunbaum 1972). To obtain the dual diagram from a star in space, take the planes through the origin, one normal to each vector of the star, and intersect that configuration of planes with a single plane not through the origin. Both the primal and dual diagrams are plane diagrams, and are related to one another via a **polarity**, a mapping which interchanges points with lines, lines with points, but maintains incidence of points with lines.* **Figure 8** shows the dual diagram for the zonohedron in **Figure 4**, and shows how the dual diagram is the topological dual of the top cap of the zonohedron. By **topological dual** we mean the figure whose vertices are the faces of the zonohedron, whose edges cross the edges of

*Note: The construction of the dual diagram need not be that stated in terms of the metric concept of "normal plane", as above. Any projectively defined polarity will do.

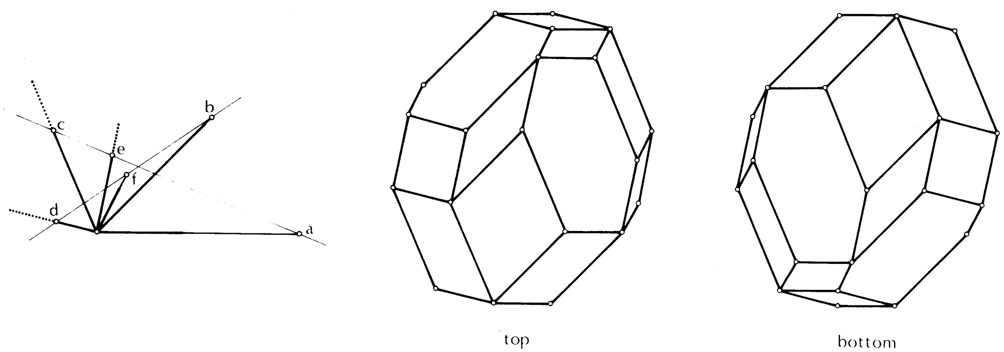

Figure 4. The projections of upper and lower caps of a zonohedron differ by a central reflection (by a half turn in the plane). Comparison of Figures 1 and 4 reveals the effect of a different choice of projective diagram, even given the same projected vectors.

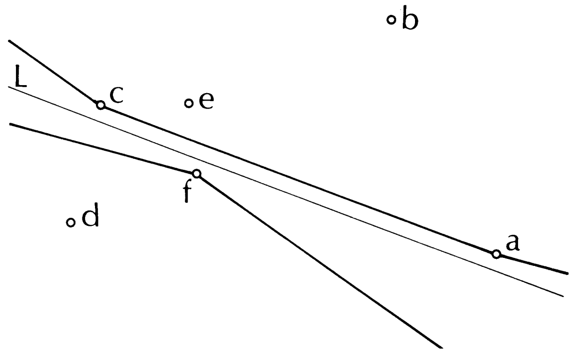

Figure 5. The projective triangle acf is the projective convex hull of the points of the diagram, relative to the separating line L. So the vertex df has neighbors adf, cdf, and d, in Figure 1.

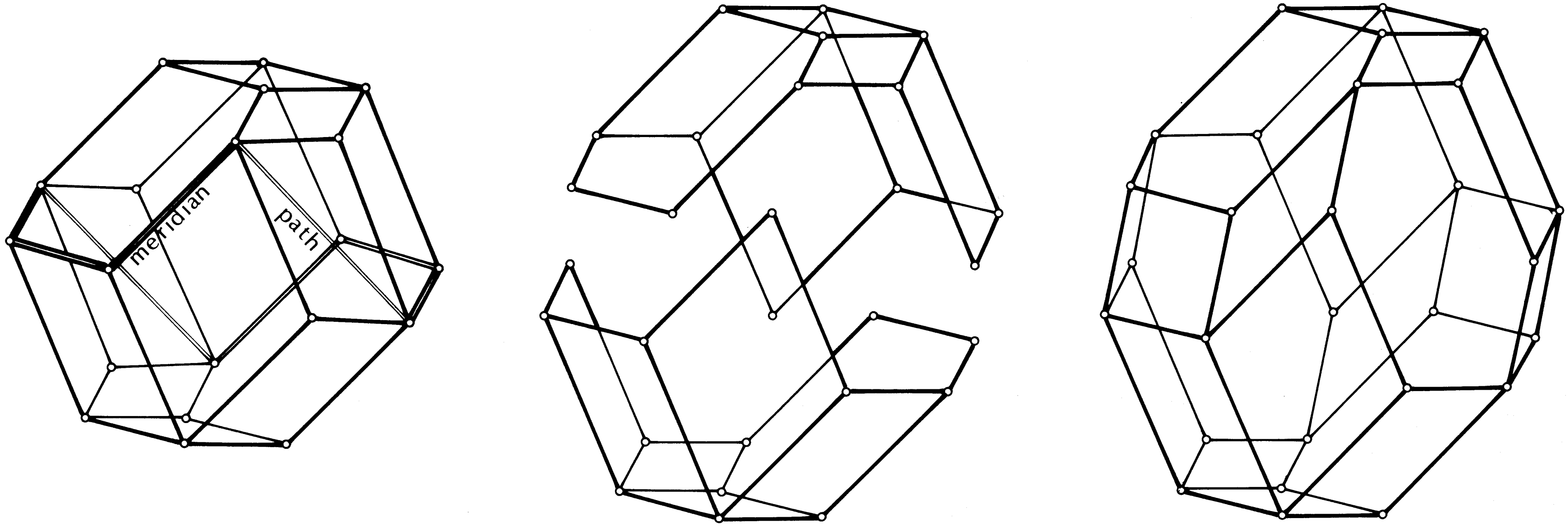

Figure 6. Separation along a meridian path produces a zonohedron with one additional zone.

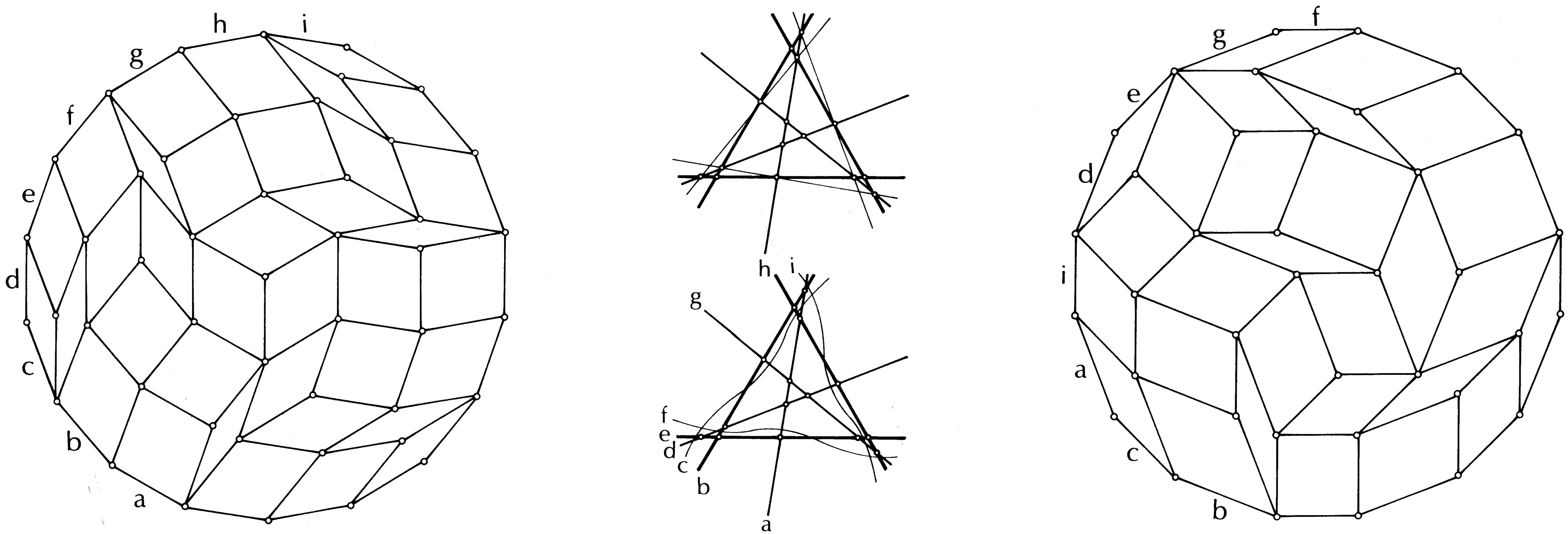

Figure 7. A non-zonohedron, its configuration of pseudo-lines derived from a proper configuration of lines, and a somewhat simpler half-zonohedron which cannot be the "top cap" with that contour.

