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AMPLITUDE OF WEIGHTED MAJORITY GAME STRICT REPRESENTATIONS

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Some real situations which may be described as weighted majority games can be modified when some players increase or decrease their weights and/or the quota is modified. Nevertheless, some of these modifications do not change the game. In the present work we shall estimate the maximal percentage variations in the weights and the quota which may be allowed without changing the game (amplitude). For this purpose, we have to use strict representations of weighted majority games.

Keywords: Simple games, weighted majority games, strict representations, linearly separable function, tolerance, amplitude.

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1. INTRODUCTION

Shareholder societies, political models and even electronical applications can be described by means of weighted majority games, which can be modified if the weights and/or the quota which define them are modified.

For instance, the increase of capital in a shareholder society makes investors to increase or decrease their shares, at the same time that it provokes a change of the quota to adapt it to the new situation, previous consensus. To assure that the new distribution does not interfere in the fight for the control of the company it is necessary to estimate the maximum percentage in the variations of weights and quota which leave invariant the game associated to the initial situation.

Analogously, in the realization of a linearly separable switching function by means of an electronic device, the components used to fix the weights and the threshold cannot be completely accurate. Hence, in determining the required accuracy of these components, it is necessary to estimate the maximum percentage errors in the weights and the threshold which may be allowed without disturbing the function to be realized.

In the reliability of systems it is interesting to know which subsets of components make an additive system to work when these subsets work, and which of them make the system to fail when all of them fail. The additive system is characterized by the weight of each component and the threshold of fail. This problem can be naturally transferred to the game theory. To do this it is only necessary to consider the components as players, and the subsets of components as coalitions. So, the additive reliability systems become weighted majority games.

In this paper we start from the tolerance, solution obtained by Hu in the resolution of this problem assigned to the field of electronics, and we improve it from the amplitude when we transfer such a problem to the field of game theory.

The paper is organized as follows: In Section 2 we explain the basical definitions that permit the pursuit of the work. In Section 3 we summarize the results on tolerance obtained by Hu and, at the same time, we improve them. In Section 4 we define the amplitude for strict representations of weighted majority games, which will be the maximum percentage in the variation of the weights and the quota which leave the game unchanged. As an immediate consequence we deduce that such a value improves the tolerance. In Section 5 we obtain the simplified expression for the amplitude for strict representations of monotonic weighted majority games. Section 6 is devoted to find the quota which allows us to find the maximum value for the amplitude when the weights are given. Finally, Section 7 includes two examples to illustrate the preceding results.

2. BASIC NOTATIONS AND DEFINITIONS

Let Q be the set $\{0,1\}$. For any given positive integer n, consider the cartesian power

$$Q^n = Q \times \cdots \times Q.$$

Thus, the elements of Q^n are the 2^n ordered n-tuples

$$(x_1, \ldots, x_n).$$

By a *switching function* of n variables, we mean a function

$$f : Q^n \longrightarrow Q$$

from the n-cube Q^n into Q.

A switching function $f : Q^n \longrightarrow Q$ is *linearly separable* if it admits a *system*

$$[T; w_1, \ldots, w_n]$$

such that for an arbitrary point

$$x = (x_1, \ldots, x_n)$$

of the n-cube Q^n we have

$$\begin{array}{lcl} w_1x_1 + \cdots + w_nx_n & \geq & T, \text{ if } f(x) = 1 \\ w_1x_1 + \cdots + w_nx_n & < & T, \text{ if } f(x) = 0. \end{array}$$

The n real numbers $w_1, \ldots, w_n$ in this system are called the *weights*, and the first real number T is referred to as the *threshold* or *quota*.

It is always possible to modify the quota in such a way that the previous definition could be rewritten using strict inequalities. In this case, the system is called a *strict separating system* for the linearly separable function f.

We will see the way in which we can transfer this concept to the field of game theory.

A *simple n-person game* is a pair (N, v) where N is described as $N = \{1, 2, \ldots, n\}$ and is called the set of *players*. Every $S \subseteq N$ is a *coalition*, $C(N)$ is the set of all coalitions and $v : C(N) \longrightarrow \{0,1\}$ such that $v(\emptyset) = 0$ is the *characteristic function*. We will suppose that *v* is not identically equal to zero. A coalition *S* is *winning* if

$v(S) = 1$ and *losing* o therwise. The set of winning coalitions is denoted by $\mathcal{W}$ and the set of losing coalitions by $\mathcal{L}$.

