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BOLSHEV'S METHOD OF CONFIDENCE LIMIT CONSTRUCTION

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Confidence intervals and regions for the parameters of a distribution are constructed, following the method due to L.N. Bolshev. This construction method is illustrated with Poisson, exponential, Bernouilli, geometric, normal and other distributions depending on parameters.

Keywords: Confidence limits, interval estimates.

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1. REGIONS, INTERVALS, CONFIDENCE LIMITS

Let $\mathbb{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ be a sample with realizations $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, $x \in \mathcal{X} \subseteq R^n$. Suppose that X_i has a density $f(x;\theta)$, $\theta = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq R^k$,with respect to the Lebesgue measure,

$$H_0 : X_i \sim f(x;\theta), \quad \theta = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq R^k.$$

Let $b = b(\theta)$ be a function $b(\cdot) : \Theta \Rightarrow B \subseteq R^m$, B^0 is the interior of B

Definition 1 *A random set $C(\mathbb{X})$, $C(\mathbb{X}) \subseteq B \subseteq R^m$ is called the confidence region for $b = b(\theta)$ with the confidence level $\gamma(0.5 < \gamma < 1)$ if*

$$\inf_{\theta\in\Theta} P_\theta\{C(\mathbb{X}) \ni b(\theta)\} = \gamma.$$

This definition implies for all $\theta \in \Theta$

$$P_\theta\{C(\mathbb{X}) \ni b(\theta)\} \geq \gamma.$$

In the case $b(\theta) \in B \subseteq R^1$ the confidence region is often an interval in R^1,

$$C(\mathbb{X}) =]b_i(\mathbb{X}), b_s(\mathbb{X})[\subseteq B \subseteq R^1,$$

and it is called the confidence interval with the confidence level γ for b. The statistics $b_i(\mathbb{X})$ and $b_s(\mathbb{X})$ are called the confidence limits of the confidence interval $C(\mathbb{X})$.

Definition 2 *A statistic $b_i(\mathbb{X})$ $(b_s(\mathbb{X}))$ is called the inferior (superior) confidence limit with the confidence level $\gamma_1(\gamma_2)$ (or inferior (superior) $\gamma_1(\gamma_2)$ - confidence limit briefly), if*

$$\inf_{\theta\in\Theta} P_\theta\{b_i(\mathbb{X}) < b\} = \gamma_1 \left(\inf_{\theta\in\Theta} P_\theta\{b_s(\mathbb{X}) > b\} = \gamma_2 \right), \quad 0.5 < \gamma_j < 1$$

The $\gamma = 1-\alpha$ confidence interval has the form $]b_i(\mathbb{X}), b_s(\mathbb{X})[$, where $b_i(\mathbb{X})$ and $b_s(\mathbb{X})$ are the $\gamma_1 = 1-\alpha_1$ inferior and $\gamma_2 = 1-\alpha_2$ superior confidence limits, respectively, such that $\alpha_1+\alpha_2 = \alpha$, $(0 < \alpha_i < 0.5)$. If $\alpha_1 = \alpha_2$, then take $\gamma_1 = \gamma_2 = 1-\alpha/2$.

Definition 3 *The intervals*

$$\{b_i(\mathbb{X}), +\infty\} \quad \textit{and} \quad \{-\infty, b_s(\mathbb{X})\}$$

are called the superior and inferior confidence intervals for b. Both intervals are unilateral.

2. THEOREM OF BOLSHEV

Lemma *(Bolshev) Let $G(t)$ be the distribution function of the random variable T. Then for all $z \in [0,1]$*

(1) $$P\{G(T) \le z\} \le z \le P\{G(T-0) < z\}.$$

If T is continuous, then

$$P\{G(T) \le z\} = z, \quad 0 \le z \le 1.$$

Proof: First, we prove the inequality

(2) $$P\{G(T) \le z\} \le z, \quad 0 \le z \le 1.$$

If $z = 1$, then $P\{G(T) \le 1\} \le 1$. Fix $z \in [0,1)$ and for this value of z consider the different cases.