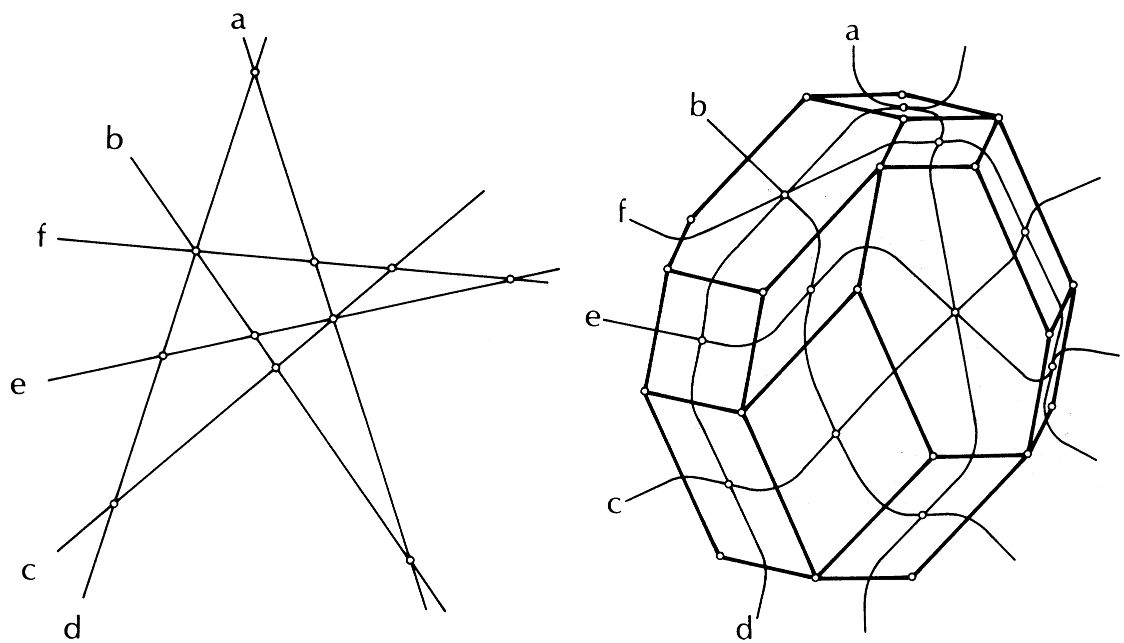

Figure 8. The dual diagram of a zonohedron, as an affine configuration, is the topological dual of its top cap.

the zonohedron (forming lines corresponding to the zones), and whose faces are the vertices of the zonohedron.

Now the process of expanding a meridian path in a plane drawing, to produce a new zone, corresponds to the process of introducing into the dual diagram a curve which crosses each line exactly once (possibly at points where two or more lines meet) and extend to infinity in both directions. Such a curve is called a **pseudo-line**. The question as to whether the resulting expanded figure is topologically equivalent to a zonohedron is equivalent to the question as to whether the arrangement of pseudo-lines can be stretched straight, to form an arrangement of straight lines, without changing the order of any sequence of crossings of lines. (Grunbaum 1972) gives a variety of examples of unstretchable arrangements of pseudo-lines, each of which yields a non-realizable zonohedral drawing. We give this warning here to prevent undue optimism regarding the universality of the three basic steps for synthesizing space-fillings, discussed below. **Figure 7B** shows a non-stretchable arrangement of nine lines for the incorrect zonohedral drawing **Figure 7A**, and also shows how the arrangement can be derived by systematically distorting a diagram of nine straight lines.

The number of topologically distinct zonohedra with a given number of zones is known for up to six zones. Considering only those zonohedra all of whose faces are parallelograms, there are only one each with 3, 4, and 5 zones, four with 6 zones, and 11 with 7 zones (Grunbaum 1967, p 394). With more general zonagons permitted as faces, there is still a unique example with 3 zones, two with 4 zones, four with 5 zones, and 17 with 6 zones (Grunbaum 1972, p 4). These numbers were obtained by counting arrangements of lines in the projective plane. In **Figure 9** we show top caps for the four different zonohedra with six zones, and only parallelograms for faces. For purposes of comparison, we have placed one of the highest valency vertices near the centre of the drawing. Even from these drawings it is evident that the four zonohedra are topologically distinct. Example A has a pair of 6-valent vertices. Examples B, C, D have two, three, and six pairs of 5-valent vertices, respectively.

Before we restrict our attention to questions of space-filling, let us pose two general problems about zonohedra. First, for purposes of dome design, it is

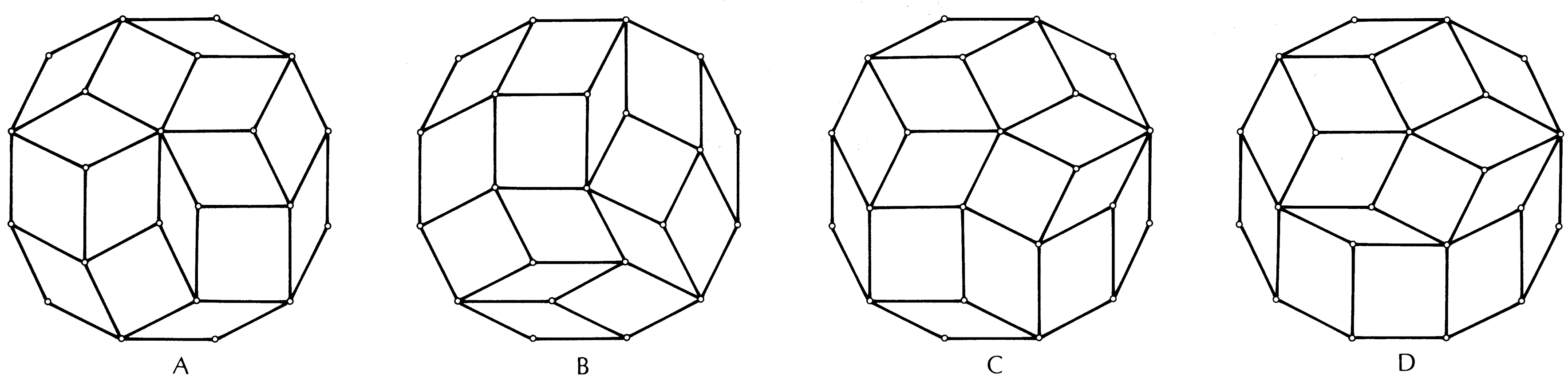

 Figure 9. The top caps of the four different 6-zone rhombic zonohedra (those with only parallelogram faces).