A simple game (N, v) is a *weighted majority game* iff there are real numbers $T, w_1, \ldots,$ w_n such that $v(S) = 1$ if $w(S) \geq T$ and $v(S) = 0$ otherwise, where $w(S) = \sum_{i \in S} w_i$. Then $[T; w_1, \ldots, w_n]$ is called a *representation* of the weighted majority game (N, v).

If T is such that $v(S) = 1$ if $w(S) > T$ and $v(S) = 0$ if $w(S) < T$, then $[T; w_1, \ldots, w_n]$ is called a *strict representation* of (N, v).

As is obvious, every weighted majority game admits a strict representation.

From $v(\emptyset) = 0$ it is clear that $T > 0$.

A simple game is *monotonic* if all subcoalitions of the losing coalitions are losing. If each proper subcoalition of a winning coalition is losing, this winning coalition is called *minimal*. It should be noted that a monotonic simple game is completely determined by its minimal winning coalitions. The set of minimal winning coalitions is denoted by $\mathcal{W}^m$. For monotonic simple games a player $i \in N$ is *null* if $i \notin S$ for all $S \in \mathcal{W}^m$. In a weighted majority game, we will denote by D the set of null players with non-positive weight (if any).

Throughout this paper, let $[T; w_1, w_2, \ldots, w_n]$ be a strict representation of a weighted majority game.

Formally, a switching function f with $f(0, \ldots, 0) = 0$ is equivalent to a simple game and a strict separating system is a strict representation of a weighted majority game without condition $T > 0$.

Gambarelli (1983) studied the effects on the game when a player increases his weight in perjudice of others, or decreases in favour. This situation can be generalized in case that there exist variations in each one of the weights and the quota, which is what we want to study. Carreras (1993) studied the effects on the Shapley value of a weighted majority game in which weights are given and the quota is modified. He particularly studied those effects on the European Parliament. These two articles have a relation with our paper in the sense that in both of them we can see variations either in weights or in the quota.

3. TOLERANCE

Throughout the section, let $f : Q^n \longrightarrow Q$ be an arbitrarily given linearly separable switching function of n variables and let $[T; w_1, \ldots, w_n]$ be a given strict separating system for f.

For each point $x = (x_1, \ldots, x_n)$ in Q^n, let

$$w(x) = w_1x_1 + \cdots + w_nx_n.$$

Then it follows from the definition of a strict separating system that

$$\begin{aligned} w(x) &> T, \text{ if } f(x) = 1, \\ w(x) &< T, \text{ if } f(x) = 0. \end{aligned}$$

Let A denote the maximum of the function $w(x)$ for all $x \in f^{-1}(0)$ and let B denote the minimum of the function $w(x)$ for all $x \in f^{-1}(1)$. If $f^{-1}(0)$ is empty, we set $A =$-∞; if $f^{-1}(1)$ is empty, we set $B = \infty$. Then we have $A < T < B$.

Adapting the definitions for strict representations of weighted majority games we have the following results.

Let A denote the maximum of $w(S)$ for all $S \in \mathcal{L}$ and let B denote the minimum of $w(S)$ for all $S \in \mathcal{W}$. Then, we have $A < T < B$ and $A \geq 0$.

Now let m denote the smallest of the two positive numbers $T - A$ and $B - T$. On the other hand, let

$$M = T + |w_1| + \cdots + |w_n|\,.$$

Let $\lambda_1, \ldots, \lambda_n$ and Λ be $n+1$ arbitrary real numbers and let

$$\begin{aligned} w_i^{'} &= (1+\lambda_i)w_i \qquad i = 1, \ldots, n \\ T^{'} &= (1+\Lambda)T. \end{aligned}$$

Then, the real numbers $\lambda_1, \ldots, \lambda_n$ and Λ represent the relative variations if we use the numbers $w_1^{'}, \ldots, w_n^{'}$ and $T^{'}$ instead of the original numbers $w_1, \ldots, w_n$ and T as weights and quota. In this paper we are going to find the maximum of those positive real numbers δ such that if

$$|\Lambda| < \delta, \quad |\lambda_i| < \delta \quad i = 1, \ldots, n$$

then $[T^{'}; w_1^{'}, w_2^{'}, \ldots, w_n^{'}]$ is still a representation to the given game. Such a positive real number δ was given by Hu (1965) for strict separating systems. He defined the number $\frac{m}{M}$ (taking $|T|$ in M instead of T), which is completely determined by the set of real numbers $[T; w_1, \ldots, w_n]$.

