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SOME RESULTS ENVOLVING THE CONCEPTS OF MOMENT GENERATING FUNCTION AND AFFINITY BETWEEN DISTRIBUTION FUNCTIONS. EXTENSION FOR r $k-$DIMENSIONAL NORMAL DISTRIBUTION FUNCTIONS

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We present a function $\rho\,(F_1,\,F_2,\,t)$ which contains Matusita's affinity and express the «affinity» between moment generating functions. An interesting result is expressed through decomposition of this «affinity» $\rho\,(F_1,\,F_2,\,t)$ when the functions considered are $k-$dimensional normal distributions. The same decomposition remains true for others families of distribution functions. Generalizations of these results are also presented.

Keywords: Affinity, moment generating functions, distance, inner product, multivariate normal distributions, probability density functions, absolutely convergent.

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1. INTRODUCTION AND PRELIMINARIES

Let F_1 and F_2 be two distribution functions defined on $\mathbb{R}$ and let us denote by $f_i(x)$ the probability density function of F_i with respect to a measure m in $\mathbb{R}$, for $i = 1,\, 2$.

We find in the literature several forms of defining distance between distributions of a same class. Matusita (1955) by making use of the distance function denoted by $d\,(F_1,\, F_2)$ and expressed by

$$d\,(F_1,\, F_2) = \left\{ \left(f_1^{1/2}(x) - f_2^{1/2}(x),\; f_1^{1/2}(x) - f_2^{1/2}(x) \right) \right\}^{1/2},$$

introduced in the statistical literature the concept of affinity between the distributions F_1 and F_2 denoted by $\rho_2\,(F_1,\, F_2)$ and defined by

$$\rho_2\,(F_1,\, F_2) = \left(f_1^{1/2}(x),\; f_2^{1/2}(x) \right)$$

which is related to $d\,(F_1,\, F_2)$ through the expression

$$d^2\,(F_1,\, F_2) = 2\,\{1 - \rho_2\,(F_1,\, F_2)\}$$

where $(f,\, g)$ denotes the inner product of $f(x)$ and $g(x)$ defined by:

$$(f,\, g) = \int_{\mathbb{R}} f(x)\, g(x)\, d\, m$$

The importance and usefulness of the notions of distance and affinity between distributions, in statistics, were stressed in a series of papers by Matusita (1954, 1955, 1956, 1961, 1964, 1967b, 1973), Matusita & Motoo (1955), Matusita & Akaike (1956), Khan & Ali (1971) and others.

Concrete expressions for the affinity between two multivariate normal distributions were established by Matusita (1966). As a following step, Matusita (1967) extended the notion of affinity to cover the case where there are r distributions involved and established concrete expressions when the r distributions are $k-$dimensional normal.

Our work is characterized by the introduction of the concept of a function, denoted by $P(t)$, functionally expressed through the moment generating functions relative to the distributions considered and another expression denoted by $\rho\,(F_1,\, F_2,\, t)$ that contains as a particular case the affinity between the distribution functions F_1 and F_2, or in other words, express the «affinity» between the moment generating functions relative to F_1 and F_2. We also present a result that express the decomposition of $\rho(F_1,\, F_2,\, \underset{\sim}{t})$ in a product of two factors identified as the affinity and the moment generating function when F_1 and F_2 are $k-$dimensional normal distributions. This result is extended to cover the case where there are r $k-$dimensional normal distributions.

In this same way, the concept of a more general function $D_r(s_1, \ldots, s_r; \underset{\sim j}{t})$ is introduced, and the results obtained through it contains those developed in this paper as those ones established by Matusita (1966, 1967a) and Campos (1978).

2. RESULTS

Definition 1. Let F_1 and F_2 be two distribution functions belonging to the same class and let $f_1(x)$ and $f_2(x)$ their respective probability density functions with respect to a measure m defined on $\mathbb{R}$. Let us suppose that there is a scalar t, $-h \leqslant t \leqslant h$ $(h > 0)$ such that the integral below, defined through the inner product, is absolutely convergent.

Now, we define:
(2.1)
$$P(t) = \left(\exp\{tx/2\}\left\{f_1^{1/2}(x) - f_2^{1/2}(x)\right\}, \exp\{tx/2\}\left\{f_1^{1/2}(x) - f_2^{1/2}(x)\right\}\right)$$

where (f, g) denotes the inner product of $f(x)$ and $g(x)$, defined by:

$$(f, g) = \int_{\mathbb{R}} f(x)\, g(x)\, d\,m$$

From (2.1), we obtain:

$$P(t) = M_1(t) + M_2(t) - 2\,\rho\,(F_1, F_2, t)$$

where:

$M_i(t)$ represent the moment generating function for the distribution F_i whose probability density function is $f_i(x)$, $i = 1, 2$;

and

(2.2) $$\rho\,(F_1, F_2, t) = \left(\exp\{tx/2\}\ f_1^{1/2}(x),\ \exp\{tx/2\}\ f_2^{1/2}(x)\right)$$

From (2.1) and (2.2) we verify that:

i) $P(0) = d^2\,(F_1, F_2)$,
ii) $F_1 = F_2$ implies $P(t) = 0$ for all $-h \leqslant t \leqslant h$
iii) $\rho\,(F_1, F_2, 0) = \rho_2\,(F_1, F_2)$
iv) $F_1 = F_2 = F$ implies $\rho(F, t) = M(t)$

where:
$$d^2\,(F_1, F_2) = \left(f_1^{1/2}(x) - f_2^{1/2}(x),\ f_1^{1/2}(x) - f_2^{1/2}(x)\right)$$

and $\rho_2\,(F_1,\,F_2)$ is the affinity between the distributions F_1 and F_2 as defined by Matusita (1966).

Teorema 1. Let F_1 and F_2 be $k-$dimensional nonsingular normal distributions, whose probability density functions are given by:

$$(2\,\pi)^{-k/2}\,|\underset{\sim}{A}|^{1/2}\,\exp\Big\{-1/2(\underset{\sim}{A}^{-1}(\underset{\sim}{x}-\underset{\sim}{a}),\,\underset{\sim}{x}-\underset{\sim}{a})\Big\}$$

and

$$(2\,\pi)^{-k/2}\,|\underset{\sim}{B}|^{-1/2}\,\exp\Big\{-1/2(\underset{\sim}{B}^{-1}(\underset{\sim}{x}-\underset{\sim}{b}),\,\underset{\sim}{x}-\underset{\sim}{b})\Big\}\,,$$

respectively, where:

$\underset{\sim}{x}$ is a $k-$dimensional (column vector);
$\underset{\sim}{A}$ and $\underset{\sim}{B}$ are covariance matrices de degree K and
$\underset{\sim}{a}, \underset{\sim}{b}$ are $k-$dimensional mean (column) vectors.

In these conditions, we have:

$$\rho(F_1,\,F_2,\underset{\sim}{t}) = \rho_2\,(F_1,\,F_2)\cdot M_G(\underset{\sim}{t})$$

where:

$\underset{\sim}{t}$ is $k-$dimensional (column) vector and
$M_G(\underset{\sim}{t})$ is the moment generating function of a $k-$dimensional normal distribution with mean vector $\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\underset{\sim}{a}+\underset{\sim}{B}^{-1}\,\underset{\sim}{b})$ and covariance matrix $2\,\underset{\sim}{C}^{-1}$, given by

(2.3) $$M_G(\underset{\sim}{t}) = \exp\Big\{(\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\,\underset{\sim}{a}+\underset{\sim}{B}^{-1}\,\underset{\sim}{b}),\,\underset{\sim}{t}) + (\underset{\sim}{C}^{-1}\,\underset{\sim}{t},\,\underset{\sim}{t})\Big\}$$

with $\underset{\sim}{C} = \underset{\sim}{A}^{-1}+\underset{\sim}{B}^{-1}$.

Proof: From (2.2), we have:

(2.4) $$\rho(F_1,\,F_2,\underset{\sim}{t}) = (2\,\pi)^{-k/2}|\underset{\sim}{A}\underset{\sim}{B}|^{-1/4}\int_{\mathbb{R}^k}\exp\{-1/4\,Q\}\;d\,x_1,\,\ldots,\,d\,x_k$$

where:

$$Q = (\underset{\sim}{A}^{-1}(\underset{\sim}{x}-\underset{\sim}{a}),\,\underset{\sim}{x}-\underset{\sim}{a}) + (\underset{\sim}{B}^{-1}(\underset{\sim}{x}-\underset{\sim}{b}),\,\underset{\sim}{x}-\underset{\sim}{b}) - 4(\underset{\sim}{x},\,,\underset{\sim}{t})$$

By working with this algebraic sum of inner products, we obtain:

(2.5) $Q = ((\underset{\sim}{A}^{-1} + \underset{\sim}{B}^{-1})\, \underset{\sim}{x}, \underset{\sim}{x}) - 2(\underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\,\underset{\sim}{t}, \underset{\sim}{x}) + (\underset{\sim}{A}^{-1}\underset{\sim}{a}, \underset{\sim}{a}) + (\underset{\sim}{B}^{-1}\,\underset{\sim}{b}, \underset{\sim}{b})$