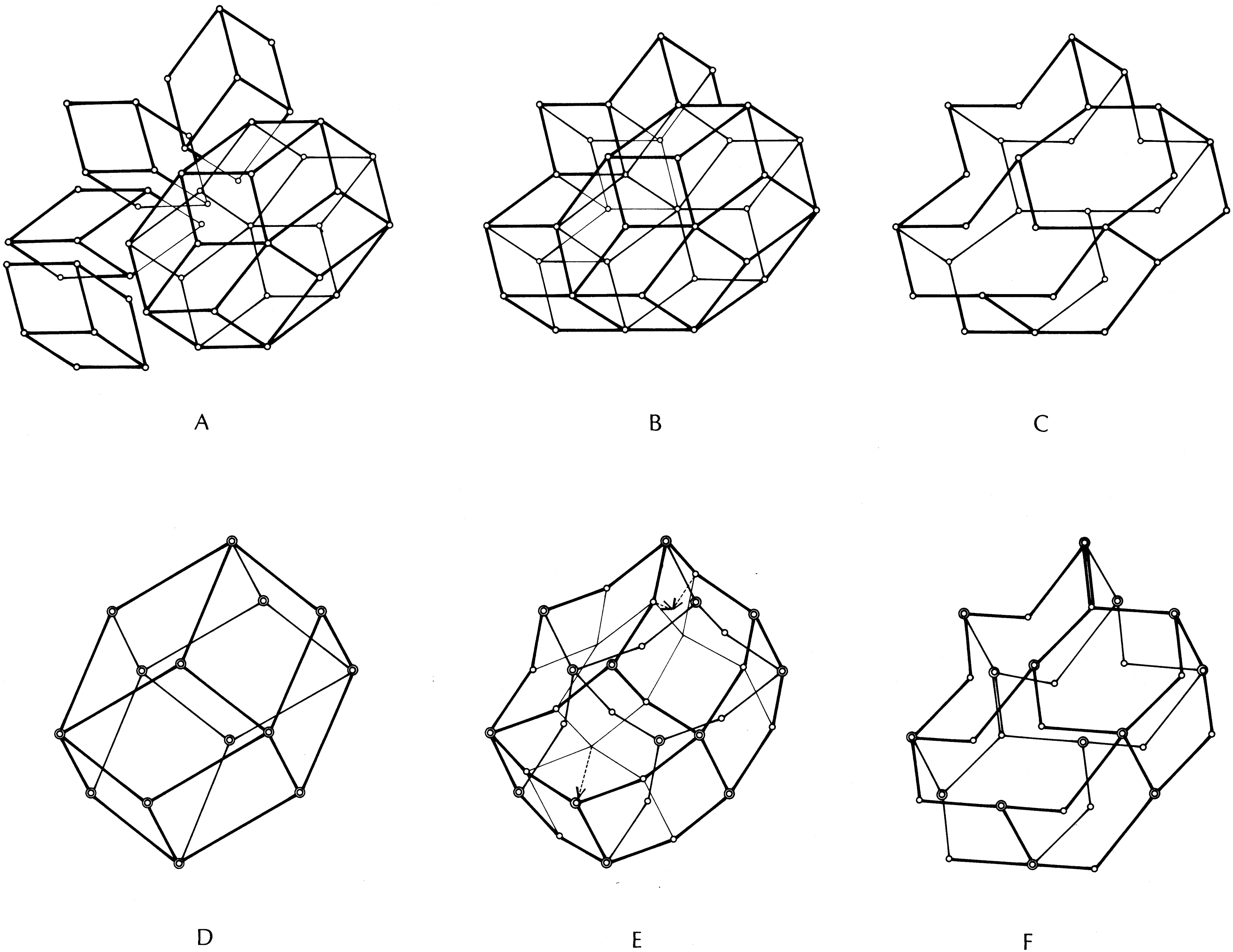

Figure 10. A divisible concave parallelohedron, with drawings to indicate how it is divisible, and how it is obtained by bending a rhombidodecahedron. Diagram F shows its topological regions of contact with adjacent concave parallelohedra.

convenient to be able to use polyhedra with inscribed spheres.

Problem 1. Which zonohedra can have inscribed spheres? The answer should be formulated as a combinatorial criterion to be satisfied by the vector star or by the projective diagram.

All zonohedra with only parallelogram faces and with up to 5 zones, and one (for sure) of the 6-zone zonohedra, can be built with all faces congruent. The zonohedron whose star is formed along the space diagonals of a regular icosahedron has only two distinct congruence classes of faces (Baer 1970). We wonder how the number of required congruence classes depends on the number of zones. In particular,

Problem 2. Using only k congruence classes of parallelogram faces, up to what value of n can some n-zone zonohedra be built? Construct the required faces, at least for n equal to 2.

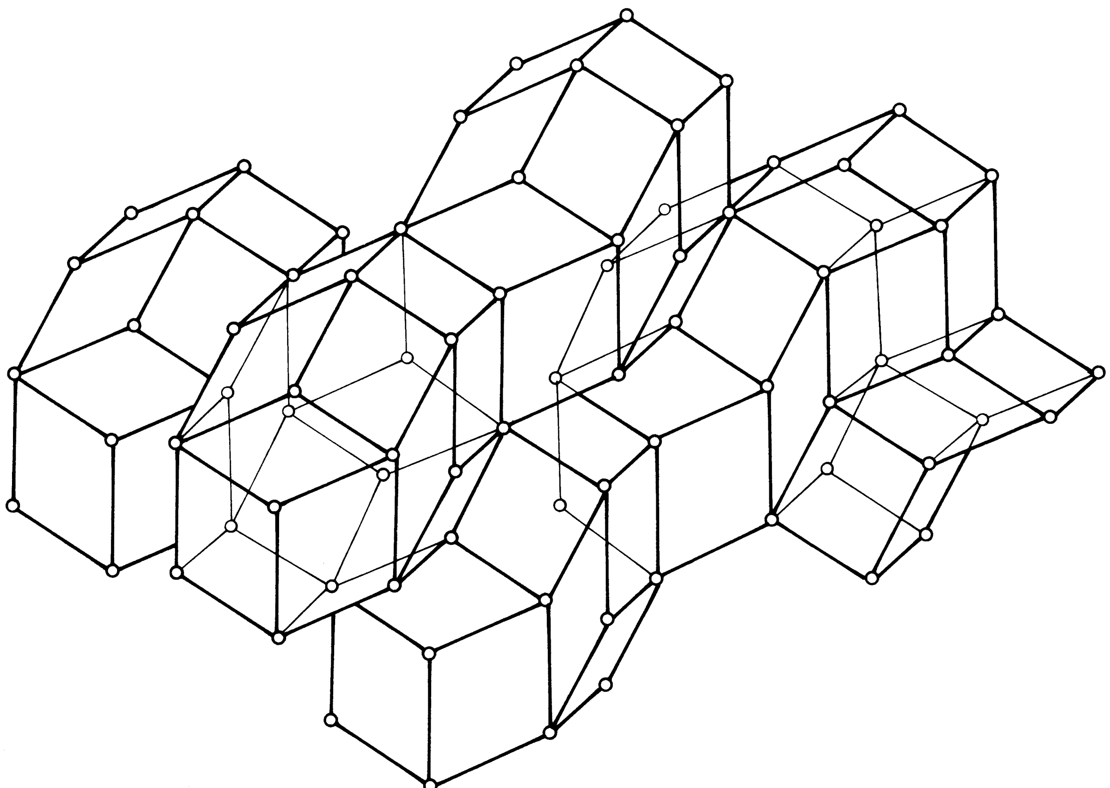

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Figure 10G shows how the five-zone zonohedra meet when the concave parallelohedron of Figure 10 fills space. This example was found by Marc Pelletier.

Concave Parallelohedra

A polyhedron (topologically a sphere, realized without self-intersection) is called a **concave parallelohedron** if and only if copies of it fill space by parallel displacement, the polyhedra meeting each other face-to-face.* We are mainly interested in those concave parallelohedra which are **divisible** into convex zonohedra. By a theorem of (Shephard 1974a), every zonohedron is further divisible into **cubes**(zonohedra with 3 zones), so a concave parallelohedron is divisible if and only if it is expressible as a juxtaposition of cubes. The concave parallelohedron in **Figure 10** is divisible into 14 cubes, or into a 5-zone zonohedron (the rhombicosihedron) and four cubes. That in **Figure 11** is divisible into 13 cubes, but is not divisible into a 5-zone zonohedron plus cubes. It does, however, have many desirable properties as a fundamental region for space-filling. Removing the 5-zone zonohedron which this fundamental region contains, we find the residue consists of a **concave** 4-zone rhombidodecahedron and a cube. In the space-fillings generated by the fundamental regions of **Figures 10 and 11**, the 5 zone zonohedra contact each other in essentially different ways. Later in this article, we shall return to study these two space-fillings and their fundamental regions in greater detail.

Given a zonohedron and its star, every subset of the set of vectors in the star is itself a star, and produces a zonohedron which we call a **relative** of the original zonohedron. The main question to be posed in the context of the article "Polyhedral Habitat"is

Problem 3. For an arbitrary (convex) zonohedron, can it be juxtaposed with some of its relatives, to produce a concave parallelohedron?

This is true for zonohedra with up to five zones, but is not known to be true for any zonohedron with six zones. However, a simple inductive argument establishes the analogous result in the plane. By **concave parallelogon** we mean a polygon (topologically a simple closed curve in the plane) which fills the plane by parallel displacement, polygons meeting each other edge-to-edge.

*Note: Adjacent concave parallelohedra may have several faces in common, but each such face of one parallelohedron meets exactly one entire face of the other parallelohedron.

Theorem. Every zonagon may be juxtaposed with some of its relatives, to form a concave parallelogon.

Figure 12 illustrates the method, each step of which involves the reduction by one of the number of elements of the star, the elements of the reduced stars being formed as **sequences** of vectors from the original star. The basic step is to replace three elements a, b, c in cyclic order by two elements: a + b and c + b. This step is applicable even if the elements a, b, c were already vector sequences formed at some earlier stage of the construction. The process continues until the number of elements is equal to three, and an underlying hexagonal tessellation appears.

Divisible concave parallelohedra are fundamental regions for tilings of space by related (convex) zonohedra. We should note that one such tiling may have a number of quite different fundamental regions. **Figure 12** indicates what sort of choice is available for a fundamental region in one tiling of the plane. From our point of view, distinct tilings are far more interesting objects of study than are distinct fundamental regions.