Theorem 3.1. *(Hu, 1965) Let $f : Q^n \longrightarrow Q$ be an arbitrarily given linearly separable switching function of n variables and let $[T; w_1, \ldots, w_n]$ be a given strict separating* *system for f. If $|\lambda_i| < \frac{m}{M}$ for each i=1, ... , n and if $|\Lambda| < \frac{m}{M}$, then $[T^{'}; w^{'}_1, \dots, w^{'}_n]$ is a strict separating system for the given linearly separable switching function.*

He called this positive number the *tolerance* of the system and denoted

$$\tau[T; w_1, \dots, w_n] = \frac{m}{M}.$$

Theorem 3.2. *(Hu, 1965) Let $f : Q^n \longrightarrow Q$ be an arbitrarily given linearly separable switching function of n variables and let $[T; w_1, \dots, w_n]$ be a given strict separating system for f. Then:*

a) $\tau[T; w_1, \dots, w_n] \leq 1$.

b) $\tau[T; w_1, \dots, w_n] \leq \tau[C; w_1, \dots, w_n]$, *where C stands for $\frac{A+B}{2}$. If $T \neq C$ the inequality is strict.*

Adapting Hu's results for strict representations of weighted majority games we have the following theorem.

Theorem 3.3. *Let $[T; w_1, \dots, w_n]$ be a strict representation of a weighted majority game. Then*

a) $\tau[T; w_1, \dots, w_n] \leq 1$.

b) $\tau[T; w_1, \dots, w_n] \leq \tau[C; w_1, \dots, w_n]$, *where C stands for $\frac{A+B}{2}$. If $T \neq C$ the inequality is strict.*

Our main objetive is to find the greatest value for δ such that if

$$|\Lambda| < \delta, \ |\lambda_i| < \delta \qquad i = 1, \dots, n$$

then $[T^{'}; w^{'}_1, \dots, w^{'}_n]$ is still a representation to the given game. We will distinguish the monotonic case from the non-monotonic case.

First we will see how the bound given by Hu for the tolerance can be improved when we are restricted to strict representations of weighted majority games.

Theorem 3.4. *Let $[T; w_1, \dots, w_n]$ be a strict representation of a weighted majority game. Then,*

$$\tau[T; w_1, \dots, w_n] \leq \frac{1}{3}.$$

Proof: From Theorem 3.3 the tolerance reaches its maximum when T is the arithmetic mean

$$T = \frac{A+B}{2}$$

of the real numbers A and B. Then it follows that $m = \frac{B-A}{2}$ and $M = \frac{A+B}{2} + |w_1| + \cdots + |w_n|$. Due to the fact that v is not identically equal to zero it exists a coalition S such that $w(S) \geq B$ and, consequently, $|w_1| + \cdots + |w_n| \geq B$.

Theorefore, we obtain

$$\begin{aligned} \tau[T; w_1, \ldots, w_n] &= \frac{m}{M} \leq \frac{\dfrac{B-A}{2}}{\dfrac{A+B}{2} + |w_1| + \cdots + |w_n|} \leq \\ &\leq \frac{B-A}{A+B+2B} \leq \frac{B}{A+3B} \leq \frac{1}{3}. \end{aligned}$$

■

The following result proves that $\frac{1}{3}$ is reached and it characterizes the strict representations of monotonic weighted majority games which reach it.

Proposition 3.5. *The set of strict representations of monotonic weighted majority games with tolerance* $\dfrac{1}{3}$ *is*

$$[T; 2T, 0, \ldots, 0].$$

Proof: Let $[T; w_1, \ldots, w_n]$ be a strict representation of a weighted majority game. Its tolerance is:

$$\tau[T; w_1, \ldots, w_n] = \frac{m}{M}$$

where $m = \min\{T-A, B-T\}$ and $M = T + |w_1| + \cdots + |w_n|$.

