BOLSHEV'S METHOD OF CONFIDENCE LIMIT CONSTRUCTION

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Confidence intervals and regions for the parameters of a distribution are constructed, following the method due to L.N. Bolshev. This construction method is illustrated with Poisson, exponential, Bernoulli, geometric, normal and other distributions depending on parameters.

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1. Regions, Intervals, Confidence Limits

Let \( X = (X_1, \ldots, X_n) \) be a sample with realizations \( x = (x_1, \ldots, x_n) \), \( x \in X \subseteq \mathbb{R}^n \). Suppose that \( X_i \) has a density \( f(x; \theta) \), \( \theta = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq \mathbb{R}^k \), with respect to the Lebesgue measure, \( H_0 : X_i \sim f(x; \theta), \ \theta = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq \mathbb{R}^k \).

Let \( b = b(\theta) \) be a function \( b(\cdot) : \Theta \rightarrow B \subseteq \mathbb{R}^m \), \( B^0 \) is the interior of \( B \).

**Definition 1** A random set \( C(X), C(X) \subseteq B \subseteq \mathbb{R}^m \) is called the confidence region for \( b = b(\theta) \) with the confidence level \( \gamma \) \((0.5 < \gamma < 1)\) if

\[
\inf_{\Theta} P_\Theta \{ C(X) \ni b(\theta) \} = \gamma.
\]

This definition implies for all \( \theta \in \Theta \)

\[
P_\Theta \{ C(X) \ni b(\theta) \} \geq \gamma.
\]

In the case \( b(\theta) \in B \subseteq \mathbb{R}^1 \) the confidence region is often an interval in \( \mathbb{R}^1 \),

\[
C(X) = [b_1(X), b_2(X)] \subseteq B \subseteq \mathbb{R}^1,
\]

and it is called the confidence interval with the confidence level \( \gamma \) for \( b \). The statistics \( b_1(X) \) and \( b_2(X) \) are called the confidence limits of the confidence interval \( C(X) \).

**Definition 2** A statistic \( b_1(X), (b_2(X)) \) is called the inferior (superior) confidence limit with the confidence level \( \gamma_1 (\gamma_2) \) (or inferior (superior) \( \gamma_1, \gamma_2 \) - confidence limit briefly), if

\[
\inf_{\Theta} P_\Theta \{ b_1(X) < b \} = \gamma_1 \left( \inf_{\Theta} P_\Theta \{ b_2(X) > b \} = \gamma_2 \right), \quad 0.5 < \gamma_j < 1
\]

The \( \gamma = 1 - \alpha \) confidence interval has the form \([b_1(X), b_2(X)]\), where \( b_1(X) \) and \( b_2(X) \) are the \( \gamma_1 = 1 - \alpha_1 \) inferior and \( \gamma_2 = 1 - \alpha_2 \) superior confidence limits, respectively, such that \( \alpha_1 + \alpha_2 = \alpha, \ (0 < \alpha_1 < 0.5). \) If \( \alpha_1 = \alpha_2, \) then take \( \gamma_1 = \gamma_2 = 1 - \alpha/2. \)

**Definition 3** The intervals

\[
\{b_1(X), +\infty\} \quad \text{and} \quad (-\infty, b_2(X))
\]

are called the superior and inferior confidence intervals for \( b \). Both intervals are unilateral.

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2. THEOREM OF BOLSHEV

Lemma (Bolshev) Let $G(t)$ be the distribution function of the random variable $T$. Then for all $z \in [0, 1]$

\[
P\{G(T) \leq z\} \leq z \leq P\{G(T - 0) < z\}.
\]

If $T$ is continuous, then

\[
P\{G(T) \leq z\} = z, \quad 0 \leq z \leq 1.
\]

Proof: First, we prove the inequality

\[
P\{G(T) \leq z\} \leq z, \quad 0 \leq z \leq 1.
\]

If $z = 1$, then $P\{G(T) \leq 1\} \leq 1$. Fix $z \in [0, 1)$ and for this value of $z$ consider the different cases.