Problem 4. For any set of related zonohedra (or zonagons) which can be juxtaposed in at least one way to form a concave parallelohedron (resp. parallelogon), enumerate the distinct tilings to which they can give rise.

We conjecture, for instance, that one decagon and four quadrilaterals, all of them being related zonagons, can form only one topological type of tiling of the plane, that shown in **Figure 12.** But there must be many distinct ways of tiling the plane with one 16-gon, an octagon, a hexagon, and 26 quadrilaterals. A fundamental region for one such tiling is shown in **Figure 13.** Note in this example how the 16-gons are separated from one another by ribbons of parallelograms and other small zonagons. It cannot be otherwise. Even for fairly small 2n-gons, it becomes impossible for them to contact one another in more than one lattice direction. (Contact in one lattice direction is always possible, no matter how many edges a zonagon has.) The same sort of separation will occur in space, once the number of zones on a

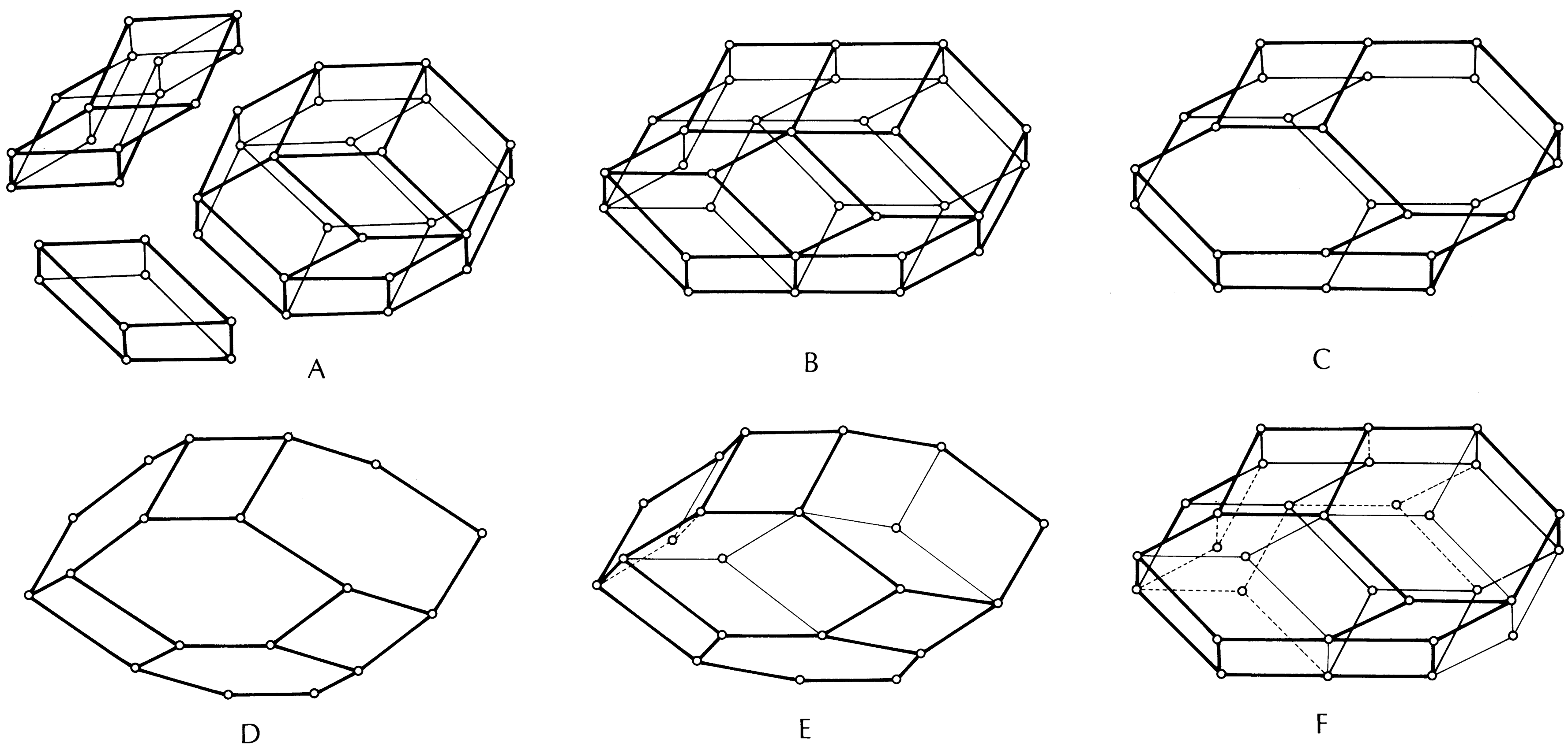


Figure 11. A divisible concave parallelohedron, with drawings to indicate how it is not divisible into a 5-zone zonohedron plus cubes, but is obtained by twisting a truncated octahedron.

basic zonohedron reaches a certain minimum value. The reader may wish to investigate this matter, and to determine certain **optimal-proximity** tilings with a given zonohedron and its relatives.

It is tempting to regard the vector sequences, which appear in **Figure 12 and 13** as broken edges of an underlying hexagonal tessellation, as the basic elements of a **star** (more generally defined) for a concave parallelogon (the fundamental region). Mathematically, such a star may be defined as an assignment (not necessarily one-to-one) of vectors to the elements of a partially-ordered set. By the **convex sum** of such a generalized star, we would mean the set of points expressible as linear combinations, using scalars between 0 and 1, but using a non-zero scalar at any location only when the scalar 1 is used at all locations earlier in the partial order. Thus, the fundamental region developed for **Figure 12** is the convex sum over a partially ordered set consisting of three chains, two of length two and one of length three, assigned the vectors (a,b), (d,e) and (-c,-b,d) respectively.

Problem 5. Sort out the relationships between generalized vector stars and concave parallelohedra. Show how concave parallelohedra are created by modifying the vector stars of the five parallelohedra.

We have some experimental evidence to indicate that concave parallelohedra may be formed from convex parallelohedra in this way. The regions of contact between adjacent fundamental regions in a space-filling are topological discs consisting of certain faces of the concave parallelohedron. First off, the number of such regions always seem to be 6, 8, 12 or 14, which are the numbers of faces of the parallelohedra. The lattice of adjacencies between fundamental regions within the space-filling seems to be the same as for one of the parallelohedral space-fillings. The topological polyhedron whose faces are these regions of contact need not be exactly a parallelohedron (see **Figure 10**, the heavy lines indicating the regions of contact), but seem always to be nearly so. Furthermore, there seems to be a way of merely "bending" the vectors in the star of one of the five parallelohedra, breaking some edge vectors into sequences of vectors, or skewing certain originally coplanar sets of vectors, so as to arrive at a generalized star for the resulting parallelohedron. The transformation required in **Figure 10** can be accomplished by bending the vectors of the star for a 4-zone zonohedron, the rhombidodecahedron. In **Figure 11** we skew sets of vectors coplanar in the star of the truncated octahedron, meanwhile bringing two vectors into alignment. In this case the topological polyhedron formed by regions of contact is just the truncated octahedron.

The above line of thought is intended to lead to a simple procedure for manipulating vector stars on the drawing board, with the aim of arriving at space-fillers suitable for any given application.

There exist in the literature a number of fascinating papers concerning space-filling by parallelotopes in spaces of arbitrary finite dimension. Coxeter conjectured that a zonotope in d-space is a space-filler by parallel transport if and only if all its natural plane projections (projections parallel to the edges of the zonotope) are tessellations (square or hexagonal) of the plane. **Figure 14** shows how the natural projection of a truncated octahedron is a hexagonal tessellation of the plane. (Shephard 1974b) verified Coxeter's conjecture for 3- and 4-dimensional space, and (McMullen 1975) completed the proof for arbitrary dimension. Their result may be stated as follows. Subsets of the set of vectors in a given vector star of rank n span various subspaces of lower rank. A set of vectors which are all those in some given subspace is called a **closed** set of vectors. Those closed sets of vectors which span subspaces of rank n-1 are called **copoints**' and those which span subspaces of rank n-2 are called **colines**. A zonotope with a star of rank n is a space-filler in (n-1)-dimensional projective space if and only if, in the