We want to determine which are the strict representations of monotonic weighted majority with tolerance $\dfrac{1}{3}$.

From Theorem 3.3 we must consider $T = \frac{A+B}{2}$.

$$\tau[T; w_1, \dots, w_n] = \frac{B - A}{A + B + 2(|w_1| + \cdots + |w_n|)} = \frac{1}{3} \Leftrightarrow$$
$$B - 2A = |w_1| + \cdots + |w_n| \,.$$

Because $|w_1|+\cdots+|w_n| \geq B$ and $A \geq 0$, we obtain that $A = 0$ and $|w_1|+\cdots+|w_n| = B$.

Therefore from $A = 0$, $B = 2T$ and $|w_1| + \cdots + |w_n| = 2T$, we can deduce that the game is:

$$[T; w_1, \dots, w_n] \equiv [T; 2T, 0, \dots, 0].$$

■

For the set of non-monotonic games this maximum is smaller.

Theorem 3.6. *Let $[T; w_1, \dots, w_n]$ be a strict representation of a non-monotonic weighted majority game. Then,*

$$\tau[T; w_1, \dots, w_n] \leq \frac{1}{5}.$$

Proof: The tolerance reaches its maximum when

$$T = \frac{A + B}{2}.$$

Then it follows that $m = \frac{B-A}{2}$ and $M = \frac{A+B}{2} + |w_1| + \cdots + |w_n|$. Due to the fact that the game is non-monotonic there exist coalitions $R \subset S$ such that $v(R) = 1$, $v(S) = 0$ and $w_i < 0 \; \forall i \in S - R$. Hence,

$$\sum_{i \in R} w_i \geq B \text{ and } \sum_{i \in S} w_i \leq A,$$

and therefore

$$\sum_{i \in S} |w_i| = \sum_{i \in R} |w_i| + \sum_{i \in S - R} |w_i| \geq B + (B - A) = 2B - A.$$

We obtain

$$\tau[T; w_1, \dots, w_n] = \frac{m}{M} \leq \frac{\frac{B-A}{2}}{\frac{A+B}{2} + |w_1| + \cdots + |w_n|} \leq \frac{B - A}{A + B + 2\sum_{i \in S} |w_i|} \leq$$

$$\leq \frac{B-A}{A+B+2(2B-A)} = \frac{B-A}{5B-A} \leq \frac{1}{5} \ .$$

■

Analogously to Proposition 3.5, we characterize the strict representations of non-monotonic weighted majority game with tolerance $\frac{1}{5}$.

Proposition 3.7. *The set of strict representations of non-monotonic weighted majority games with tolerance* $\frac{1}{5}$ *is*

$$[T; 2T, 0, \dots, 0, -2T].$$

Proof: Let $[T; w_1, \dots, w_n]$ be a strict representation of a non-monotonic weighted majority game. We want to determine which representations of this type have tolerance $\frac{1}{5}$.

From Theorem 3.3 we must consider $T = \frac{A+B}{2}$.

$$\tau[T; w_1, \dots, w_n] = \frac{B-A}{B+A+2(|w_1| + \dots + |w_n|)} = \frac{1}{5} \Leftrightarrow$$

$$2B - 3A = |w_1| + \dots + |w_n| \, .$$

In the proof of Theorem 3.6 we showed that $|w_1| + \dots + |w_n| \geq 2B - A$, and since $A \geq 0$ we obtain that $A = 0$ and $|w_1| + \dots + |w_n| = 2B$.

Therefore from $A = 0$, $B = 2T$ and $|w_1| + \dots + |w_n| = 4T$ we can deduce that:

$$[T; w_1, \dots, w_n] \equiv [T; 2T, 0, \dots, 0, -2T].$$

■

4. AMPLITUDE FOR STRICT REPRESENTATIONS OF WEIGHTED MAJORITY GAMES

Our main objetive in the present section is to find, for strict representations of weighted majority games, the greatest positive real number δ such that if

$$|\Lambda| < \delta, \quad |\lambda_i| < \delta \ \ i = 1, 2, \dots, n$$

then $[T^{'}; w_1^{'}, \ldots, w_n^{'}]$ is equivalent to $[T; w_1, \ldots, w_n]$.