1) There exists a solution y of the equation $G(y)$ = z. Note

$$y_0 = sup\{y : G(y) = z\}.$$

It can be:

a)$G(y_0) = z$. In this case

$$P\{G(T) \le z\} \le P\{T \le y_0\} = G(y_0) = z.$$

b) $G(y_0) > z$. Then

$$P\{G(T) \le z\} \le P\{T < y_0\} = G(y_0 - 0) \le z.$$

2) A solution of the equation $G(y) = z$ does not exist. In this case there exists y such that

$$G(y) > z \quad \text{et} \quad G(y-0) < z,$$

so

$$P\{G(T) \le z\} \le P\{T < y\} = G(y-0) < z.$$

The inequality (2) is proved.

We prove now the second inequality in (1) :

$$z \le P\{G(T-0) < z\}, \quad 0 \le z \le 1. \tag{3}$$

Consider the statistic $-T$. Its distribution function is

$$G^{-}(y) = P\{-T \le y\} = P\{T \ge -y\} = 1 - G(-y-0\}.$$

Replacing

$$T, z, G \quad \text{by} \quad -T, 1-z \quad \text{and} \quad G^{-}$$

in the inequality (2) we have:

$$P\{G^{-}(-T) \le 1-z\} \le 1-z, \quad 0 \le z \le 1.$$

This implies

$$P\{1 - G(T-0) \le 1-z\} \le 1-z,$$
$$P\{G(T-0) \ge z\} \le 1-z,$$
$$P\{G(T-0) < z\} \ge z, \quad 0 \le z \le 1.$$

If T is continuous, then $G(t-0) = G(t)$, and (2) and (3) imply $P\{G(T) \le z\} = z$ for all $z \in [0,1]$.

The lemma is proved.

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Theorem (Bolshev) *Suppose that the random variable* $T = T(\mathbb{X}, b)$, $b \in B$, *is such that its distribution function*

$$G(t;b) = P_\theta\{T \le t\}$$

depends only on b *for all* $t \in R$ *and the functions*

$$I(b;x) = G(T(x,b)-0;b) \quad \textit{and} \quad S(b;x) = G(T(x,b);b)$$

are decreasing and continuous in b *for all fixed* $x \in \mathcal{X}$. *In this case:*

1) *the statistic* $b_i(\mathbb{X})$ *such that*

$$b_i = b_i(\mathbb{X}) = sup\{b : I(b;\mathbb{X}) \ge \gamma, b \in B\}, \quad \textit{if } \textbf{this supremum} \textit{ exists}, \tag{4}$$

or

(5) $$b_i = b_i(\mathbb{X}) = inf\, B, \ otherwise$$

is the inferior confidence limit for $b \in B^0$ *with confidence level larger or equal to* γ ;

2) *the statistic* $b_s(\mathbb{X})$ *such that*

(6) $$b_s = b_s(\mathbb{X}) = inf\{b : S(b;\mathbb{X}) \leq 1-\gamma, \quad b \in B\}, \quad if \ \textbf{this infimum} \ exists,$$

or

(7) $$b_s = b_s(\mathbb{X}) = sup\, B, \ otherwise$$

is the superior confidence limit for $b \in B^0$ *with the confidence level larger or equal to* γ;
3) *if* $x \in \mathcal{X}$, *is such that the functions* $I(b;x)$ *and* $S(b;x)$ *are strongly decreasing with respect* to b, *then* $b_i(x)$ *and* $b_s(x)$ *are the roots of the equations*

(8) $$I(b_i(x);x) = \gamma \quad and \quad S(b_s(x);x) = 1-\gamma.$$

Proof: Denote $D = D(\mathbb{X})$ the event

$$D = \{there \ exists \quad b \, such \ that \quad I(b;\mathbb{X}) \geq \gamma\}.$$

Then for the true value $b \in B^0$ we have (using Bolshev's lemma)

$$P\{b_i < b\} = P\{(b_i < b)\bigcap D\} + P\{(b_i < b)\bigcap \bar{D}\} =$$

$$P\{((sup\, b^* : I(b^*;\mathbb{X}) \geq \gamma, b^* \in B) < b)\bigcap D\} + P\left\{(inf\, B < b)\bigcap \bar{D}\right\} =$$

$$= P\{(I(b;\mathbb{X}) < \gamma)\bigcap D\} + P\{\bar{D}\} \geq P\{(I(b;\mathbb{X}) < \gamma)\bigcap D\} + P\{(I(b;\mathbb{X}) < \gamma)\bigcap \bar{D}\} =$$
$$= P\{I(b;\mathbb{X}) < \gamma\} \geq \gamma.$$

The theorem is proved.