1) There exists a solution $y$ of the equation $G(y) = z$. Note

\[y_0 = \sup\{y : G(y) = z\}.
\]

It can be:

a) $G(y_0) = z$. In this case

\[P\{G(T) \leq z\} \leq P\{T \leq y_0\} = G(y_0) = z.
\]

b) $G(y_0) > z$. Then

\[P\{G(T) \leq z\} \leq P\{T < y_0\} = G(y_0 - 0) \leq z.
\]

2) A solution of the equation $G(y) = z$ does not exist. In this case there exists $y$ such that

\[G(y) > z \quad \text{et} \quad G(y - 0) < z,
\]

so

\[P\{G(T) \leq z\} \leq P\{T < y\} = G(y - 0) < z.
\]

The inequality (2) is proved.
We prove now the second inequality in (1):

\begin{equation}
(3) \quad z \leq P\{ G(T - 0) < z \}, \quad 0 \leq z \leq 1.
\end{equation}

Consider the statistic \(-T\). Its distribution function is

\[ G^-(y) = P\{ -T \leq y \} = P\{ T \geq -y \} = 1 - G(-y - 0). \]

Replacing \( T, z, G \) by \(-T, 1 - z\) and \( G^-\) in the inequality (2) we have:

\[ P\{ G^-(-T) \leq 1 - z \} \leq 1 - z, \quad 0 \leq z \leq 1. \]

This implies

\begin{align*}
P\{ 1 - G(T - 0) \leq 1 - z \} & \leq 1 - z, \\
P\{ G(T - 0) \geq z \} & \leq 1 - z, \\
P\{ G(T - 0) < z \} & \geq z, \quad 0 \leq z \leq 1.
\end{align*}

If \( T \) is continuous, then \( G(t - 0) = G(t) \), and (2) and (3) imply \( P\{ G(T) \leq z \} = z \) for all \( z \in [0, 1] \).

The lemma is proved.

\[ \blacksquare \]

**Theorem** (Bolshiev) Suppose that the random variable \( T = T(\mathcal{X}, b), b \in B \), is such that its distribution function

\[ G(t; b) = P_b\{ T \leq t \} \]

depends only on \( b \) for all \( t \in \mathcal{R} \) and the functions

\[ I(b; x) = G(T(x, b) - 0; b) \quad \text{and} \quad S(b, x) = G(T(x, b); b) \]

are decreasing and continuous in \( b \) for all fixed \( x \in \mathcal{X} \). In this case:

1) the statistic \( b_1(\mathcal{X}) \) such that

\begin{equation}
(4) \quad b_1 = b_1(\mathcal{X}) = \sup\{ b : I(b; \mathcal{X}) \geq \gamma, b \in B \}, \quad \text{if this supremum exists,}
\end{equation}
or

\[ b_i = \inf B \text{, otherwise} \]

is the inferior confidence limit for \( b \in B^0 \) with confidence level larger or equal to \( \gamma \);

2) the statistic \( b_s(\mathcal{X}) \) such that

\[ b_s = \inf \{ b : S(b;\mathcal{X}) \leq 1 - \gamma, \ b \in B \} \text{, if this infimum exists,} \]

or

\[ b_s = \sup B \text{, otherwise} \]

is the superior confidence limit for \( b \in B^0 \) with the confidence level larger or equal to \( \gamma \);

3) if \( x \in \mathcal{X} \), is such that the functions \( I(b;x) \) and \( S(b;x) \) are strongly decreasing with respect to \( b \), then \( b_i(x) \) and \( b_s(x) \) are the roots of the equations

\[ I(b_i(x);x) = \gamma \quad \text{and} \quad S(b_s(x);x) = 1 - \gamma \]

**Proof:** Denote \( D = D(\mathcal{X}) \) the event

\[ D = \{ \text{there exists } b \text{ such that } I(b;\mathcal{X}) \geq \gamma \}. \]