We will call this constant amplitude of the representation and we will see that it is the maximum of rate one in the variation of weights and in the quota, so as the game remains invariant. As the tolerance provides a bound which guarantees that the game remains invariant, the tolerance has to be smaller than, or equal to, the amplitude.

Given a strict representation of a weighted majority game $[T; w_1, \ldots, w_n]$, for each coalition $S \subseteq N$ let

$$\begin{array}{rcl} a(S) & = & |w(S) - T| \\ b(S) & = & T + \sum_{i \in S} |w_i| \,. \end{array}$$

Note that these are positive numbers. Take

$$P = \min_{S \subseteq N} \frac{a(S)}{b(S)}.$$

We call this number the *amplitude* of the representation $[T; w_1, \ldots, w_n]$ and denote

$$\mu[T; w_1, \ldots, w_n] = P.$$

The minimum P is attained for, at least, some coalition, namely from now on S_0.

Theorem 4.1. *If $|\lambda_i| < P$ for each $i = 1, 2, \ldots, n$ and $|\Lambda| < P$, then $[T^{'}; w_1^{'}, \ldots, w_n^{'}]$ is equivalent to $[T; w_1, \ldots, w_n]$ and P is the greatest upper bound for the constants $\lambda_1, \ldots, \lambda_n, \Lambda$.*

Proof: First of all, we observe that, from the definition, $P < 1$ and because $T^{'} = (1 + \Lambda)T$, if $|\Lambda| < P$ we obtain that $T^{'} > 0$.

For each coalition $S \subseteq N$, let

$$w^{'}(S) = \sum_{i \in S} w_i^{'} \,.$$

For the first part it suffices to prove that $w^{'}(S) > T^{'}$ for every $S \in \mathcal{W}$ and $w^{'}(S) < T^{'}$ for every $S \in \mathcal{L}$.

First, let us assume that $S \in \mathcal{W}$. Then we have

$$a(S) = w(S) - T.$$

By definition of $w^{'}(S)$, we have

$w^{'}(S) - T^{'} = \sum_{i \in S} w_i^{'} - T^{'} = \sum_{i \in S}(1+\lambda_i)w_i - (1+\Lambda)T = [w(S) - T] +$

$+[\sum_{i \in S} \lambda_i w_i - \Lambda T]$.

Since $w(S) - T = a(S)$ and

$$\left|\sum_{i \in S} \lambda_i w_i - \Lambda T\right| \leq \sum_{i \in S} |\lambda_i|\,|w_i| + |\Lambda|\,T < P[\sum_{i \in S} |w_i| + T] = \frac{a(S_0)}{b(S_0)} b(S) \leq a(S),$$

it follows that $w^{'}(S) - T^{'} > 0$ and hence $w^{'}(S) > T^{'}$.

Next, let us assume that $S \in \mathcal{L}$. Then we have

$$a(S) = T - w(S)\ .$$

As above, we have

$$T^{'} - w^{'}(S) = (1+\Lambda)T - \sum_{i \in S}(1+\lambda_i)w_i = [T - w(S)] + [\Lambda T - \sum_{i \in S} \lambda_i w_i].$$

Since $T - w(S) = a(S)$ and

$$\left|\Lambda T - \sum_{i \in S} \lambda_i w_i\right| \leq |\Lambda|\,T + \sum_{i \in S} |\lambda_i|\,|w_i| < P[T + \sum_{i \in S} |w_i|] = \frac{a(S_0)}{b(S_0)} b(S) \leq a(S)$$

it follows that $T^{'} - w^{'}(S) > 0$ and hence $w^{'}(S) < T^{'}$.

For the second part we will suppose $Q > P$, and then we will demonstrate that the game given by

$$[T(1+\Lambda); (1+\lambda_1)w_1, \ldots, (1+\lambda_n)w_n]$$

is not equivalent to $[T; w_1, \ldots, w_n]$ for all Λ and λ_i with $|\Lambda| < Q$ and $|\lambda_i| < Q$ for each $i = 1, \ldots, n$.