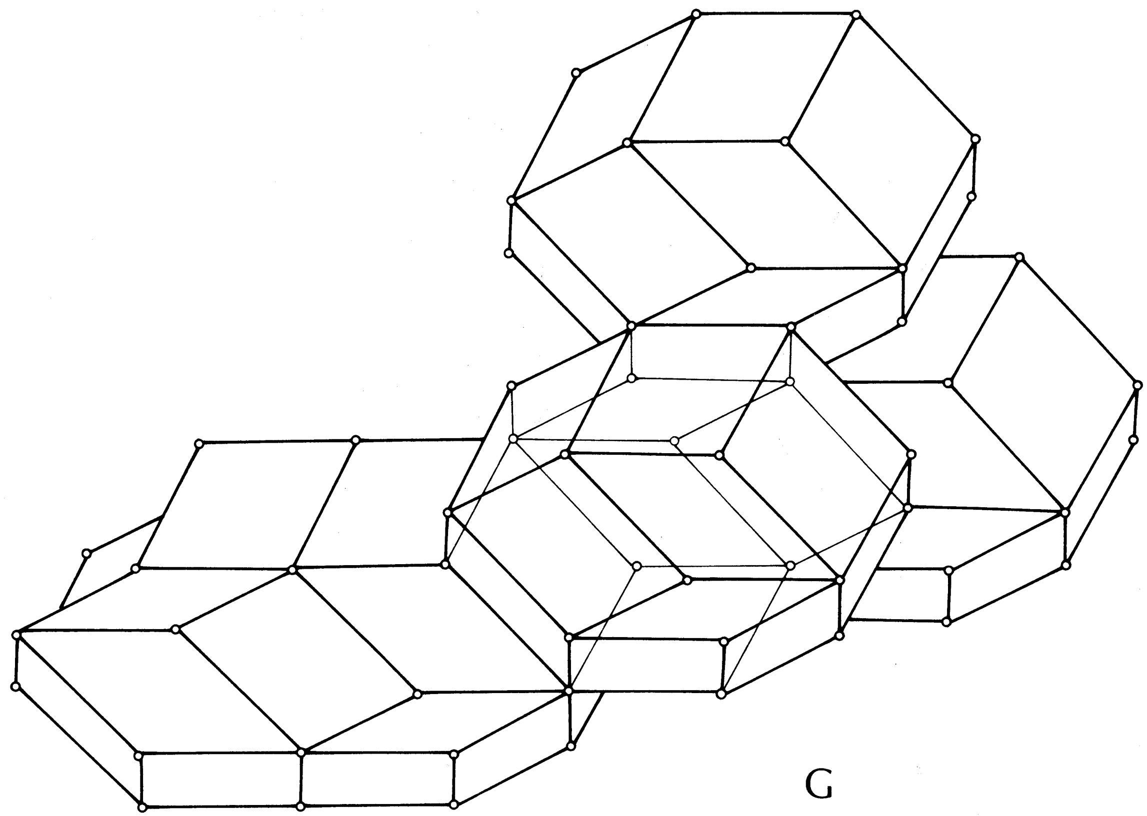


Figure 11G shows how the five-zone zonohedra meet each other when the concave parallelohedron of Figure 11 fills space. (The voids are not divisible into cubes.) This example was found by Janos Baracs.

geometry of its star, each coline is contained in at most 3 copoints. That is, the vector geometry of its star is binary (see Tutte 1958), being representable over a field with only two elements, 0 and 1.

For any facet of a zonotope, there is a translation vector which carries the facet onto the opposite facet. This translation vector is expressible as a linear combination of those vectors of the star which do **not** lie in the facet, using only the scalars, 1, -1. The rank of the matrix of these coefficients is always greater than or equal to the rank of the star. The crucial step in McMullen's proof is to realize that the rank is **equal** to the rank of the star if and only if the zonotope is a space-filler. In our opinion, these same methods are directly applicable to the questions at hand, concerning concave parallelohedra, and should yield immediate results.

The columns of the matrix displayed in **Figure 14** give the directions in which the edges are traveled in going from a face to its opposite, and this for each pair of faces on the truncated octahedron. The calculation is illustrated for two such pairs of faces. Each pair of faces comes from a line in the primal projective diagram. The split between positive and negative sign for vectors **not** on a pair of faces is that between points or one side or the other of the corresponding line in the projective diagram. The definition of "sides" of a line is relative to a choice of line at infinity, which must be appropriate to the given drawing. The truncated octahedron is a space-filler, a parallelohedron, so the rows of the matrix will give a natural coordinatization of the vectors of the star, in a **3-dimensional** subspace of 7 dimensional space. To verify the dimension, take abc as a basis, and check that d = c-b, e = c-a, f = b-a.

The vector star for the truncated octahedron reveals a further essential feature of proper plane drawings of zonohedra. As we said before, we may freely choose lengths for each parallel family of edges in such a drawing, and may also choose the directions of these edges within certain projective limits. What becomes clear from this example is that these projective limits may be very restrictive indeed. The six directions of the edges for the truncated octahedron must intersect a plane in the correct primal diagram, a diagram formed by all points of intersection of four (unrestricted) lines abf, ace, bcd, def. The six directions must thus be the six directions of the edges of a tetrahedron whose faces intersect the plane in those four lines, such as the tetrahedron shown in projection with vertices acde. (Think of abf as the line at infinity, so the six points of intersection along that line are the six "directions" of the vectors in the star. The six points form a "quadrilateral set", as it is called in projective geometry.) Any five of these six directions may be freely selected, but the sixth is then **completely determined**. If the six directions in a plane drawing of a truncated octahedron do not satisfy this condition, the drawing will have no proper spatial realization as a polyhedron, let alone a zonohedron! This has a bearing on what follows.

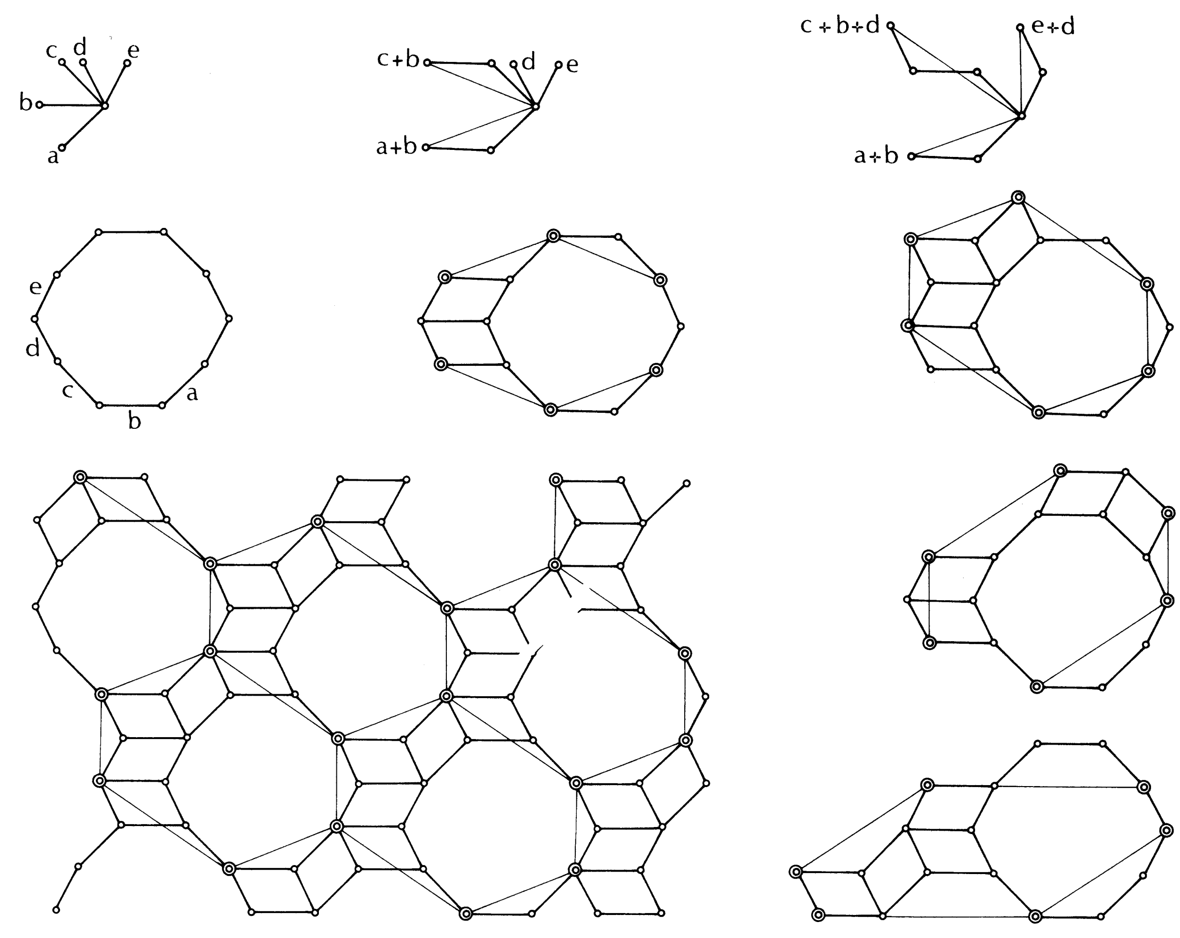


Figure 12. How a 10-sided zonagon fills the plane in combination with four parallelogons.