Then for the true value \( b \in B^0 \) we have (using Bolshev’s lemma)

\[
P\{b_i < b \} = P\{(b_i < b) \cap D\} + P\{(b_i < b) \cap \bar{D}\} = \
\]

\[
P\{(\sup b^* : I(b^*;\mathcal{X}) \geq \gamma, b^* \in B) < b \} \cap D\} + P\{(\inf B < b) \cap \bar{D}\} = \
\]

\[
= P\{(I(b;\mathcal{X}) < \gamma) \cap D\} + P\{I(b;\mathcal{X}) < \gamma\} \cap D\} + P\{(I(b;\mathcal{X}) < \gamma) \cap \bar{D}\} = \
\]

\[
= P\{I(b;\mathcal{X}) < \gamma\} \geq \gamma \]

The theorem is proved.

\[ \square \]

**Remark:** Often, instead of the statistic \( T \) a sufficient statistic or some function of a sufficient statistic for a parameter \( b \) can be taken.
3. EXAMPLES

1. Let \( X = (X_1, \ldots, X_n)^T \) be a sample and suppose that \( X_i \) has a Poisson distribution with a parameter \( \theta \):

\[
X_i \sim f(x; \theta) = \frac{\theta^x}{x!} e^{-\theta}, \ x \in X = \{0, 1, \ldots\}, \ \theta \in \Theta = [0, \infty[.
\]

Denote

\[ T = X_1 + \ldots + X_n. \]

a) Show that the statistics

\[
\theta_i = \frac{1}{2n} \chi^2_{2-\gamma_1}(2T) \quad \text{and} \quad \theta_s = \frac{1}{2n} \chi^2_{2+\gamma_2}(2T + 2)
\]
are the inferior and superior confidence limits for \( \theta \) with confidence levels larger or equal to \( \gamma_1 \) and \( \gamma_2 \) respectively; \( \chi^2_{n}(n) \) denotes the \( \alpha \)-quantile of a chi-square distribution with \( n \) degrees of freedom.

b) Find a confidence interval for \( \theta \) with confidence level larger or equal to \( \gamma \).

Solution. The sufficient statistic \( T \) follows the Poisson distribution with parameter \( n\theta \). Then

\[
G(k; \theta) = P_{\theta}\{T \leq k\} = \sum_{i=0}^{k} \frac{(n\theta)^i}{i!} e^{-n\theta} = P\{\chi^2_{2k+1} \geq 2n\theta\} = \mathcal{P}(2n\theta, 2k+2), \ k = 0, 1, \ldots
\]

and

\[
G(k-0; \theta) = P_{\theta}\{T < k\} = \sum_{i=0}^{k-1} \frac{(n\theta)^i}{i!} e^{-n\theta} = \mathcal{P}(2n\theta, 2k), \ k = 1, 2, \ldots
\]

\[
G(k-0; \theta) = 0, \ k = 0.
\]

The functions \( I \) and \( S \) are

\[
I(\theta; X) = \begin{cases} \mathcal{P}(2n\theta, 2T), & \text{if } X \neq 0, \\ 0, & \text{if } X = 0, \end{cases}
\]

\[
S(\theta; X) = \mathcal{P}(2n\theta, 2T + 2).
\]

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The function \( S \) is strictly decreasing for all \( T, T \geq 0 \), and \( I \) is strictly decreasing for all \( T \neq 0 \). In these cases the theorem of Bolshev implies (see (8)):

\[
P(2n\theta_1, 2T) = \gamma_1 \quad P(2n\theta, 2T + 2) = 1 - \gamma_2,
\]

from which it follows

\[
(9) \quad \theta_i = \frac{1}{2n} \chi^2_{1-\gamma_1} (2T), \quad \theta_s = \frac{1}{2n} \chi^2_{1-\gamma_2} (2T + 2).
\]