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Remark: Often, instead of the statistic T a sufficient statistic or some function of a sufficient statistic for a parameter b can be taken.

□

3. EXAMPLES

1. Let $\mathbb{X} = (X_1, \dots, X_n)^{\mathrm{T}}$ be a sample and suppose that X_i has a Poisson distribution with a parameter θ:

$$X_i \sim f(x;\theta) = \frac{\theta^x}{x\,!} e^{-\theta}, \, x \in \mathcal{X} = \{0, 1, \dots\}, \, \theta \in \Theta =]0, \infty[.$$

Denote

$$T = X_1 + \ldots + X_n.$$

a) Show that the statistics

$$\theta_i = \frac{1}{2n} \chi^2_{1-\gamma_1}(2T) \text{ and } \theta_s = \frac{1}{2n} \chi^2_{\gamma_2}(2T+2)$$

are the inferior and superior confidence limits for θ with confidence levels larger or equal to γ_1 and γ_2 respectively; $\chi^2_\alpha(n)$ denotes the α-quantile of a chi-square distribution with n degrees of freedom.
b) Find a confidence interval for θ with confidence level larger or equal to γ.

Solution. The sufficient statistic T follows the Poisson distribution with parameter $n\theta$. Then

$$G(k;\theta) = P_\theta\{T \le k\} = \sum_{i=0}^{k} \frac{(n\theta)^i}{i!} e^{-n\theta} = P\{\chi^2_{2k+1} \ge 2n\theta\} = \mathcal{P}(2n\theta, 2k+2), \, k = 0, 1, \ldots$$

and

$$G(k-0;\theta) = P_\theta\{T < k\} = \sum_{i=0}^{k-1} \frac{(n\theta)^i}{i!} e^{-n\theta} = \mathcal{P}(2n\theta, 2k), \, k = 1, 2, \ldots,$$

$$G(k-0;\theta) = 0, k = 0.$$

The functions I and S are

$$I(\theta;\mathbb{X}) = \begin{cases} \mathcal{P}(2n\theta, 2T), & \text{if } \mathbb{X} \neq 0, \\ 0, & \text{if } \mathbb{X} = 0, \end{cases}$$

$$S(\theta;\mathbb{X}) = \mathcal{P}(2n\theta, 2T+2).$$

The function S is strictly decreasing for all T, $T \geq 0$, and I is strictly decreasing for all $T \neq 0$. In these cases the theorem of Bolshev implies (see (8)):

$$\mathcal{P}(2n\theta_i, 2T) = \gamma_1 \quad \mathcal{P}(2n\theta_s, 2T+2) = 1-\gamma_2,$$

from which it follows

$$\theta_i = \frac{1}{2n}\chi^2_{1-\gamma_1}(2T), \quad \theta_s = \frac{1}{2n}\chi^2_{\gamma_2}(2T+2). \tag{9}$$

If $T = 0$ then $I(\theta;\mathbb{X}) = 0$. There is no such θ that

$$I(\theta;\mathbb{X}) = \gamma_1 > \frac{1}{2}.$$

The formula (5) implies

$$\theta_i = \inf_{\theta>0} \theta = \inf]0,+\infty[= 0.$$

b) The interval $]\theta_i, \theta_s[$ is the confidence interval for θ with a confidence level larger or equal to $\gamma = 1-\alpha$, if $\gamma_1 = 1-\alpha_1$, $\gamma_2 = 1-\alpha_2$, $\alpha_1+\alpha_2 = \alpha$. If $\alpha_1 = \alpha_2$, take $\gamma_1 = \gamma_2 = 1-\alpha/2$.

2. Let $\mathbb{X} = (X_1,\ldots,X_n)^{\mathrm{T}}$ be a sample and suppose that X_i has an exponential distribution with mean $\theta, \theta > 0$:

$$X_i \sim f(x;\theta) = \frac{1}{\theta}\exp\{-\frac{x}{\theta}\}1_{(x>0)}. \tag{10}$$

a) Find γ-confidence limits for θ.

b) Let $\mathbb{X}_n^{(r)} = (X_{(1)},\ldots,X_{(r)})^{\mathrm{T}}$ be a type II censored sample from the distribution (10).