If we define the transformation:

$$\underset{\sim}{y} = \underset{\sim}{C}^{1/2}\,\underset{\sim}{x}$$

with $\underset{\sim}{C} = \underset{\sim}{A}^{-1} + \underset{\sim}{B}^{-1}$ and Jacobian equal to $\mod |\underset{\sim}{C}|^{-1/2}$, (2.5) may be written as follows:

$$Q = (\underset{\sim}{y},\, \underset{\sim}{y}) - 2(\underset{\sim}{C}^{-1/2}(\underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\underset{\sim}{t}),\, \underset{\sim}{y}) + (\underset{\sim}{A}^{-1}\underset{\sim}{a}, \underset{\sim}{a}) + (\underset{\sim}{B}^{-1}\underset{\sim}{b},\, \underset{\sim}{b})$$

That is,

(2.6) $Q = Q_1 - (\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\underset{\sim}{t}), \underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\underset{\sim}{t}) + (\underset{\sim}{A}^{-1}\underset{\sim}{a}, \underset{\sim}{a}) + (B^{-1}\underset{\sim}{b}, \underset{\sim}{b})$

where:

$$Q_1 = (\underset{\sim}{y} - \underset{\sim}{C}^{-1/2}(\underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\underset{\sim}{t}),\, \underset{\sim}{y} - \underset{\sim}{C}^{-1/2}(\underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\underset{\sim}{t}))$$

We have also:

$$(\underset{\sim}{A}^{-1}\,\underset{\sim}{a}, \underset{\sim}{a}) = (\underset{\sim}{A}^{-1}\underset{\sim}{a},\, \underset{\sim}{C}^{-1}\,\underset{\sim}{A}^{-1}\,\underset{\sim}{a}) + (\underset{\sim}{A}^{-1}\,\underset{\sim}{a},\, \underset{\sim}{C}^{-1}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{a})$$

and

$$(\underset{\sim}{B}^{-1}\,\underset{\sim}{b}, \underset{\sim}{b}) = (\underset{\sim}{B}^{-1}\underset{\sim}{b},\, \underset{\sim}{C}^{-1}\,\underset{\sim}{A}^{-1}\,\underset{\sim}{b}) + (\underset{\sim}{B}^{-1}\,\underset{\sim}{b},\, \underset{\sim}{C}^{-1}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{b})$$

By using these results in (2.6), we obtain, after same algebraic manipulations:

(2.7)
$$\begin{aligned} Q \;=\;& Q_1 - (\underset{\sim}{C}^{-1}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{b},\, \underset{\sim}{A}^{-1}\,\underset{\sim}{a}) - (\underset{\sim}{C}^{-1}\,\underset{\sim}{A}^{-1}\,\underset{\sim}{a},\, \underset{\sim}{B}^{-1}\,\underset{\sim}{b}) + (\underset{\sim}{A}^{-1}\,\underset{\sim}{a},\, \underset{\sim}{C}^{-1}\,\underset{\sim}{B}^{-1}\,\underset{\sim}{a}) + \\ & (\underset{\sim}{B}^{-1}\underset{\sim}{b}, \underset{\sim}{C}^{-1}\underset{\sim}{A}^{-1}\underset{\sim}{b}) + (\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}a + \underset{\sim}{B}^{-1}\underset{\sim}{b} + 2\underset{\sim}{t}), 2\underset{\sim}{t}) - (2\,\underset{\sim}{C}^{-1}\underset{\sim}{t},\, \underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\underset{\sim}{b}) \end{aligned}$$

From (2.7), it follows that:

$$\begin{aligned} Q \;=\;& Q_1 + ((\underset{\sim}{B}\,\underset{\sim}{C}\,\underset{\sim}{A})^{-1}\underset{\sim}{b},\, \underset{\sim}{b} - \underset{\sim}{a}) - ((\underset{\sim}{A}\,\underset{\sim}{C}\,\underset{\sim}{B})^{-1}\underset{\sim}{a},\, \underset{\sim}{b} - \underset{\sim}{a}) - \\ - \;& 4(\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\underset{\sim}{a} + \underset{\sim}{B}^{-1}\,\underset{\sim}{b}),\, \underset{\sim}{t}) - 4(\underset{\sim}{C}^{-1}\,\underset{\sim}{t},\, \underset{\sim}{t}) \end{aligned}$$

Since $\underset{\sim}{C} = \underset{\sim}{A}^{-1} + \underset{\sim}{B}^{-1}$, we have:

$$\underset{\sim}{A}\,\underset{\sim}{C}\,\underset{\sim}{B} = \underset{\sim}{B}\,\underset{\sim}{C}\,\underset{\sim}{A} = \underset{\sim}{A} + \underset{\sim}{B}$$

Or,

(2.8) $Q = Q_1 + ((\underset{\sim}{A}+\underset{\sim}{B})^{-1}(\underset{\sim}{b}-\underset{\sim}{a}),\, \underset{\sim}{b}-\underset{\sim}{a}) - 4(\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\underset{\sim}{a}+\underset{\sim}{B}^{-1}\,\underset{\sim}{b}),\, \underset{\sim}{t}) - 4(\underset{\sim}{C}^{-1}\,\underset{\sim}{t},\, \underset{\sim}{t})$

