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STATISTICAL MODELS STRONGLY–INVARIANT UNDER THE ACTION OF A GROUP

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Some basic results about invariance are given using quotient σ–fields. A strong kind of invariance is considered. Under appropiate conditions we obtain a sufficient statistics for models with such a invariance property.

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1. NOTATIONS AND BASIC RESULTS

The use of invariance considerations has been profitable for many statistical problems. These considerations are used for some particular problems by Hotelling and Pitman in the thirties. The relationship between sufficiency and invariance has been studied by C. Stein before 1950. The reader can find in Hall, Wijsman and Ghosh (1965) the basic definitions and results on invariance as well as some results on sufficiency and invariance. The paper by Landers and Rogge (1973) also deals with these notions. This paper is mainly concerned with statistical models which are not simply invariant under the action of a group of transformations but even strongly invariant (see below). The main result yields a way to calculate conditional expectations with respect to the σ–field of the G–invariant events and a sufficient statistics for such a model.

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We fix some notations to be used throughout the paper. Let $(\Omega, \mathfrak{U})$ and $(\Omega', \mathfrak{U}')$ be two measurable spaces and $f : \Omega \longrightarrow \Omega'$ a map. If $f^{-1}(A') \in \mathfrak{U}$ for all $A' \in \mathfrak{U}'$, we shall say that f is a random variable (r.v.) or a $(\mathfrak{U}, \mathfrak{U}')$–measurable map if we want to keep the σ–fields considered. If P is a probability measure on $(\Omega, \mathfrak{U})$ we shall write P^f for the probability law of the r.v. f, i.e., $P^f(A') = P(f \in A')$, $A' \in \mathfrak{U}'$. Let G be a group of one–to–one measurable maps of $(\Omega, \mathfrak{U})$ onto itself. G being a group, every $g \in G$ is bimeasurable. An event $A \in \mathfrak{U}$ is said to be G–invariant if $g(A) \subset A$ for all $g \in G$ (and then, $g(A) = A, \forall g \in G$). We shall write $\mathfrak{U}_G$ for the class of this events. It is easy to see that $\mathfrak{U}_G$ is a σ–field. A r.v. f is said G–invariant if $f(g(\omega)) = f(\omega)$ for every $g \in G$ and every observation $\omega \in \Omega$. With this notation we have the following

Proposition 1

Suppose that $\{\omega'\} \in \mathfrak{U}'$ for all $\omega' \in \Omega'$. The r.v. f is G–invariant if and only if f is a $(\mathfrak{U}_G, \mathfrak{U}')$–measurable map.

Proof

Suppose firts that f is G–invariant and let $A' \in \mathfrak{U}'$ and $g \in G$. For $\omega \in f^{-1}(A')$ we have that $f(g(\omega)) = f(\omega) \in A'$ and then $g(\omega) \in f^{-1}(A')$. This shows that $g(f^{-1}(A')) \subset f^{-1}(A')$. Since $f^{-1}(A') \in \mathfrak{U}$ we have that f is $(\mathfrak{U}_G, \mathfrak{U}')$–measurable.

Now suppose that f is a $(\mathfrak{U}_G, \mathfrak{U}')$–measurable map. Let $\omega \in \Omega$ and $g \in G$. Writing $\omega' = f(\omega)$ it follows that

$$g(\omega) \in g\ (f^{-1}(\{\omega'\})) = f^{-1}(\{\omega'\})$$

since $\{\omega'\} \in \mathfrak{U}'$ and f is $(\mathfrak{U}_G, \mathfrak{U}')$–measurable. This shows that f is G–invariant. ∎

Next let us describe a r.v. generating the σ–field $\mathfrak{U}_G$. The group G gives in a natural way an equivalence relation $\approx$ in Ω; namely: $\omega_1 \approx \omega_2$ if and only if there exists $g \in G$ such that $g(\omega_1) = \omega_2$. Each class of equivalence in Ω is called a orbit and it has the form $\{g(\omega)/g \in G\}$. We shall write Ω/G for the quotient set and ρ for the quotient map from Ω onto Ω/G. We write $\mathfrak{U}/G$ for the greatest σ–field making ρ measurable and we call it the quotient σ–field on Ω/G. ρ is a maximal invariant r.v., i.e., it is invariant and it takes

different values on different orbits. The following result shows that ρ is the r.v. generating $\mathfrak{U}_G$.

Proposition 2

a) $\mathfrak{U}/G = \{B \subset \Omega/G/\rho^{-1}(B) \in \mathfrak{U}\}$.

b) $\mathfrak{U}_G = \rho^{-1}(\mathfrak{U}/G)$.