Find a γ-confidence interval for θ and the survival function

$$S(x;\theta) = P_\theta\{X_1 > x\}.$$

Solution. a). Denote

$$T = X_1 + \ldots + X_n.$$

The sufficient statistic T follows a gamma distribution $G(n;\frac{1}{\theta})$ with parameters n and $1/\theta$:

$$P\{T \leq t\} = \frac{1}{(n-1)!\theta^n}\int_0^t u^{n-1}e^{-u/\theta}du,\ t \geq 0,$$

and hence T/θ follows the gamma distribution $G(n;1)$, and

$$\frac{2T}{\theta} = \chi^2_{2n}.$$

In this example the functions I and S can be taken as

$$I(\theta;\mathbb{X}) = S(\theta;\mathbb{X}) = 1 - \mathcal{P}\left(\frac{2T}{\theta}, 2n\right).$$

These functions are decreasing in θ and the formula (8) implies

$$1 - \mathcal{P}\left(\frac{2T}{\theta_i}, 2n\right) = \gamma \quad \text{and} \quad 1 - \mathcal{P}\left(\frac{2T}{\theta_s}, 2n\right) = 1 - \gamma,$$

from where we obtain

$$\frac{2T}{\theta_i} = \chi^2_{\gamma}(2n) \quad \text{and} \quad \frac{2T}{\theta_s} = \chi^2_{1-\gamma}(2n),$$

and hence

$$\theta_i = \frac{2T}{\chi^2_{\gamma}(2n)} \quad \text{and} \quad \theta_s = \frac{2T}{\chi^2_{1-\gamma}(2n)}.$$

b) As it is well known the statistic

$$T_r = \sum_{k-1}^{r} X_{(k)} + (n-r)X_{(r)}$$

follows a gamma distribution $G(r;\frac{1}{\theta})$, and hence the $\gamma = 1-\alpha$-confidence interval for θ is $]\theta_i, \theta_s[$, where

$$\theta_i = \frac{2T_r}{\chi^2_{1-\alpha/2}(2r)} \quad \text{and} \quad \theta_s = \frac{2T_r}{\chi^2_{1-\alpha/2}(2r)}.$$

Since the survival function $S(x;b) = e^{-x/\theta}$, $x > 0$, is increasing in θ, we have the γ-confidence interval $]S_i, S_s[$ for $S(x;\theta)$, where

$$S_i = e^{-x/\theta_i} \quad \text{and} \quad S_s = e^{-x/\theta_s}.$$

3. Let $\mathbb{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ be a sample from Bernoulli distribution with parameter θ:

$$X_i \sim f(x;\theta) = \theta^x(1-\theta)^{1-x},\, x \in \mathcal{X} = \{0,1\},\, \theta \in \Theta =]0,1[.$$

Find the limits of confidence for θ with the confidence levels larger or equal to γ_1.

Solution. It is clear that the sufficient statistic

$$T = \sum_{i=1}^{n} X_i$$

follows the binomial distribution $B(n, \theta)$ with parameters n and θ. Then

$$G(k;\theta) = P_\theta\{T \le k\} = \sum_{i=0}^{k} \binom{n}{i} \theta^i (1-\theta)^{n-i} =$$

$$I_{1-\theta}(n-k, k+1) = 1 - I_\theta(k+1, n-k),\, k = 0, 1, \ldots, n-1,$$

$$G(k;\theta) = 1, \text{ if } k = n,$$

where $I_x(a,b)$ is the beta distribution function with parameters a and b, and

$$G(k-0;\theta) = \sum_{i=0}^{k-1} \binom{n}{i} \theta^i (1-\theta)^{n-i} =$$

$$1 - I_\theta(k, n-k+1),\, k = 1, 2, \ldots, n,$$

$$G(k-0;\theta) = 0, \text{ if } k = 0.$$

The functions I and S are

$$I(\theta;\mathbb{X}) = \begin{cases} I_{1-\theta}(n-T+1, T), & \text{if } T \neq 0 \\ 0, & \text{otherwise} \end{cases},$$

$$S(\theta;\mathbb{X}) = \begin{cases} I_{1-\theta}(n-T, T+1), & \text{if } T \neq n \\ 1, & \text{if } T = n. \end{cases}$$