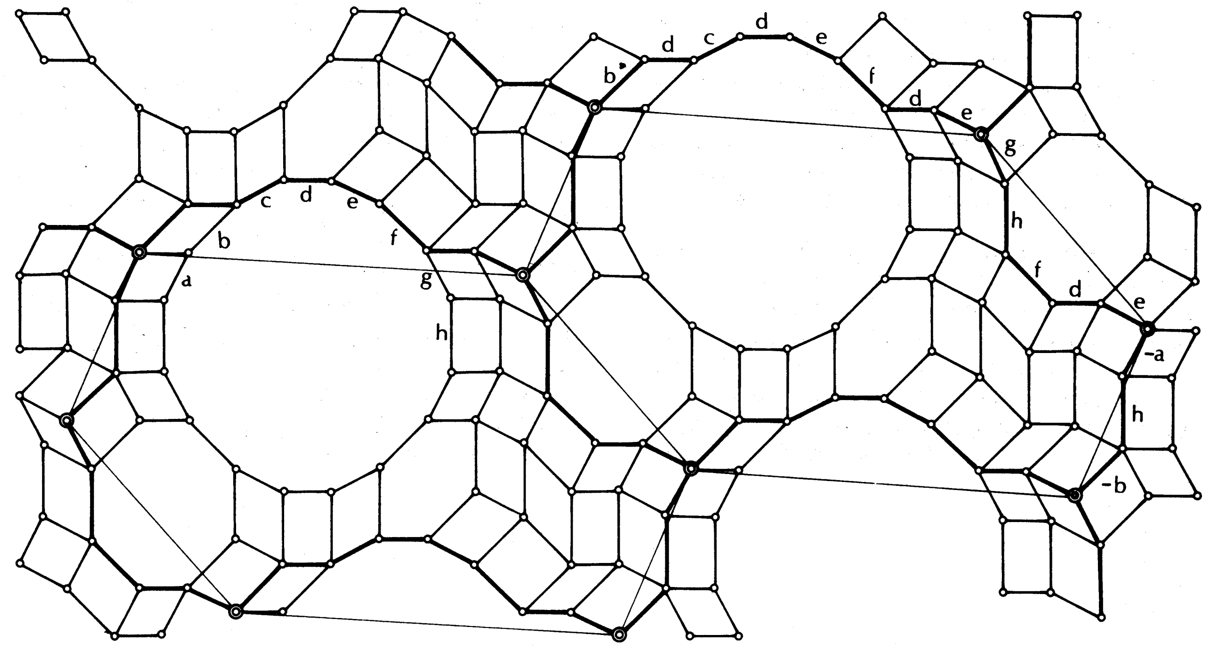
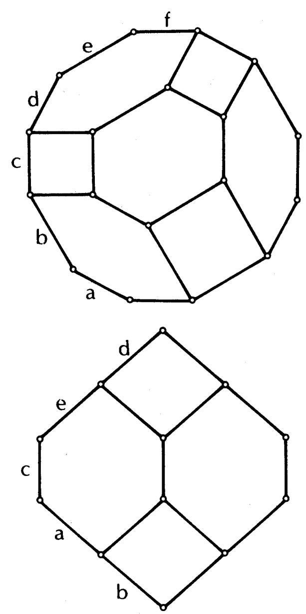
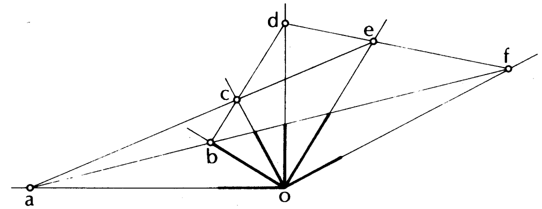
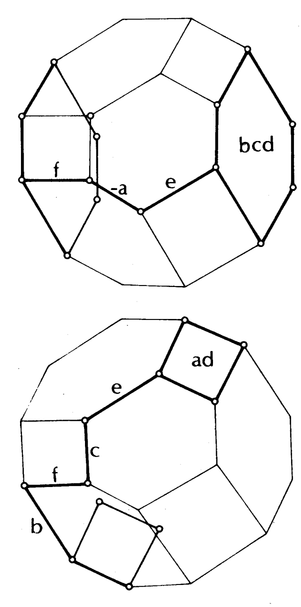


Figure 13. A 16-gon needs many relatives, in order to tile the plane.







	a b f	a c e	a d	b c d	b e	c f	d e f
a	0	0	0	-1	-1	1	1
b	0	1	1	0	0	1	1
c	-1	0	1	0	-1	0	1
d	-1	-1	0	0	-1	-1	0
e	-1	0	1	1	0	-1	0
f	0	1	1	1	1	0	0

Figure 14. Each natural projection of a parallelohedron is always a square or hexagonal tessellation. Paths between opposite faces yield a natural coordinatization matrix of rank 3.

Splitting

Given a tessellation of the plane by a concave parallelogon divisible into (convex) zonagons, we wish to determine all possible splittings which will result in further such tessellations. The method is suggested by Plates 6 and 7 in the paper "Polyhedral Habitat", but is not spelled out in complete detail.

Some necessary conditions on the **cuts**(there indicated by broken lines along polygonal arcs) are evident. Each cut is itself periodic, and is determined by a segment cut from a vertex A to some other vertex A' to which A is carried by some translational symmetry of the tessellation. Thus the vector from one end point of the segment to the other is one of the lattice vectors of the tessellation. The cut goes along certain edges of the tessellation, and goes diagonally across certain of its faces. Secondly, all the other cuts in the splitting are obtained by translation of the first cut by all multiples of some single lattice vector, a lattice vector independent of that used to translate from A to A'. See **Figure 15**.

When we try to split a tessellation along such a system of cuts, we may find there is no direction in which the cuts may be opened so that the resulting faces are convex and the resulting tessellation is free of self-intersection. **Figure 15** shows the computation of permissible directions of opening a split, along a segment of a cut. For each edge of the tessellation traversed by the cut, there is a half-space of permissible directions. For each zonagon crossed by the cut, there is an acute angle of permissible directions, namely the exterior angle between the two edges of the zonagon incident at the vertex where the cut enters the zonagon. For a segment of a cut to permit some direction of opening, all these (open) half spaces and (open) acute angles must have some direction in common. Thus in **Figure 15A** there remains an acute angle of permissible directions. In **Figure 15B** there are none. **Figure 15B** shows what happens if you try to open a cut in an impermissible direction. In this case we find one polygonal face S of the new tessellation is concave. There is no choice of direction of opening which will make both S and T into proper convex polygons.

The choice of a cut for splitting a tessellation is important for reasons which will become apparent