Using (2.8) in (2.4), we obtain:

(2.9)

$$\begin{aligned} \rho(F_1,\, F_2, \underset{\sim}{t}) \;\; &= \;\; (2\pi)^{-k/2} |\underset{\sim}{A}\underset{\sim}{B}|^{-1/4} \exp\left\{ -1/4((\underset{\sim}{A}+\underset{\sim}{B})^{-1}\,(\underset{\sim}{b}-\underset{\sim}{a}),\, \underset{\sim}{b}-\underset{\sim}{a}) \right\} \\ & \exp\left\{ (\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\,\underset{\sim}{a}+\underset{\sim}{B}^{-1}\,\underset{\sim}{b}),\, \underset{\sim}{t}) + (\underset{\sim}{C}^{-1}\,\underset{\sim}{t},\, \underset{\sim}{t}) \right\} \\ \cdot \quad & \int_{\mathbb{R}^k} \exp\left\{ -\frac{1}{4}\, Q_1 \right\} |\underset{\sim}{C}|^{-1/2}\, d\,y_1,\, \ldots,\, d\,y_k \end{aligned}$$

We easily verify that:

$$\int_{\mathbb{R}^k} \exp\left\{ -\frac{1}{4}\, Q_1 \right\} d\,y_1,\, \ldots,\, d\,y_k = 2^{k/2}\,(2\pi)^{k/2}$$

Or,

(2.10)

$$\begin{aligned} \rho(F_1,\, F_2,\, \underset{\sim}{t}) \;\; &= \;\; |\underset{\sim}{A}\underset{\sim}{B}|^{1/4} \left| \frac{1}{2}(\underset{\sim}{A}+\underset{\sim}{B}) \right|^{-1/2} \exp\left\{ -1/4((\underset{\sim}{A}+\underset{\sim}{B})^{-1}\,(\underset{\sim}{b}-\underset{\sim}{a}),\, \underset{\sim}{b}-\underset{\sim}{a}) \right\} \cdot \\ \cdot \quad & \exp\left\{ (\underset{\sim}{C}^{-1}(\underset{\sim}{A}^{-1}\,\underset{\sim}{a}+\underset{\sim}{B}^{-1}\,\underset{\sim}{b}),\, \underset{\sim}{t}) + (\underset{\sim}{C}^{-1}\,\underset{\sim}{t},\, \underset{\sim}{t}) \right\} \end{aligned}$$

since

$$|\underset{\sim}{C}|^{-1/2} = |\underset{\sim}{A}|^{1/2}\,|\underset{\sim}{A}+\underset{\sim}{B}|^{-1/2}\,|\underset{\sim}{B}|^{1/2}$$

It follows from theorem demonstrated by Matusita and (2.3) that (2.10) may be written as:

$$\rho(F_1,\, F_2,\, \underset{\sim}{t}) = \rho_2(F_1,\, F_2)\, M_G(\underset{\sim}{t})$$

Corolary 1. When $\underset{\sim}{A} = \underset{\sim}{B}$ it follows that

$$\rho(F_1,\, F_2, \underset{\sim}{t}) = \exp\left\{ -1/8(\underset{\sim}{A}^{-1}(\underset{\sim}{b}-\underset{\sim}{a}), \underset{\sim}{b}-\underset{\sim}{a}) \right\} M_G(\underset{\sim}{t})$$

where:

$$M_G(\underset{\sim}{t}) = \exp\left\{(1/2(\underset{\sim}{a}+\underset{\sim}{b}),\, \underset{\sim}{t}) + 1/2(\underset{\sim}{A}\,\underset{\sim}{t},\, \underset{\sim}{t})\right\}$$

Corolary 2. If $\underset{\sim}{a} = \underset{\sim}{b}$ it follows that:

$$\rho(F_1,\, F_2, \underset{\sim}{t}) = \left|\underset{\sim}{A}\,\underset{\sim}{B}\right|^{1/4} \left|\frac{1}{2}\left(\underset{\sim}{A}+\underset{\sim}{B}\right)\right|^{-1/2} \exp\left\{(\underset{\sim}{a}, \underset{\sim}{t}) + \left(\underset{\sim}{C}^{-1}\,\underset{\sim}{t}, \underset{\sim}{t}\right)\right\}$$

Corolary 3. For $F_1 = F_2 = F$, we have:

$$\rho(F,\, \underset{\sim}{t}) = \exp\left\{(\underset{\sim}{a},\, \underset{\sim}{t}) + 1/2(\underset{\sim}{A}\,\underset{\sim}{t},\, \underset{\sim}{t})\right\} = M_X(\underset{\sim}{t})$$

where $M_X(\underset{\sim}{t})$ is the moment generating function of F.