We remark that $S(\theta;\mathbb{X})$ is strictly decreasing in θ for $T \neq n$, and $I(\theta;\mathbb{X})$ is strictly decreasing in θ for $T \neq 0$, and hence from the formula (8) it follows that

$$I_{1-\theta_i}(n-T+1, T) = \gamma_1 \quad \text{for} \quad T \neq 0$$

and

$$\theta_i = 0, \quad \text{if} \quad T = 0,$$

$$I_{1-\theta_s}(n-T, T+1) = 1 - \gamma_1 \quad \text{for} \quad T \neq n$$

and

$$\theta_s = 1, \quad \text{if} \quad T = n.$$

Hence,

$$\theta_i = \begin{cases} 1 - x(\gamma_1; n-T+1, T), & \text{if } T \neq 0 \\ 0, & \text{if } T = 0, \end{cases}$$

$$\theta_s = \begin{cases} 1 - x(1-\gamma_1; n-T, T+1), & \text{if } T \neq n \\ 1, & \text{if } T = n, \end{cases}$$

where $x(\gamma_1; a, b)$ is the γ_1-quantile of the beta distribution with parameters a and b.

4. Let X be a discret random variable with the cumulative distribution function

$$F(x;\theta) = P_\theta\{X \leq x\} = (1-\theta^{[x]})1_{]0,+\infty[}(x), \quad x \in R^1, \quad \theta \in \Theta =]0,1[.$$

Find a γ-confidence interval for θ, if $X = 1$.

Solution. In this case

$$I(X;\theta) = F(X-0;\theta) \quad \text{and} \quad S(X;\theta) = F(X;\theta).$$

If $X = 1$ then

$$I(1;\theta) = F(1-0;\theta) = F(0;\theta) = 0$$

and according to the formula (5) we have that the inferior confidence limit θ_i for θ with confidence level larger or equal to γ_1 is

$$\theta_i = \inf \theta = \inf]0,1[= 0.$$

If $\gamma_1 = 1$ then $P\{\theta_i \leq \theta\} = \gamma_1$, so $\theta_i = 0$ is 1-confidence inferior limit for θ. On the other hand the function

$$S(1;\theta) = F(1;\theta) = 1-\theta$$

is decreasing in θ, and hence according to the formula (8) we have

$$S(1;\theta_s) = 1-\gamma_2,$$

from where $\theta_s = \gamma_2$, so the $\gamma_1 = 1$ and γ_2 confidence limits for θ are 0 and γ_2, and a *gamma*-confidence interval for θ is $]0,\gamma[$, since for $\gamma_1 = 1$ the equality $\gamma = \gamma_1 + \gamma_2 - 1$ is true when $\gamma_2 = \gamma$.

5. Let X_1 and X_2 be two independent random variables,

$$X_i \sim f(x;\theta) = e^{-(x-\theta)}1_{[\theta,\infty[}(x), \; \theta \in \Theta = R^1.$$

Find the smallest γ-confidence interval for θ.

Solution. The likelihood function $L(\theta)$ for X_1 and X_2 is

$$L(\theta) = exp\{-(X_1+X_2-2\theta)\}1_{[\theta,\infty[}(X_{(1)}),$$

from where it follows that $X_{(1)} = \min(X_1, X_2)$ is the minimal sufficient statistic for θ and $\hat{\theta} = X_{(1)}$ is the maximum of the function

$$l(\theta) = ln L(\theta) = (2\theta - X_1 - X_2) 1_{[\theta,\infty[}(X_{(1)}),$$

which is increasing in θ on the interval $]-\infty, X_{(1)}]$. Since for any $x \geq 0$

$$P_\theta\{X_{(1)} > x\} = P_\theta\{X_1 > x, X_2 > x\} = \left(\int_x^\infty e^{-(t-\theta)} dt \right)^2 = e^{-2(x-\theta)},$$

we have

$$P_\theta\{X_{(1)} \leq x\} = G(x;\theta) = \left(1 - e^{-2(x-\theta)}\right) 1_{[\theta,\infty[}(x), \quad x \in R^1.$$

In this example the functions $I(\theta; X_{(1)})$ and $S(\theta; X_{(1)})$ are

$$I(\theta; X_{(1)}) = S(\theta; X_{(1)}) = G(X_{(1)}; \theta) = 1 - e^{-2(X_{(1)} - \theta)}.$$