If \( T = 0 \) then \( I(\theta; X) = 0 \). There is no such \( \theta \) that

\[
I(\theta; X) = \gamma_1 > \frac{1}{2}.
\]

The formula (5) implies

\[
\theta_i = \inf_{\theta > 0} \theta = \inf_{\theta > 0} (0, +\infty) = 0.
\]

b) The interval \( [\theta_i, \theta_s] \) is the confidence interval for \( \theta \) with a confidence level larger or equal to \( \gamma = 1 - \alpha \), if \( \gamma_1 = 1 - \alpha_1, \gamma_2 = 1 - \alpha_2, \alpha_1 + \alpha_2 = \alpha \). If \( \alpha_1 = \alpha_2 \), take \( \gamma_1 = \gamma_2 = 1 - \alpha/2 \).

2. Let \( X = (X_1, \ldots, X_n)^T \) be a sample and suppose that \( X_i \) has an exponential distribution with mean \( \theta, \theta > 0 \):

\[
X_i \sim f(x; \theta) = \frac{1}{\theta} \exp\left\{ -\frac{x}{\theta} \right\} 1_{(x>0)}.
\]

a) Find \( \gamma \)-confidence limits for \( \theta \).

b) Let \( X^{(r)} = (X_{(1)}, \ldots, X_{(r)})^T \) be a type II censored sample from the distribution (10).

Find a \( \gamma \)-confidence interval for \( \theta \) and the survival function

\[
S(x; \theta) = P_\theta \{ X_1 > x \}.
\]

Solution. a). Denote

\[
T = X_1 + \ldots + X_n.
\]

The sufficient statistic \( T \) follows a gamma distribution \( G(n; \frac{1}{\theta}) \) with parameters \( n \) and \( 1/\theta \):

\[
P(T \leq t) = \frac{1}{(n-1)!\theta^n} \int_0^t u^{n-1} e^{-u/\theta} du, \quad t \geq 0,
\]

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and hence \( T/\theta \) follows the gamma distribution \( G(n; 1) \), and 
\[
\frac{2T}{\theta} = \chi^2_{2n}
\]

In this example the functions \( I \) and \( S \) can be taken as
\[
I(\theta; X) = S(\theta; X) = 1 - P \left( \frac{2T}{\theta}, 2n \right).
\]

These functions are decreasing in \( \theta \) and the formula (8) implies
\[
1 - P \left( \frac{2T}{\theta_i}, 2n \right) = \gamma \quad \text{and} \quad 1 - P \left( \frac{2T}{\theta_s}, 2n \right) = 1 - \gamma,
\]
from where we obtain
\[
\frac{2T}{\theta_i} = \chi^2_{1-\gamma}(2n) \quad \text{and} \quad \frac{2T}{\theta_s} = \chi^2_{1-\gamma}(2n),
\]
and hence
\[
\theta_i = \frac{2T}{\chi^2_{1-\gamma}(2n)} \quad \text{and} \quad \theta_s = \frac{2T}{\chi^2_{1-\gamma}(2n)}.
\]

b) As it is well known the statistic
\[
T_r = \sum_{k=1}^{r} X_{(k)} + (n-r)X_{(r)}
\]
follows a gamma distribution \( G(r; \frac{1}{\theta}) \), and hence the \( \gamma = 1 - \alpha \)-confidence interval for \( \theta \) is \( [\theta_i, \theta_s] \), where
\[
\theta_i = \frac{2T_r}{\chi^2_{1-\alpha/2}(2r)} \quad \text{and} \quad \theta_s = \frac{2T_r}{\chi^2_{1-\alpha/2}(2r)}.
\]
Since the survival function \( S(x; b) = e^{-x/\theta}, x > 0, \) is increasing in \( \theta \), we have the \( \gamma \)-confidence interval \([S_i, S_s]\) for \( S(x; \theta) \), where
\[
S_i = e^{-x/\theta_i} \quad \text{and} \quad S_s = e^{-x/\theta_s}.
\]

3. Let \( X = (X_1, \ldots, X_n)^T \) be a sample from Bernoulli distribution with parameter \( \theta \):
\[
X_i \sim f(x; \theta) = \theta^x(1 - \theta)^{1-x}, \quad x \in \{0, 1\}, \theta \in \Theta = [0, 1].
\]

Find the limits of confidence for \( \theta \) with the confidence levels larger or equal to \( \gamma \).