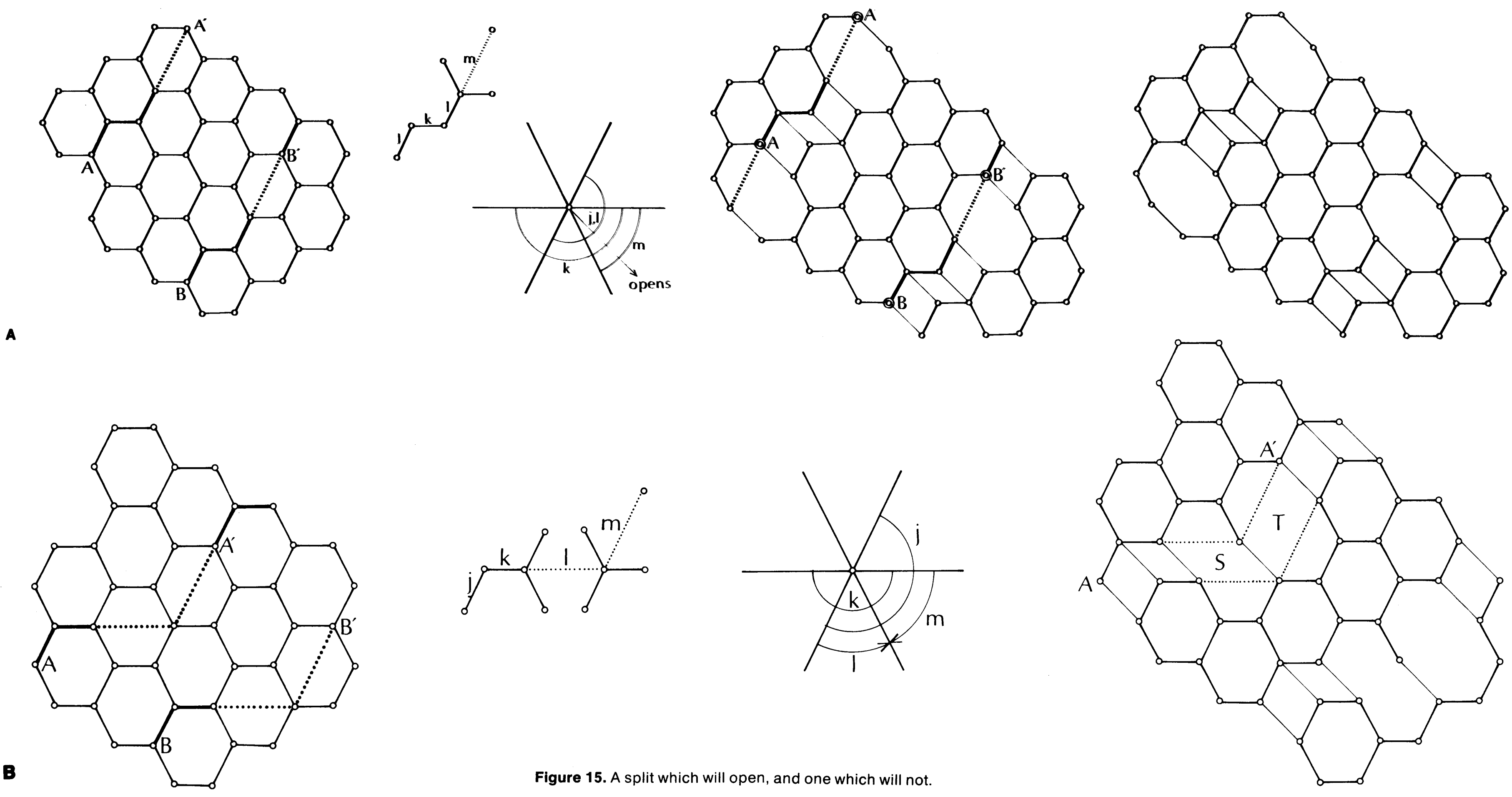


Figure 15. A split which will open, and one which will not.

when we try to **stagger** the resulting tessellation, and when we try to **lift** the resulting double tessellation to a space-filling. That is, the simpler tessellations have a number of essentially different staggerings which lift to space-fillings with convex regions. We should choose cuts which do not eliminate too many of these possible staggerings, or which perhaps even introduce a new parameter of choice of staggering. This is not an easy matter, as we shall see in the next section. Finally, for simplicity of design and fabrication, a cut should be chosen so as to keep the number of different convex polygons small.

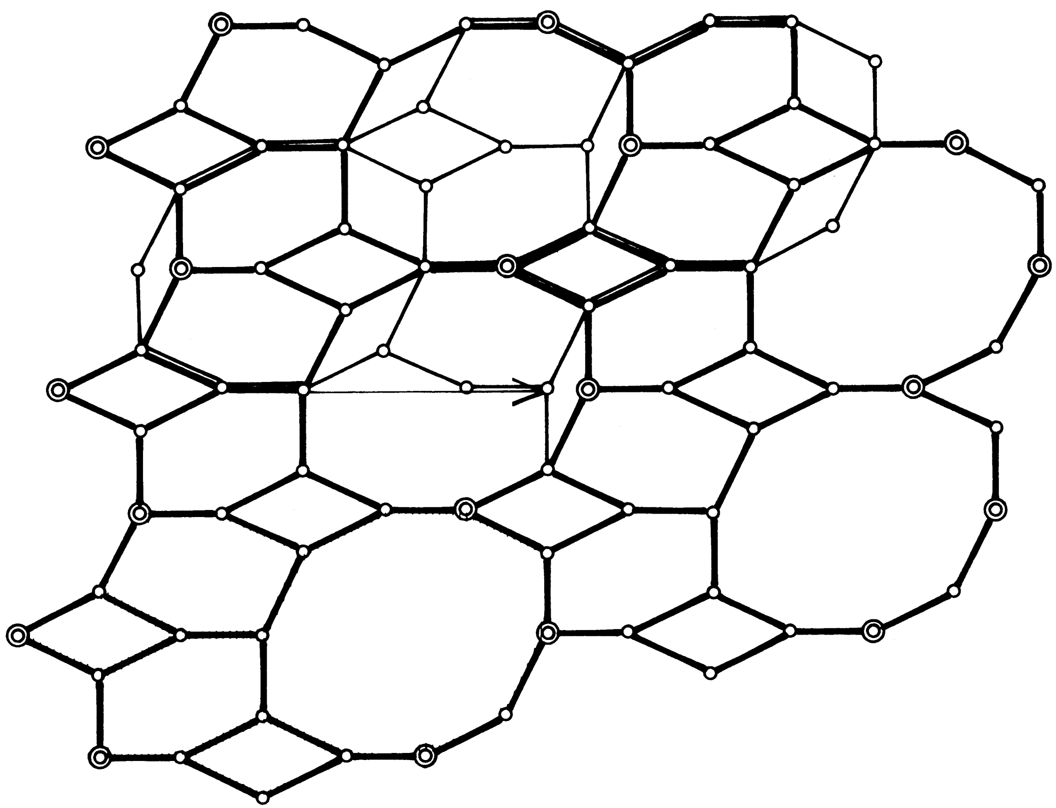


Figure 16. A tessellation with only one non-trivial staggering.

Staggering

Even for such simple tessellations as shown in Plates 6 and 7 of "Polyhedral Habitat", the number of possible staggerings of a plane tessellation can be quite small. For instance, the tessellation in Plate 7 (section 4D) has only one non-trivial staggering, that shown in **Figure 16**. And here we are only trying to satisfy the condition that all residual regions of the staggered pair be (convex) zonagons.

Problem 6. Given a tessellation, find all possible staggerings. If possible, reduce this search to a consideration of the star of its fundamental region. (This will have to be a generalized star, because the fundamental region may be concave.)

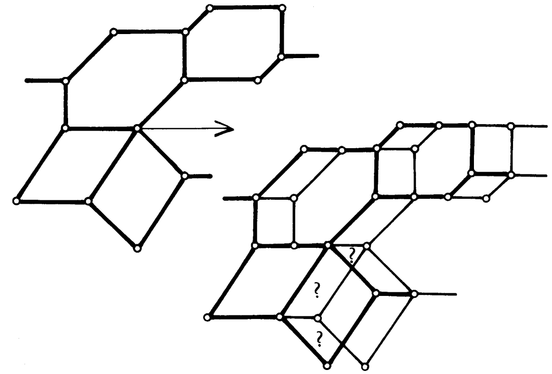


Figure 17. To suggest local valency conditions for the existence of one-parameter families of staggerings.

Some staggerings, such as that in **Figure 16**, are discrete. Others fall into continuous one-parameter families which contain the zero staggering. For this latter type of staggering to exist, certain local valency conditions must be satisfied, as suggested by **Figure 17**.

Lifting

In much of the practical work to date on space filling by concave parallelohedra, work leading to model building and design projects, the step of lifting a staggered pair to a space-filling has been carried out by inspection of the plane drawing, and spatial intuition. A mathematical treatment will hopefully reveal how many free choices are involved in this spatial construction over the given projection, and may establish that spatial realizations exist whose regions are convex. Let's see.

Since each of the residual regions in the staggered pair is a zonagon, we have merely to establish a change in altitude for each family of related segments (pieces of edges of one tessellation, cut off by edges of the other tessellation). This relation of segments, a partition of the segments into parallel families, is generated by the pairing of segments as **opposites** across the various residual zonagons of the staggered pair. Any closed path along the segments is a symmetric difference of zonagonal face cycles. Since altitude changes are chosen to be the same on related segments, the total change of altitude around any zonagonal face is zero, and consequently the total change of altitude around any closed path is zero. Thus if there is some way to choose the altitude changes for each class so that not all the resulting spatial vectors are coplanar, we will always arrive at a consistent assignment of heights for a non-trivial space-filling.

Thus: Most staggered pairs of plane tessellation have a consistent spatial realization as a space-filling, but the regions of the space-filling are not necessarily convex.

The only way we can be prevented from making the spatial vectors noncoplanar is if too many vectors are forced to be coplanar by being together as edges of a hexagonal or higher 2n-gonal face of the staggered pair.