They are decreasing in θ and hence from the theorem of Bolshev we have

$$1 - e^{-2(X_{(1)} - \theta_i)} = \gamma_1, \quad \text{and} \quad 1 - e^{-2(X_{(1)} - \theta_s)} = 1 - \gamma_2,$$

thus

$$\theta_i = X_{(1)} + \frac{1}{2} ln(1 - \gamma_1), \quad \text{and} \quad \theta_s = X_{(1)} + \frac{1}{2} ln \gamma_2.$$

The interval $]\theta_i, \theta_s[$ is the γ-confidence interval for θ if $\gamma = \gamma_1 + \gamma_2 - 1$.

The length of this interval is

$$\theta_s - \theta_i = \frac{1}{2}[ln \gamma_2 - ln(1 - \gamma_1)].$$

We have to find γ_1 and γ_2 such that $\gamma_1 + \gamma_2 = 1 + \gamma$, $0.5 < \gamma_i \leq 1$ $(i = 1, 2)$ and the interval $]\theta_i, \theta_s[$ is the shortest. We consider $\theta_s - \theta_i$ as the function of γ_2. In this case

$$(\theta_s - \theta_i)' = \frac{1}{2}[ln \gamma_2 - ln \gamma_2 - \gamma]' =$$

$$\frac{1}{2}\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_2 - \gamma} \right) < 0,$$

and hence $\theta_s - \theta_i$ is decreading in γ_2 $(0.5 < \gamma_2 \leq 1)$ and the minimal value of $\theta_s - \theta_i$ occurs when $\gamma_2 = 1$ and $\gamma_1 = 1 + \gamma - \gamma_2 = \gamma$. Since in this case

$$\theta_i = X_{(1)} + \frac{1}{2} ln(1 - \gamma) \quad \text{and} \quad \theta_s = X_{(1)}$$

$$\min(\theta_s - \theta_i) = -\frac{1}{2}\ln(1-\gamma) - \ln\sqrt{1-\gamma}.$$

6. Let X_1 and X_2 be two independent random variables uniformly distributed on $[\theta-1,\theta+1]$, $\theta \in R^1$. Find the shortest γ-confidence interval for θ.

Solution. It is clear that $Y_i - \theta$ is uniformly distributed on [-1,1], from where it follows that the distribution of the random variable

$$T = X_1 + X_2 - 2\theta = Y_1 + Y_2$$

does not depend on θ. It is easy to show that

$$G(y) = P\{T \le y\} = \begin{cases} 0, & y \le -2, \\ \frac{1}{8}(y+2)^2, & -2 \le y \le 0, \\ 1 - \frac{(y-2)^2}{8}, & 0 \le y \le 2, \\ 1, & y \ge 2. \end{cases}$$

The function

$$G(T) = G(X_1 + X_2 - 2\theta),\ \theta \in R^1,$$

is decreasing in θ. From (8) it follows that the inferior and the superior confidence limits with the confidence levels γ_1 and γ_2 correspondingly ($0.5 < \gamma_i \le 1$) satisfy the equations

$$G(X_1 + X_2 - 2\theta_i) = \gamma_1 \quad \text{and} \quad G(X_1 + X_2 - 2\theta_s) = 1 - \gamma_2,$$

from where we find

$$\theta_i = \frac{X_1 + X_2}{2} - 1 + \sqrt{2(1-\gamma_1)} \quad \text{and} \quad \theta_s = \frac{X_1 + X_2}{2} + 1 - \sqrt{2(1-\gamma_2)}.$$

It is easy to show that for given $\gamma = \gamma_1 + \gamma_2 - 1$ the function

$$\theta_s - \theta_i = 2 - \sqrt{2(1-\gamma_1)} - \sqrt{2(1-\gamma_2)}$$

has its minimal value (considered as function of γ_1, $0.5 < \gamma_1 \le 1$) when

$$\gamma_1 = \frac{1+\gamma}{2}.$$

In this case $\gamma_2 = \frac{1-\gamma}{2}$, so the shortest γ-confidence interval for θ is $]\theta_i, \theta_s[$ where

$$\theta_i = \frac{X_1 + X_2}{2} - 1 + \sqrt{1-\gamma} \quad \text{and} \quad \theta_s = \frac{X_1 + X_2}{2} + 1 - \sqrt{1-\gamma}.$$

7. Suppose that T is the number of shots until the first success. Find the $\gamma = 0.9$ confidence intervals for the probability p of success, if
a). $T = 1$; b). $T = 4$; c). $T = 10$.