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Solution. It is clear that the sufficient statistic

\[ T = \sum_{i=1}^{n} X_i \]

follows the binomial distribution \( B(n, \theta) \) with parameters \( n \) and \( \theta \). Then

\[ G(k; \theta) = P_{\theta}(T \leq k) = \sum_{i=0}^{k} \binom{n}{i} \theta^i (1-\theta)^{n-i} = \]

\[ I_{1-\theta}(n-k, k+1) = 1 - I_{\theta}(k+1, n-k), k = 0, 1, \ldots, n-1, \]

\[ G(k; \theta) = 1, \text{ if } k = n, \]

where \( I_x(a, b) \) is the beta distribution function with parameters \( a \) and \( b \), and

\[ G(k-0; \theta) = \sum_{i=0}^{k-1} \binom{n}{i} \theta^i (1-\theta)^{n-i} = \]

\[ 1 - I_{\theta}(k, n-k+1), k = 1, 2, \ldots, n, \]

\[ G(k-0; \theta) = 0, \text{ if } k = 0. \]

The functions \( I \) and \( S \) are

\[ I(\theta; x) = \begin{cases} 
I_{1-\theta}(n-T+1, T), & \text{if } T \neq 0 \\
0, & \text{otherwise}
\end{cases} \]

\[ S(\theta; x) = \begin{cases} 
I_{1-\theta}(n-T, T+1), & \text{if } T \neq n \\
1, & \text{if } T = n.
\end{cases} \]

We remark that \( S(\theta; x) \) is strictly decreasing in \( \theta \) for \( T \neq n \), and \( I(\theta; x) \) is strictly decreasing in \( \theta \) for \( T \neq 0 \), and hence from the formula (8) it follows that

\[ I_{1-\theta}(n-T+1, T) = \gamma_1 \text{ for } T \neq 0 \]

and

\[ \theta_i = 0, \text{ if } T = 0, \]

\[ I_{1-\theta}(n-T, T+1) = 1 - \gamma_1 \text{ for } T \neq n \]

and

\[ \theta_s = 1, \text{ if } T = n. \]

Hence,

\[ \theta_i = \begin{cases} 
1 - x(\gamma_i; n-T+1, T), & \text{if } T \neq 0 \\
0, & \text{if } T = 0,
\end{cases} \]

\[ \theta_s = \begin{cases} 
1 - x(1-\gamma_i; n-T, T+1), & \text{if } T \neq n \\
1, & \text{if } T = n,
\end{cases} \]
where \( x(\gamma_1; a, b) \) is the \( \gamma_1 \)-quantile of the beta distribution with parameters \( a \) and \( b \).

4. Let \( X \) be a discrete random variable with the cumulative distribution function

\[
F(x; \theta) = P_\theta \{ X \leq x \} = (1 - \theta^x) 1_{[0, +\infty)}(x), \quad x \in R^1, \quad \theta \in \Theta = [0, 1[.
\]

Find a \( \gamma \)-confidence interval for \( \theta \), if \( X = 1 \).