If we insist on convexity of the zonohedral cells of the lifting, a serious problem arises, a problem which must be avoided already at the stage of staggering. Say in one tessellation there is a large 2n-gon, which under staggering becomes subdivided by certain faces of the other tessellation. This subdivision will have to be the top cap of a zonohedron. The staggered image of the large 2n-gon is also subdivided by a portion of the original tessellation, and this subdivision must be a translate of the bottom cap of the zonohedron. But we know that the plane projection of the two caps of a zonohedron differ by a central inversion. See **Figure 18** for an example found by Walter Whiteley. In that example the staggering is not correct. One subdivision of the decagon differs from the central-symmetric image of the other by an incorrect choice of two vertices. This possibility of error imposes a crucial condition on staggerings of tessellations, a condition rather more subtle than those we have considered heretofore. Furthermore, our demand that each tessellation admit at least some non-trivial staggering which lifts with convex cells, carries this condition back to the stage at which the tessellation itself is produced, by splitting. That is (and the following will be very rough, but I believe it points in the right direction), the central symmetry of any zonagon should extend to at least some larger region on opposite sides of the zonagon, so that what lies to the "left" of the zonagon (that which will subdivide the zonagon, when the tessellation is staggered to the "right") is centrally symmetric with that which lies to the "right" of the zonagon (that which subdivides the staggered zonagon).

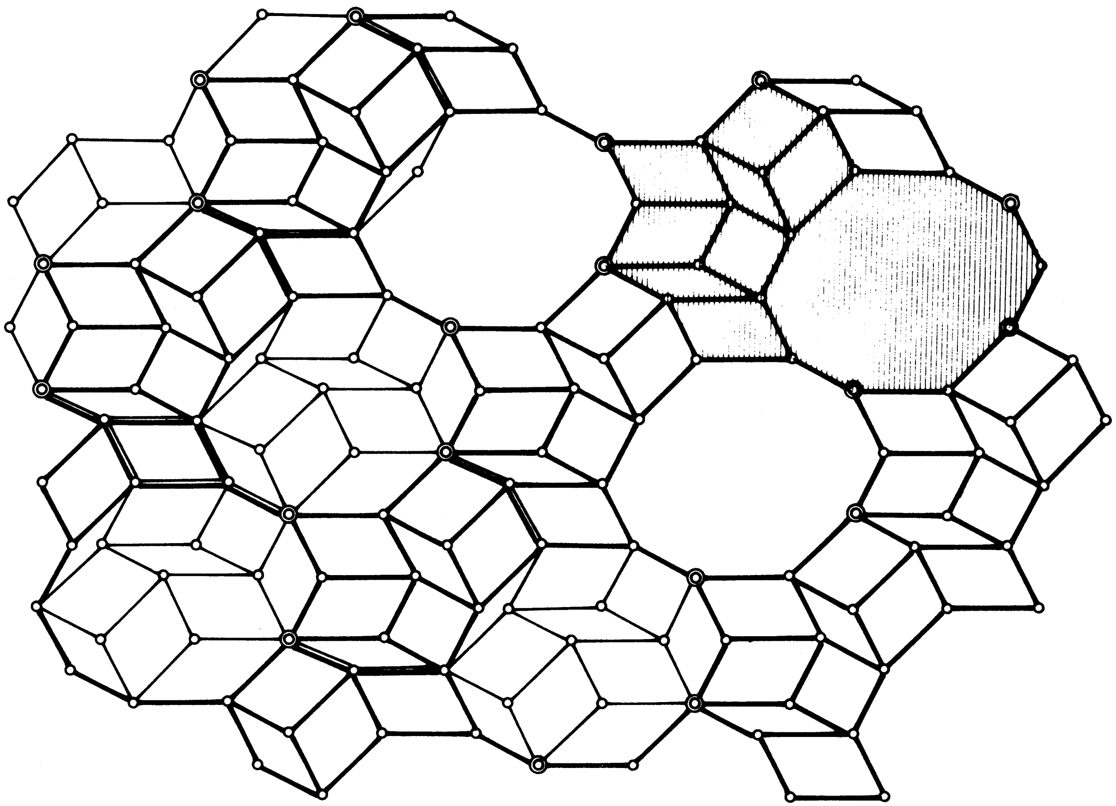


Figure 18. This staggering does not give rise to a space-filling by related zonohedra, because the two subdivisions of the 10-gons are not centrally symmetric to one another.

A further difficulty with lifting to form tilings by related zonahedra has to do with the unrealizable zonahedral drawings (arising from non-stretchable arrangements of pseudo-lines) discussed above. This difficulty is not likely to arise at the level of architectural practice, where a small number of zones will usually suffice, or where slight concavity of cells may not be a problem. But it poses a real obstacle to the mathematically correct formulation of a general method for synthesis of space-fillings. The problem is that a plane tessellation developed in a quite reasonable-looking sequence of splits may, in some staggering, form a grid no lifting of which has all its cells convex. And the reasons for which this is the case need not be evident, even upon close inspection of the plane drawing.

We emphasize the subtlety of these difficulties by giving a final example. In **Figure 19A** we show a correct plane projection of a layer of interfaces in the space-filling by truncated octahedra. We know from the discussion of **Figure 14** that the six directions in this drawing form a quadrilateral set, being the six directions of the edges of a projected tetrahedron. So no one direction in the drawing may be altered without altering some others. In **Figures 19B and 19C** we change the direction d to d', and show how the resulting diagram could easily be obtained by staggering a reasonable-looking tiling by a decagon, two hexagons and three parallelograms. Yet it has no proper spatial interpretation as a surface composed of plane polygons.

There are several attitudes toward this situation, but we think the most positive is seriously to investigate the process of **spatial splitting** of the space-fillings by (convex) parallelohedra. This requires the careful definition of a **grid** of meridian paths, the determination of a possible periodicity of these grids through the space-filling, and a study of the directions in which an opening can be made along the system of grids. In the envelope at the back of (Fejes-Toth 1964) the reader will find a lovely stereoscopic drawing of such a grid, opened up like a slice through a honeycomb. The spatial splitting method has the advantages that we are less likely to arrive at unrealizable configurations, and that more general (in fact all) space-fillings by clusters of related zonohedra become accessible. (Those obtained by the method of planar splitting and staggering are singled out by the fact that they are composed of alternate sheets of interfaces, between which there is only a single layer of zonohedral cells.) Such an approach has the disadvantage that the synthesis of space-fillings must be carried out directly in 3-space, thus forcing the use of models rather than drawings.

In closing, we assure our readers that any correspondence, even the most informal, on any of the many questions here raised concerning space-filling, will be most gratefully received and considered by the members of the Structural Topology research group, whose experimentation and study have provided the basis of this report.

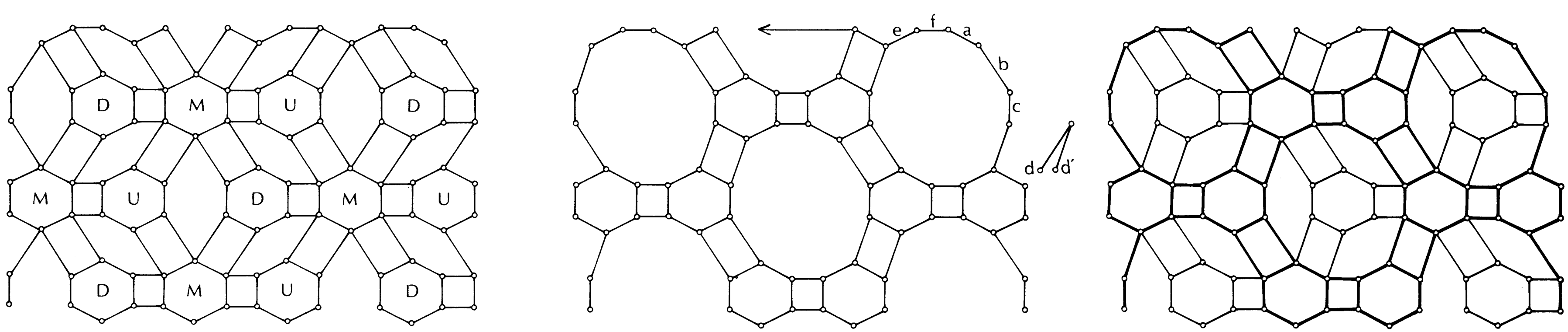


Figure 19. An interface layer of the truncated octahedral space-filling, and a projectively incorrect drawing C which is easily obtained by staggering a plane tiling, but which will not lift to a flat-faced polyhedral surface in space.

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