The conclusion (result) of theorem 1 is naturally generalized for r $k-$dimensional normal distributions. To accomplish this objective, we first generalize the concept of $\rho(F_1,\, F_2,\, \underset{\sim}{t})$, by considering r distributions $F_1,\, \ldots,\, F_r$, defined over the same space $\mathbb{R}$, with probability density functions $f_1(x),\, \ldots,\, f_r(x)$ with respect to a measure on $\mathbb{R}$, and let us suppose that the integral below be absolutely convergent. Then we define:

Definition 2

$$\rho(F_1,\, \ldots,\, F_r,\, t) = \int_{\mathbb{R}} \left\{\prod_{j=1}^{r} \exp(tx)\, f_j(x)\right\}^{1/r} dm$$

If $F_1,\, \ldots,\, F_r$ denote r $k-$ dimensional non singular normal distributions whose probability density functions are given for $j = 1,\, \ldots,\, r$ by

(2.11) $$(2\pi)^{-k/2}\, |\underset{\sim}{A}_j|^{-1/2} \exp\left\{-1/2(\underset{\sim}{A}_j^{-1}(\underset{\sim}{x} - \underset{\sim}{a}_j),\, \underset{\sim}{x} - \underset{\sim}{a}_j)\right\}$$

where $\underset{\sim}{A}_j$ is the covariance matrix and $\underset{\sim}{a}_j$ the mean vector of F_j, respectively, we have the following result whose proof we omit:

Theorem 2

$$\rho(F_1,\, \ldots,\, F_r,\, \underset{\sim}{t}) = \rho_r(F_1,\, \ldots,\, F_r)\, M_G(\underset{\sim}{t})$$

where

$$\rho_r(F_1, \ldots, F_r) = \left\{ \prod_{j=1}^{r} |\underset{\sim}{A}_j|^{-1/2r} \right\} \left| \frac{1}{r} \sum_{j=1}^{r} \underset{\sim}{A}_j^{-1} \right|^{-1/2}$$

$$\cdot \quad \exp \left\{ -1/2r \left\{ \sum_{\substack{j=2 \\ l \leqslant i < j}}^{r} ((\underset{\sim}{A}_j \underset{\sim}{D} \underset{\sim}{A}_i)^{-1} \underset{\sim}{a}_j, \underset{\sim}{a}_j - \underset{\sim}{a}_i - \right.\right.$$

$$\left.\left. - \sum_{\substack{i=1 \\ i < j \leqslant r}}^{r-1} ((\underset{\sim}{A}_i \underset{\sim}{D} \underset{\sim}{A}_j)^{-1} \underset{\sim}{a}_i, \underset{\sim}{a}_j - \underset{\sim}{a}_i) \right\} \right\}$$

is the affinity between r $k-$dimensional normal (non singular) distributions obtained by Matusita (1967a) and expressed in another form by Campos (1978); and

$$M_G(\underset{\sim}{t}) = \exp \left\{ \left(\underset{\sim}{D}^{-1} \left(\sum_{j=1}^{r} \underset{\sim}{A}_j^{-1} \underset{\sim}{a}_j \right), \underset{\sim}{t} \right) + \frac{1}{2} \left(r \, \underset{\sim}{D}^{-1} \underset{\sim}{t}, \underset{\sim}{t} \right) \right\}$$

is the moment generating functions of a $k-$dimensional normal distribution with mean vector $\underset{\sim}{D}^{-1} \left(\sum_{j=1}^{r} \underset{\sim}{A}_j^{-1} \underset{\sim}{a}_j \right)$ and covariance matrix $r\,\underset{\sim}{D}^{-1}$ with $\underset{\sim}{D} = \sum_{j=1}^{r} \underset{\sim}{A}_j^{-1}$ and $\underset{\sim}{t}$ a $k-$dimensional (column) vector.

With the objective of generalizing these results, as those obtained by Matusita (1966, 1967a) or Campos (1978) we introduce the following definition.

Definition 3. Let $F_1, \ldots, F_r$ be multivariate distributions defined on the same space $\mathbb{R}$ and let $f_1(x), \ldots, f_r(x)$ be their respective probability density functions. Let us suppose that there are r scalars $s_1, \ldots, s_r$ such that:

$$\sum_{j=1}^{r} s_j = 1 \quad \text{and} \quad 0 \leqslant s_j \leqslant 1 \quad \text{for} \quad j = 1, \ldots, r.$$

In these conditions, and if the integral below is absolutely convergent, we define:

$$D_r(s_1, \ldots, s_r, \underset{\sim}{t}_j) = \int_{\mathbb{R}^k} \prod_{j=1}^{r} \exp\left\{ s_j(\underset{\sim}{x}, \underset{\sim}{t}_j) \right\} f_j^{s_j}(x)\, d\,x_1 \ldots d\,x_k$$

where $\underset{\sim}{t}_j$ is a $k-$dimensional (column) vector with components $t_{j_1}, \ldots, t_{j_k}, j = 1, \ldots, r$.