**Solution.** In this case

\[
I(X; \theta) = F(X - 0; \theta) \quad \text{and} \quad S(X; \theta) = F(X; \theta).
\]

If \( X = 1 \) then

\[
I(1; \theta) = F(1 - 0; \theta) = F(0; \theta) = 0
\]

and according to the formula (5) we have that the inferior confidence limit \( \theta_i \) for \( \theta \) with confidence level larger or equal to \( \gamma_1 \) is

\[
\theta_i = \inf \theta = \inf [0, 1[ = 0.
\]

If \( \gamma_1 = 1 \) then \( P\{ \theta_i \leq \theta \} = \gamma_1 \), so \( \theta_i = 0 \) is 1-confidence inferior limit for \( \theta \). On the other hand the function

\[
S(1; \theta) = F(1; \theta) = 1 - \theta
\]

is decreasing in \( \theta \), and hence according to the formula (8) we have

\[
S(1; \theta_s) = 1 - \gamma_2,
\]

from where \( \theta_s = \gamma_2 \), so the \( \gamma_1 = 1 \) and \( \gamma_2 \) confidence limits for \( \theta \) are 0 and \( \gamma_2 \), and a gamma-confidence interval for \( \theta \) is \( ]0, \gamma_2[ \), since for \( \gamma_1 = 1 \) the equality \( \gamma = \gamma_1 + \gamma_2 - 1 \) is true when \( \gamma_2 = \gamma \).

5. Let \( X_1 \) and \( X_2 \) be two independent random variables,

\[
X_i \sim f(x; \theta) = e^{-(x-\theta)} 1_{[0,\infty)}(x), \theta \in \Theta = R^1.
\]

Find the smallest \( \gamma \)-confidence interval for \( \theta \).

**Solution.** The likelihood function \( L(\theta) \) for \( X_1 \) and \( X_2 \) is

\[
L(\theta) = \exp \{ -(X_1 + X_2 - 2\theta) \} 1_{[0,\infty)}(X_1),
\]

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from where it follows that $X_{(1)} = \min(X_1, X_2)$ is the minimal sufficient statistic for $\theta$ and $\hat{\theta} = X_{(1)}$ is the maximum of the function

$$I(\theta) = \ln L(\theta) = (2\theta - X_1 - X_2)1_{\theta = \gamma}(X_{(1)}),$$

which is increasing in $\theta$ on the interval $]-\infty, X_{(1)}]$. Since for any $x \geq 0$

$$P_{\theta}\{X_{(1)} > x\} = P_{\theta}\{X_1 > x, X_2 > x\} = \left(\int_x^\infty e^{-(t-\theta)}dt\right)^2 = e^{-2(x-\theta)},$$

we have

$$P_{\theta}\{X_{(1)} \leq x\} = G(x; \theta) = \left(1 - e^{-2(x-\theta)}\right)1_{\theta = \gamma}(x), \quad x \in R^1.$$

In this example the functions $I(\theta; X_{(1)})$ and $S(\theta; X_{(1)})$ are

$$I(\theta; X_{(1)}) = S(\theta; X_{(1)}) = G(X_{(1)}; \theta) = 1 - e^{-2(X_{(1)} - \theta)}.$$ 

They are decreasing in $\theta$ and hence from the theorem of Bolshev we have

$$1 - e^{-2(X_{(1)} - \theta_1)} = \gamma_1, \quad \text{ and } \quad 1 - e^{-2(X_{(1)} - \theta_2)} = 1 - \gamma_2,$$

thus

$$\theta_1 = X_{(1)} + \frac{1}{2} \ln (1 - \gamma_1), \quad \text{ and } \quad \theta_2 = X_{(1)} + \frac{1}{2} \ln \gamma_2.$$ 

The interval $[\theta_1, \theta_2]$ is the $\gamma$-confidence interval for $\theta$ if $\gamma = \gamma_1 + \gamma_2 - 1$.

The length of this interval is

$$\theta_2 - \theta_1 = \frac{1}{2} \left[\ln \gamma_2 - \ln (1 - \gamma_1)\right].$$