Let $S_0 \subseteq N$ such that $\frac{a(S_0)}{b(S_0)} = P$. If $S_0 \in \mathcal{W}$, taking

$$\Lambda = \epsilon \quad \text{and} \quad \lambda_i = \begin{cases} -\epsilon & \text{if } w_i \geq 0 \\ \epsilon & \text{if } w_i < 0 \end{cases}$$

with $P < \epsilon < Q$, we will obtain a contradiction concerning S_0.

$w'(S_0) - T' = [w(S_0) - T] - \epsilon[T + \sum_{i \in S_0} |w_i|] = a(S_0) - \epsilon b(S_0) < a(S_0) - \frac{a(S_0)}{b(S_0)} b(S_0) = 0$

and hence $S_0 \notin \mathcal{W}$.

Analogously, is $S_0 \in \mathcal{L}$, taking

$$\Lambda = -\epsilon \qquad \text{and} \qquad \lambda_i = \begin{cases} \epsilon & \text{if } w_i \geq 0 \\ -\epsilon & \text{if } w_i < 0 \end{cases}$$

with $P < \epsilon < Q$, we will obtain a contradiction concerning S_0.

$T' - w'(S_0) = [T - w(S_0)] - \epsilon[T + \sum_{i \in S_0} |w_i|] = a(S_0) - \epsilon b(S_0) < a(S_0) - \frac{a(S_0)}{b(S_0)} b(S_0) = 0$

and hence $S_0 \notin \mathcal{L}$.

As a consequence of the maximality of the amplitude, we can deduce that the tolerance is smaller than, or equal to, the amplitude.

5. AMPLITUDE OF MONOTONIC WEIGHTED MAJORITY GAME STRICT REPRESENTATIONS

When we are restricted to strict representations of monotonic weighted majority games, the weight of each no null player is positive (we only have to compare a minimal winning coalition, which possesses this no null player, with the same coalition without him; the resulting inequality tells us the weight must be positive). Thus, a weight may only be non-positive if it belongs to a null player. Taking into account these facts, for monotonic games we are going to find a simpler expression for the amplitude.

Theorem 5.1. *If $[T; w_1, \ldots, w_n]$ is a strict representation of a monotonic weighted majority game with amplitude P, then*

$$P = \min\left\{\frac{B - T}{B + T - 2w(D)}, \frac{T - A}{T + A}\right\}$$

where D is the set of null players with negative weight (if any).

Proof: First we are going to demonstrate that

$$P \geq \min\left\{\frac{B-T}{B+T-2w(D)}, \frac{T-A}{T+A}\right\}.$$

Taking into account that $P = \min_{S\subseteq N} \frac{a(S)}{b(S)}$, it suffices to prove for $S \in \mathcal{L}$ that

$$\frac{a(S)}{b(S)} \geq \frac{T-A}{T+A}$$

and for $S \in \mathcal{W}$ that

$$\frac{a(S)}{b(S)} \geq \frac{B-T}{B+T-2w(D)}.$$

For this purpose we have to check for $S \in \mathcal{L}$ that

$$2AT + A\left(\sum_{i\in S}|w_i| - w(S)\right) - T\left(\sum_{i\in S}|w_i| + w(S)\right) \geq 0.$$

Since $\sum_{i\in S}|w_i| - w(S) = -2w(S\cap D)$, $\sum_{i\in S}|w_i| + w(S) = 2w(S-D)$ and by definition of null player $w(S-D) \leq A$, it follows that

$$2[T(A - w(S-D)) - Aw(S\cap D)] \geq 0.$$

Next, let us assume that $S \in \mathcal{W}$. Then, taking into account $\sum_{i\in S}|w_i| - w(S) = -2w(S\cap D)$ and $\sum_{i\in S}|w_i| + w(S) = 2w(S-D)$, we have to check

$$2[-BT + Tw(D) - w(D)w(S) + Bw(S\cap D) + Tw(S-D)] \geq 0.$$

Regrouping terms it is clear that we have to check

$$2[T(w(D) + w(S-D) - B) - w(D)w(S) + Bw(S\cap D)] \geq 0.$$

Taking into account $w(S) \geq B$, it is enough to check

$$2[T(w(D) + w(S-D) - B) + B(w(S\cap D) - w(D))] \geq 0.$$

The first member of the sum is positive because for any winning coalition S we have that $(S - D) \cup D$ is also winning. The second member of the sum is non-negative because $w(S \cap D) \geq w(D)$. So, it is clear that:

$$2[T(w(D) + w(S - D) - B) + B(w(S \cap D) - w(D))] \geq 0.$$

Therefore,

$$P \geq \min \left\{ \frac{B - T}{B + T - 2w(D)}, \frac{T - A}{T + A} \right\}.$$

Now, we are going to demonstrate that $P \leq \min \left\{ \frac{B-T}{B+T-2w(D)}, \frac{T-A}{T+A} \right\}$.