If $F_1, \ldots, F_r$ denote $k-$dimensional normal (non singular) distributions defined as (2.11) we establish the following result:

Theorem 3

(2.12) $$D_r(s_1, \ldots, s_r; \underset{\sim}{t}_j) = D_r(s_1, \ldots, s_r)\; M_G\left(\sum_{j=1}^{r} s_j\, \underset{\sim}{t}_j\right)$$

where:

(2.13)
$$\begin{aligned} D_r(s_1, \ldots, s_r) \;=\; & \left\{\prod_{j=1}^{r} |\underset{\sim}{A}_j|^{-s_j/2}\right\} \left|\sum_{j=1}^{r} s_j\, \underset{\sim}{A}_j^{-1}\right|^{-1/2} \\ & \cdot \exp\left\{-1/2\left\{\sum_{\substack{j=2 \\ l \leqslant i < j}}^{r} (s_i\, s_j (\underset{\sim}{A}_j\, \underset{\sim}{C}\, \underset{\sim}{A}_i)^{-1}\, \underset{\sim}{a}_j, \underset{\sim}{a}_j - \underset{\sim}{a}_i) - \right.\right. \\ & \left.\left. - \sum_{\substack{i=1 \\ i < j \leqslant r}}^{r} (s_i\, s_j (\underset{\sim}{A}_i\, \underset{\sim}{C}\, \underset{\sim}{A}_j)^{-1}\, \underset{\sim}{a}_i, \underset{\sim}{a}_j - \underset{\sim}{a}_i)\right\}\right\} \end{aligned}$$

and $M_G\left(\sum_{j=1}^{r} s_j\, \underset{\sim}{t}_j\right)$ is the moment generating function for a $k-$dimensional normal distribution with mean vector $\underset{\sim}{C}^{-1}\left(\sum_{j=1}^{r} s_j\, \underset{\sim}{A}_j^{-1}\, \underset{\sim}{a}_j\right)$ and covariance matrix $\underset{\sim}{C}^{-1}$, expressed by:

(2.14)
$$\begin{aligned} M_G\left(\sum_{j=1}^{r} s_j\, \underset{\sim}{t}_j\right) \;=\; & \exp\left\{\left(\underset{\sim}{C}^{-1}\left(\sum_{j=1}^{r} s_j\, \underset{\sim}{A}_j^{-1}\, \underset{\sim}{a}_j\right), \sum_{j=1}^{r} s_j\, \underset{\sim}{t}_j\right) + \right. \\ & \left. + \left(1/2\, \underset{\sim}{C}^{-1}\left(\sum_{j=1}^{r} s_j\, \underset{\sim}{t}_j\right), \sum_{j=1}^{r} s_j\, \underset{\sim}{t}_j\right)\right\} \end{aligned}$$

with $\underset{\sim}{C} = \sum_{j=1}^{r} s_j \underset{\sim}{A}_j^{-1}$.

Proof: By the definition 3, we have:

(2.15)
$$D_r(s_1, \ldots, s_r; \underset{\sim}{t}_j) = (2\pi)^{-k/2} \prod_{j=1}^{r} |\underset{\sim}{A}_j|^{-s_j/2} \int_{\mathbb{R}^k} \exp\left\{-\frac{1}{2} Q\right\} d\,x_1 \ldots d\,x_k$$

where

$$Q = \sum_{j=1}^{r} s_j \left(\underset{\sim}{A}_j^{-1} (\underset{\sim}{x} - \underset{\sim}{a}_j), \, \underset{\sim}{x} - \underset{\sim}{a}_j \right) - 2 \sum_{j=1}^{r} s_j (\underset{\sim}{x}, \, \underset{\sim}{t}_j)$$