Solution. The distribution of T is geometric :

$$P\{T=k\}=p(1-p)^{k-1},\ k=1,2,\dots.$$

The values of the distribution function of T in the points k are

$$G(k;p)=\sum_{i=1}^{k}p(1-p)^{i-1}=1-(1-p)^{k-1},\ k=1,2,\dots.$$

The functions I ans S are

$$I(p;T)=1-(1-p)^{T-1},\quad S(p;T)=1-(1-p)^{T}.$$

The functions $I(p;T)$ and $S(p;T)$ are increasing in p if $T>1$ and $T\geq 1$, respectively. So they are decreasing in $q=1-p$.

It follows from the formula (8) that γ_1 lower and upper confidence limits satisfy the equations

$$1-q_i^{T-1}=\gamma_1\quad\text{for}\quad T>1,$$
$$1-q_s^{T}=1-\gamma_1\quad\text{for}\quad T\geq 1.$$

So

$$q_i=(1-\gamma_1)^{\frac{1}{T-1}}\quad\text{for}\quad T>1, q_s=\gamma_1^{\frac{1}{T}}\quad\text{for}\quad T\geq 1$$

and

$$p_i=1-q_s=1-\gamma_1^{\frac{1}{T}}\quad\text{for}\quad T\geq 1, p_s=1-q_i=1-(1-\gamma_1)^{\frac{1}{T-1}}\quad\text{for}\quad T>1.$$

If $T=1$, then $q_i=\inf]0,1[=0,\ p_s=1$.

To find the $\gamma=1-\alpha=0.9$ confidence interval we take $\gamma_1=1-\alpha/2=\frac{1+\gamma}{2}=0.95$.

So the $\gamma=0.9$ confidence interval for p is (p_i,p_s), where

$$p_i=0.05,\ p_s=1\quad\text{for}\quad T=1,$$

$$p_i=1-0.95^{\frac{1}{4}}=0.01274,\quad p_s=1-0.05^{1/3}=0.6316\quad\text{for}\quad T=4,$$
$$p_i=1-0.95^{\frac{1}{10}}=0.005116,\quad p_s=1-0.05^{1/9}=0.2831\quad\text{for}\quad T=10.$$

8. Let $\mathbb{X}=(X_1,\dots,X_n)^T$ be a sample and suppose that X_i has the normal distribution: $X_i\sim N(\mu,\sigma^2)$. Find a γ confidence interval for μ.

Solution. The sufficient statistic is $(\bar{X},S^2)$

$$\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i,\quad S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

Consider the statistic

$$T(\mathbb{X},\mu) = \frac{\sqrt{n}(\overline{X}-\mu)}{S}.$$

The random variable $T(\mathbb{X},\mu)$ has the Student distribution with $n-1$ degrees of freedom and distribution function $F_{t_{n-1}}$. So

$$I(\mu,\mathbb{X}) = S(\mu,\mathbb{X}) = F_{t_{n-1}}(T(\mathbb{X},\mu)).$$

The functions I and S are decreasing with respect to μ, so by the theorem of Bolshev

$$F_{t_{n-1}}(T(\mathbb{X},\mu_i)) = \gamma_1 = \frac{1+\gamma}{2}$$

$$F_{t_{n-1}}(T(\mathbb{X},\mu_s)) = 1-\gamma_1 = \frac{1-\gamma}{2}$$

and

$$\mu_i = \overline{X} - \frac{S}{\sqrt{n}} t_{n-1}\left(\frac{1+\gamma}{2}\right),$$

$$\mu_s = \overline{X} + \frac{S}{\sqrt{n}} t_{n-1}\left(\frac{1+\gamma}{2}\right),$$

where $t_{n-1}(\alpha)$ is the α-quantile of the Student distribution with $n-1$ degrees of freedom.

Confidence intervals for the variance, for the difference of two means, for the ratio of two variances, etc., can be obtained in a similar way.

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