Let $S_0 \in \mathcal{W}$ be such that $w(S_0) = B$. Then $D \cap S_0 = D$ and

$$P = \min_{S \subseteq N} \frac{a(S)}{b(S)} \leq \frac{a(S_0)}{b(S_0)} = \frac{B - T}{T + B - 2w(D)}.$$

Let $S_0 \in \mathcal{L}$ such that $w(S_0) = A$. Then $D \cap S_0 = \emptyset$ and

$$P = \min_{S \subseteq N} \frac{a(S)}{b(S)} \leq \frac{a(S_0)}{b(S_0)} = \frac{T - A}{T + A}.$$

Therefore,

$$P \leq \min \left\{ \frac{B - T}{B + T - 2w(D)}, \frac{T - A}{T + A} \right\}.$$

■

Shareholder societies and most models in political science can be described by specifying non-negative weights for the voters and a positive quota. These situations give rise to $w(D) = 0$ and therefore the amplitude is $P = \min \left\{ \frac{B-T}{B+T}, \frac{T-A}{T+A} \right\}$.

Notice that from the definition of P the amplitude satisfies $0 < \mu < 1$, and for each number $x \in (0, 1)$ it exists a 2-game

$$\left[\frac{1 + x}{1 - x}; \left(\frac{1 + x}{1 - x} \right)^2, 1 \right]$$

whose amplitude is x.

6. MAXIMUM AMPLITUDE

In this section we want to determine the maximum value that the amplitude for a strict representation of a weighted majority game can reach, when the weights of players are invariable.

For a given strict representation of a weighted majority game $[T; w_1, \dots, w_n]$, the quota T may be any real number between A and B. Let us suppose $A > 0$, this is,

$$0 < \max_{S \in \mathcal{L}} w(S) = A < T < B = \min_{S \in \mathcal{W}} w(S).$$

We define the function

$$f(S,T) = \frac{a(S,T)}{b(S,T)} \qquad \text{if } S \subseteq N \quad \text{and} \quad T \in (A,B),$$

where $a(S,T) = |w(S) - T|$, $b(S,T) = T + \sum_{i \in S} |w_i|$ and let

$$\begin{aligned} F(T) &= \min_{S \in \mathcal{L}} f(S,T), \\ G(T) &= \min_{S \in \mathcal{W}} f(S,T). \end{aligned}$$

Then $f(S,T) \leq 1$, and using $A > 0$ it follows that $F(T) < 1$.

It turns out that the amplitude $\mu[T; w_1, \dots, w_n]$ reaches its maximum when T is the unique number such that

$$F(T) = G(T).$$

Precisely, we have the following theorem.

Theorem 6.1. *For every strict representation of a weighted majority game*

$[T; w_1, \dots, w_n]$ *with* $0 < A < T < B$ *we have*

$$\mu[T; w_1, \dots, w_n] \leq \mu[T^*; w_1, \dots, w_n]$$

where T^* *stands for the unique number such that* $F(T) = G(T)$. *If* $T \neq T^*$ *the inequality is strict.*

Proof: For every coalition S, the function $f(S,T)$ is continuous and derivable with respect to T in (A,B), and consequently $F(T)$ and $G(T)$ are continuous too. Fixing S, the derivative of $f(S,T)$ with respect to T is

$$\frac{\sum_{i\in S}|w_i| + w_i}{\left(T + \sum_{i\in S}|w_i|\right)^2} \geq 0 \qquad \text{if} \quad S \in \mathcal{L},$$

$$-\frac{\sum_{i\in S}|w_i| + w_i}{\left(T + \sum_{i\in S}|w_i|\right)^2} < 0 \qquad \text{if} \quad S \in \mathcal{W}.$$