We have to find $\gamma_1$ and $\gamma_2$ such that $\gamma_1 + \gamma_2 = 1 + \gamma$. $0.5 < \gamma_i \leq 1$ ($i = 1, 2$) and the interval $[\theta_1, \theta_2]$ is the shortest. We consider $\theta_2 - \theta_1$ as the function of $\gamma_2$. In this case

$$(\theta_2 - \theta_1)' = \frac{1}{2} \left[\ln \gamma_2 - \ln \gamma_2 - \gamma_2'\right] = \frac{1}{2} \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_2 - \gamma}\right) < 0,$$

and hence $\theta_2 - \theta_1$ is decreasing in $\gamma_2$ ($0.5 < \gamma_2 \leq 1$) and the minimal value of $\theta_2 - \theta_1$ occurs when $\gamma_2 = 1$ and $\gamma_1 = 1 + \gamma - \gamma_2 = \gamma$. Since in this case

$$\theta_1 = X_{(1)} + \frac{1}{2} \ln (1 - \gamma) \quad \text{and} \quad \theta_2 = X_{(1)},$$

$$\gamma_1 = 1 + \gamma - \gamma_2 = \gamma.$$
\[
\min (\theta_i - \theta) = -\frac{1}{2} \ln (1 - \gamma) - \ln \sqrt{1 - \gamma}
\]

6. Let \( X_1 \) and \( X_2 \) be two independent random variables uniformly distributed on \([\theta - 1, \theta + 1], \theta \in R^1\). Find the shortest \( \gamma \)-confidence interval for \( \theta \).

**Solution.** It is clear that \( Y_i - \theta \) is uniformly distributed on \([-1,1]\), from where it follows that the distribution of the random variable
\[
T = X_1 + X_2 - 2\theta = Y_1 + Y_2
\]
does not depend on \( \theta \). It is easy to show that
\[
G(y) = P\{T \leq y\} = \begin{cases} 
0, & y \leq -2, \\
\frac{1}{8}(y+2)^2, & -2 \leq y \leq 0, \\
1 - \frac{(y-2)^2}{8}, & 0 \leq y \leq 2, \\
1, & y \geq 2.
\end{cases}
\]
The function
\[
G(T) = G(X_1 + X_2 - 2\theta), \theta \in R^1,
\]
is decreasing in \( \theta \). From (8) it follows that the inferior and the superior confidence limits with the confidence levels \( \gamma_1 \) and \( \gamma_2 \) correspondingly \((0.5 < \gamma_i \leq 1)\) satisfy the equations
\[
G(X_1 + X_2 - 2\theta_i) = \gamma_1 \quad \text{and} \quad G(X_1 + X_2 - 2\theta_s) = 1 - \gamma_2,
\]
from where we find
\[
\theta_i = \frac{X_1 + X_2}{2} - 1 + \sqrt{2(1 - \gamma_1)} \quad \text{and} \quad \theta_s = \frac{X_1 + X_2}{2} + 1 - \sqrt{2(1 - \gamma_2)}.
\]
It is easy to show that for given \( \gamma = \gamma_1 + \gamma_2 - 1 \) the function
\[
\theta_i - \theta_i = 2 - 2\sqrt{2(1 - \gamma_1)} - \sqrt{2(1 - \gamma_2)}
\]
has its minimal value (considered as function of \( \gamma_1 \), \( 0.5 < \gamma_1 \leq 1 \)) when
\[
\gamma_1 = \frac{1 + \gamma}{2}.
\]
In this case \( \gamma_2 = \frac{1 - \gamma}{2} \), so the shortest \( \gamma \)-confidence interval for \( \theta \) is \( ]\theta_i, \theta_s[ \) where
\[
\theta_i = \frac{X_1 + X_2}{2} - 1 + \sqrt{1 - \gamma} \quad \text{and} \quad \theta_s = \frac{X_1 + X_2}{2} + 1 - \sqrt{1 - \gamma}.
\]

7. Suppose that \( T \) is the number of shots until the first success. Find the \( \gamma = 0.9 \) confidence intervals for the probability \( p \) of success, if
a). \( T = 1 \); b). \( T = 4 \); c). \( T = 10 \).
Solution. The distribution of $T$ is geometric:

$$P\{T = k\} = p(1 - p)^{k-1}, \quad k = 1, 2, \ldots.$$ 

The values of the distribution function of $T$ in the points $k$ are

$$G(k; p) = \sum_{i=1}^{k} p(1 - p)^{i-1} = 1 - (1 - p)^{k-1}, \quad k = 1, 2, \ldots.$$ 

The functions $I$ and $S$ are

$$I(p; T) = 1 - (1 - p)^{T-1}, \quad S(p; T) = 1 - (1 - p)^{T}.$$ 

The functions $I(p; T)$ and $S(p; T)$ are increasing in $p$ if $T > 1$ and $T \geq 1$, respectively. So they are decreasing in $q = 1 - p$.