That is,

$$Q = (\underset{\sim}{C}\, \underset{\sim}{x}, \, \underset{\sim}{x}) - 2(\underset{\sim}{b}, \, \underset{\sim}{x}) - 2(\underset{\sim}{t}, \, \underset{\sim}{x}) + \sum_{j=1}^{r} \left(s_j \underset{\sim}{A}_j^{-1} \underset{\sim}{a}_j, \, \underset{\sim}{a}_j \right) \tag{2.16}$$

with

$$\underset{\sim}{b} = \sum_{j=1}^{r} s_j \underset{\sim}{A}_j^{-1} \underset{\sim}{a}_j$$

and

$$\underset{\sim}{t} = \sum_{j=1}^{r} s_j \underset{\sim}{t}_j$$

After the transformation

$$\underset{\sim}{y} = \underset{\sim}{C}^{1/2} \underset{\sim}{x}$$

whose Jacobian is $\mod |\underset{\sim}{C}|^{-1/2}$ and same intermediate steps, (2.16) may be written

$$\begin{aligned} Q \;&=\; \left(\underset{\sim}{y} - \underset{\sim}{C}^{-1/2}(\underset{\sim}{b} + \underset{\sim}{t}), \, \underset{\sim}{y} - \underset{\sim}{C}^{-1/2}(\underset{\sim}{b} + \underset{\sim}{t}) \right) - \left(\underset{\sim}{C}^{-1} \underset{\sim}{b}, \, \underset{\sim}{t} \right) - \\ &-\; \left(\underset{\sim}{C}^{-1} \underset{\sim}{t}, \, \underset{\sim}{b} \right) - \left(\underset{\sim}{C}^{-1} \underset{\sim}{t}, \, \underset{\sim}{t} \right) + \sum_{j=1}^{r} \left(s_j \underset{\sim}{A}_j^{-1} \underset{\sim}{a}_j, \, \underset{\sim}{a}_j \right) - \left(\underset{\sim}{C}^{-1} \underset{\sim}{b}, \, \underset{\sim}{b} \right) \end{aligned} \tag{2.17}$$

One may also prove that:

$$\sum_{j=1}^{r}\left(s_j \underset{\sim}{A}_j^{-1}\underset{\sim}{a}_j,\ \underset{\sim}{a}_j\right)-\left(\underset{\sim}{C}^{-1}\underset{\sim}{b},\ \underset{\sim}{b}\right)=\sum_{\substack{j\,=\,2\\ l\,\leqslant\, i\,<\,j}}^{r}\left(s_i\, s_j(\underset{\sim}{A}_j\,\underset{\sim}{C}\,\underset{\sim}{A}_i)^{-1}\underset{\sim}{a}_j,\ \underset{\sim}{a}_j-\underset{\sim}{a}_i\right)-$$

$$-\sum_{\substack{i\,=\,1\\ i\,<\,j\,\leqslant\, r}}^{r-1}\left(s_i\, s_j(\underset{\sim}{A}_i\,\underset{\sim}{C}\,\underset{\sim}{A}_j)^{-1}\underset{\sim}{a}_i,\ \underset{\sim}{a}_j-\underset{\sim}{a}_i\right)$$

On applying this result, (2.17) is expressed as:

$$Q=Q_3+Q_1-Q_2-2\left(\underset{\sim}{C}^{-1}\underset{\sim}{b},\ \underset{\sim}{t}\right)-\left(\underset{\sim}{C}^{-1}\underset{\sim}{t},\ \underset{\sim}{t}\right) \tag{2.18}$$

where:

$$Q_3 = \left(\underset{\sim}{y}-\underset{\sim}{C}^{-1/2}(\underset{\sim}{b}+\underset{\sim}{t}),\ \underset{\sim}{y}-\underset{\sim}{C}^{-1/2}(\underset{\sim}{b}+\underset{\sim}{t})\right)$$

$$Q_1 = \sum_{\substack{j\,=\,2\\ l\,\leqslant\, i\,<\,j}}^{r}\left(s_i\, s_j(\underset{\sim}{A}_j\,\underset{\sim}{C}\,\underset{\sim}{A}_i)^{-1}\underset{\sim}{a}_j,\ \underset{\sim}{a}_j-\underset{\sim}{a}_i\right) \quad \text{and}$$

$$Q_2 = \sum_{\substack{i\,=\,1\\ i\,<\,j\,\leqslant\, r}}^{r-1}\left(s_i\, s_j(\underset{\sim}{A}_i\,\underset{\sim}{C}\,\underset{\sim}{A}_j)^{-1}\underset{\sim}{a}_i,\ \underset{\sim}{a}_j-\underset{\sim}{a}_i\right)$$

By substitution of (2.18) in (2.15), we obtain:
(2.19)

$$\begin{aligned} D_r(s_1,\ \ldots,\ s_r;\ \underset{\sim}{t}_j) &= (2\pi)^{-k/2}\prod_{j=1}^{r}|\underset{\sim}{A}_j|^{-s_j/2}\cdot \\ &\cdot\ \exp\left\{-\frac{1}{2}(Q_1-Q_2)\right\}\exp\left\{\left(\underset{\sim}{C}^{-1}\underset{\sim}{b},\ \underset{\sim}{t}\right)+\left(\frac{1}{2}\,\underset{\sim}{C}^{-1}\underset{\sim}{t},\ \underset{\sim}{t}\right)\right\}\cdot \\ &\cdot\ \int_{\mathbb{R}^k}\exp\left\{-\frac{1}{2}\,Q_3\right\}|\underset{\sim}{C}|^{-1/2}\,d\,y_1\ \ldots\ d\,y_k \end{aligned}$$