Then, $F(T)$ is a nondecreasing function and $G(T)$ is a strictly decreasing function. To obtain $T \in (A, B)$ which defines the maximum amplitude for the given representation, we consider the function

$$P(T) = \min\{F(T), G(T)\} \text{ for } A < T < B$$

and we demonstrate there is just one number T^* which reaches the $\max_{T\in(A,B)} P(T)$. The uniqueness is due to the above considerations about nondecreasing behaviour of F and decreasing behaviour of G. The existence can be proved using Bolzano's Theorem:

$$\lim_{T \longrightarrow B^-} F(T) - G(T) = \lim_{T \longrightarrow B^-} F(T) > 0$$

and due to the fact that $A > 0$, it follows that $\lim_{T \longrightarrow A^+} F(T) = 0$. Then

$$\lim_{T \longrightarrow A^+} F(T) - G(T) = \lim_{T \longrightarrow A^+} -G(T) < 0 \ .$$

■

In particular, for monotonic games the amplitude $P = \min\left\{\frac{B-T}{B+T-2w(D)}, \frac{T-A}{T+A}\right\}$ reaches its maximum when $\frac{B-T}{B+T-2w(D)} = \frac{T-A}{T+A}$ and therefore

$$T = \frac{w(D) + \sqrt{(w(D))^2 + 4AB - 4Aw(D)}}{2}.$$

If $w(D) = 0, T$ is the geometric mean of the real numbers A and B.

7. APPLICATIONS

To illustrate the ideas of this paper, two examples of amplitude of strict representations of weighted majority games are presented.

Example 7.1. A town signed a biannual agreement with three gas companies, X, Y and Z for which the supply of gas to the town is guaranteed by the collaboration of at least two of them. The first year the needs of the town were of 75 Km^3, and each one of the firms offered a fixed quantity of 60 Km^3, 30 Km^3 and 60 Km^3, respectively.

This situation can be described by the strict representation of the weighted majority game

$$[75; 60, 30, 60].$$

As it can be seen any coalition made by two or more of the firms is sufficient for the town needs, that's to say: $\mathcal{W}^m = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $A = 60,\ B = 90.$

Apart from the coalition formed, the amplitude of the representation is

$$\mu = \min\left\{\frac{T-A}{T+A}, \frac{B-T}{B+T}\right\} = \frac{1}{11}.$$

From this, we can assure that for the second year, bearing in mind that the town necessities and the disponibilities of the companies will slightly vary, we can describe this situation as follows:

$$[75(1+\Lambda); 60(1+\lambda_1), 30(1+\lambda_2), 60(1+\lambda_3)].$$

From the amplitude which we obtain, we can assure that the maximum percentage of such variations is a 9.09% so as to guarantee the fulfilment of the agreement.

But, if we had used the tolerance, $\tau = \dfrac{1}{15}$, the maximum estimated percentage in the possible modifications would have been just of a 6.66%.

□

Example 7.2. We suppose that a shareholder society is formed by three majority shareholders (each one of them has respectively 50.000, 25.000 and 25.000 shares) and an ocean of small shareholders which possess a total of 5.000 shares. A bill of the company is passed if the sum of the shares belonging to the holders which vote in favour is more than 60.000 shares. It can be foreseen that at the end of the year there will be a variation of the capital which will affect the distribution of the actions as well as the quota. If we exclude the possibility of the entry of new investors, the situation can be described with the following game:

$[60000(1 + \Lambda); 50000(1 + \lambda_1), 25000(1 + \lambda_2), 25000(1 + \lambda_3), w_4(1 + \lambda_4), \ldots, w_n(1 + \lambda_n)]$,

where the subindices 4, ... , n represent the smaller players, $\sum_{i=4}^{n} w_i = 5.000$ and $w_i > 0$ for $i = 4, \ldots, n$.

As the amplitude is $\frac{1}{23}$, any variation such as $|\Lambda| < \frac{1}{23}$, $|\lambda_i| < \frac{1}{23}$ for each

$i = 1, 2, \ldots, n$, assures that the process of taking decisions in the company will not vary.

For example, the distribution

$$[62.500; 47.916, 26.041, 23.958, w_4(1 + \lambda_4), \ldots, w_n(1 + \lambda_n)]$$

represents the same situation as the first game.

□

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