It follows from the formula (8) that $\gamma_1$ lower and upper confidence limits satisfy the equations

$$1 - q_i^{T-1} = \gamma_1 \quad \text{for} \quad T > 1,$$

$$1 - q_s^T = 1 - \gamma_1 \quad \text{for} \quad T \geq 1.$$ 

So

$$q_i = (1 - \gamma_1)^{\frac{1}{T-1}} \quad \text{for} \quad T > 1, \quad q_s = \gamma_1^T \quad \text{for} \quad T \geq 1$$

and

$$p_i = 1 - q_s = 1 - \gamma_1^T \quad \text{for} \quad T \geq 1, \quad p_s = 1 - q_i = 1 - (1 - \gamma_1)^{\frac{1}{T-1}} \quad \text{for} \quad T > 1.$$ 

If $T = 1$, then $q_i = \inf \{0, 1\} = 0$, $p_s = 1$.

To find the $1 - \alpha = 0.9$ confidence interval we take $\gamma_1 = 1 - \alpha/2 = \frac{1 + \gamma}{2} = 0.95$.

So the $0.9$ confidence interval for $p$ is $(p_i, p_s)$, where

$$p_i = 0.05, \quad p_s = 1 \quad \text{for} \quad T = 1,$$

$$p_i = 1 - 0.95^{\frac{1}{4}} = 0.01274, \quad p_s = 1 - 0.05^{1/3} = 0.6316 \quad \text{for} \quad T = 4,$$

$$p_i = 1 - 0.95^{\frac{1}{9}} = 0.005116, \quad p_s = 1 - 0.05^{1/9} = 0.2831 \quad \text{for} \quad T = 10.$$ 

8. Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a sample and suppose that $X_i$ has the normal distribution: $X_i \sim N(\mu, \sigma^2)$. Find a $\gamma$ confidence interval for $\mu$.

Solution. The sufficient statistic is $(\bar{X}, S^2)$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$ 

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Consider the statistic

\[ T(\bar{X}, \mu) = \frac{\sqrt{n}(\bar{X} - \mu)}{S}. \]

The random variable \( T(\bar{X}, \mu) \) has the Student distribution with \( n - 1 \) degrees of freedom and distribution function \( F_{n-1} \). So

\[ I(\mu, \bar{X}) = S(\mu, \bar{X}) = F_{n-1}(T(\bar{X}, \mu)). \]

The functions \( I \) and \( S \) are decreasing with respect to \( \mu \), so by the theorem of Bolshev

\[ F_{n-1}(T(\bar{X}, \mu_i)) = \gamma_1 = \frac{1 + \gamma}{2}, \]

\[ F_{n-1}(T(\bar{X}, \mu_s)) = 1 - \gamma_1 = \frac{1 - \gamma}{2}, \]

and

\[ \mu_i = \bar{X} - \frac{S}{\sqrt{n}} t_{n-1} \left( \frac{1 + \gamma}{2} \right), \]

\[ \mu_s = \bar{X} + \frac{S}{\sqrt{n}} t_{n-1} \left( \frac{1 + \gamma}{2} \right), \]

where \( t_{n-1}(\alpha) \) is the \( \alpha \)-quantile of the Student distribution with \( n - 1 \) degrees of freedom.

Confidence intervals for the variance, for the difference of two means, for the ratio of two variances, etc., can be obtained in a similar way.

REFERENCES