The integral of (2.19) after the transformation

$$\underset{\sim}{z}=\underset{\sim}{y}-\underset{\sim}{C}^{-1/2}(\underset{\sim}{b}+\underset{\sim}{t})$$

becomes equal

$$|\underset{\sim}{C}|^{-1/2}(2\pi)^{k/2}$$

By using this result in (2.19), we obtain, in accord with (2.13) and (2.14) the result that we have established through theorem 3, that is,

$$D_r(s_1, \ldots, s_r; \underset{\sim}{t}_j) = D_r(s_1, \ldots, s_r) \cdot M_G \left(\sum_{j=1}^{r} s_j \underset{\sim}{t}_j \right)$$

This result is a generalization in the sense of summarize the results established through theorems 1 and 2 as those obtained by Matusita (1966, 1967a) or Campos (1978).

REFERENCES

Ahmad, I.A. (1980). «Nonparametric estimation of an affinity measure between two absolutely continuous distributions with hypotheses testing applications». *Annals of the Institute of Statistical Mathematics*, 32, 223-240.

Ahmad, I.A. (1980). «Nonparametric estimation of Matusita's measure of affinity between absolutely continuous distributions». *Annals of the Institute of Statistical Mathematics*, 32, 241-246.

Campos, A.D. (1978). «Afinidade entre distribuições normais multivariadas». *Atas do 3.º Simpósio Nacional de Probabilidade e Estatística* (ed. IME-USP), 75-79.

Cuadras, C.M. (1988). «Statistical distances». *Estadística Española*, 119, 295-378.

Cuadras, C.M. (1989). «Distance analysis in discrimination and classification using both continous and categorical variables». In *Statistical Data Analysis and Inference*, (Y. Dodge, ed.), Elsevier, Amsterdam, 459-473.

Cuadras, C.M. and Fortiana, J. (1993). «Applying distances in statistics». *Qüestiió*, 17, 39-74.

Khan, A.H. & Ali, S.M. (1971). «A new coefficient of association». *Annals of the Institute of Statistical Mathematics*, 23, 41-50.

Krzanowski, W.J. (1983). «Distance between populations using mixed continuous and categorical variables». *Biometrika*, 70, 235-243.

Matusita, K. (1954). «On the estimation by the minimum distance method». *Annals of the Institute of Statistical Mathematics*, 5, 59-65.

Matusita, K. (1955). «Decision rules based on the distance for problems of fit, two samples and estimation». *Annals of the Institute of Statistical Mathematics*, 26, 631-640.

Matusita, K. (1956). «Decision rule based on the distance for the classification problem». *Annals of the Institute of Statistical Mathematics*, 8, 67-77.

Matusita, K. (1961). «Interval estimation based on the notion of affinity». *Bulletin of the International Statistical Institute*, 38, 4, 241-244.

Matusita, K. (1964). «Distance and decision rules». *Annals of the Institute of Statistical Mathematics*, 16, 305-315.

Matusita, K. (1966). «A distance and related statistics in multivariate analysis». *Multivariate Analysis - Proceedings of an International Symposium* (ed. P.R. Krishnaiah). Academic Press New York, 187-200.

Matusita, K. (1967a). «On the notion of affinity of several distributions and some of its applications». *Annals of the Institute of Statistical Mathematics*, 19, 181-192.

Matusita, K. (1967b). «Classification based on distance in multivariate Gaussian cases». *Proceedings of the Fifth Berkeley Simposium on Mathematical Statistics and Probability*. University California Press, 1, 299-304.

Matusita, K. (1973). «Correlation and affinity in Gaussian cases». *Multivariate Analysis - III - Proceedings of the Third International - Symposium* (ed. P.R. Krishnaiah), Academic Press, New Yor, 345-349.

Matusita, K. (1973). «Discrimination and the affinity of distributions». In: *Discriminant Analysis and Applications*. (T. Cacoullos, ed.), Academic Press, N.Y., 213-223.

Matusita, K. (1977). «Cluster analysis and affinity of distributions. Recent Developments in Statistics». *Proceedings of the 1976 European Meeting of Statisticians*, 537-544.

Matusita, K. & Motoo, M. (1955). «On the fundamental theorem for the decision rule based distance II.»*Annals of the Institute of Statistical Mathematics*, 7, 137-142.

Matusita, K. & Akaike, H. (1956). «Decision rules based on the distance for the problems of independence, invariance and two samples». *Annals of the Institute of Statistical Mathematics*, 7, 67-80.