Proof

a) It is easy to see that $\mathcal{C} = \{B \subset \Omega/G/\rho^{-1}(B) \in \mathfrak{U}\}$ is a σ–field. Since ρ is $(\mathfrak{U}, \mathcal{C})$–measurable we have that $\mathcal{C} \subset \mathfrak{U}/G$. On the other hand it is clear that $\mathcal{C}$ contains every σ–field on Ω/G making ρ measurable and then it contains $\mathfrak{U}/G$.

b) For $B \in \mathfrak{U}/G$ and $g \in G$ we have $\rho^{-1}(B) \in \mathfrak{U}$ and $g(\rho^{-1}(B)) = (\rho \circ g^{-1})^{-1}(B) = \rho^{-1}(B)$, ρ being G–invariant. This shows that $\rho^{-1}(\mathfrak{U}/G) \subset \mathfrak{U}_G$. Now let $A \in \mathfrak{U}_G$. We will show that $A = \rho^{-1}(\rho(A))$; this will suffices since then $\rho(A) \in \mathfrak{U}/G$ (see a)) and hence $A \in \rho^{-1}(\mathfrak{U}/G)$. It is always true that $A \subset \rho^{-1}(\rho(A))$. On the other hand, if $\omega \in \rho^{-1}(\rho(A))$ there exists $\omega' \in A$ such that $\rho(\omega) = \rho(\omega')$. Since $g(A) = A$ for all $g \in G$ we have that $\omega \in A$ and this gives the proof.

■

2. THE MAIN RESULT

A probability measure P on $(\Omega, \mathfrak{U})$ will be said G–invariant if $P^g = P$ for all $g \in G$ where P^g denotes the probability law of g with respect to P defined for $A \in \mathfrak{U}$ by

$$P^g(A) = P(g^{-1}(A)).$$

A statistical model or experiment is a triplet $(\Omega, \mathfrak{U}, \mathcal{P})$ where $\mathcal{P}$ is a family of probability measures on $(\Omega, \mathfrak{U})$. Such a family $\mathcal{P}$ (or the model) will be said to be strongly G–invariant if every $P \in \mathcal{P}$ is G–invariant.

In the rest of this paragraph let us suppose the group G endowed with a σ–field $\mathfrak{S}$. For $g' \in G$ we shall write $\tau_{g'}$ for the right–translation on G defined by

$$\tau_{g'}(g) = g \circ g', \qquad g \in G.$$

A probability measure μ on $(G, \mathfrak{S})$ satisfying

$$\mu(\tau_g(\Gamma)) = \mu(\Gamma)$$

for every $g \in G$ and every $\Gamma \in \mathfrak{S}$ will be called a right–invariant probability measure. We are now ready to show the main result of this paper.

Theorem

Let P a G–invariant probability measure on $(\Omega, \mathfrak{U})$, $f : (\Omega, \mathfrak{U}) \longrightarrow \Re$ a bounded r.v. and μ a right–invariant probability measure on $(G, \mathfrak{S})$. Suppose also that the map $(g,\omega) \in G \times \Omega \longrightarrow g(\omega) \in \Omega$ is $(\mathfrak{S} \times \mathfrak{U}, \mathfrak{U})$–measurable. We have then

$$E_P(f|\mathfrak{U}_G)(\omega) = \int_G f(g(\omega))\, d\mu(g) \quad P - \text{a.e.}$$

Proof:

Under the hypotheses,

$$\tilde{f} : (g,\omega) \in G \times \Omega \longrightarrow \tilde{f}(g,\omega) = f(g(\omega)) \in \Re$$

is a bounded Borel measurable map with respect to the product σ–field $\mathfrak{S} \times \mathfrak{U}$. Hence it is $\mu \times P$–integrable. Fubini's theorem shows that

$$h_f : \omega \in (\Omega, \mathfrak{U}) \longrightarrow h_f(\omega) = \int_G f(g(\omega))\, d\mu(g)$$

is Borel–measurable. For $\omega \in \Omega$ let us write $\hat{\omega}$ for the map $g \in G \longrightarrow g(\omega) \in \Omega$. We can write then

$$h_f(\omega) = \int_G f(\hat{\omega}(g))\, d\mu(g).$$

For $g' \in G$ and $\omega \in \Omega$ the right–invariance of μ gives that

$$\begin{aligned} h_f(g'(\omega)) &= \int_G f(g(g'(\omega)))\, d\mu(g) = \int_G f(\hat{\omega})((\tau_{g'}(g)))\, d\mu(g) = \\ &= \int_G f(\hat{\omega}(g))\, d\mu^{\tau_{g'}}(g) = h_f(\omega). \end{aligned}$$

This shows that h is G–invariant and then $\mathfrak{U}_G$–measurable in view of the proposition in the precedent paragraph.

Moreover for $A \in \mathfrak{U}_G$ Fubini's theorem shows that

$$\begin{aligned} \int_A h_f\, dP &= \int_A \int_G f(g(\omega))\, d\mu(g)\, dP(\omega) = \int_G \int_A f(g(\omega))\, dP(\omega)\, d\mu(g) = \\ &= \int_G \int_{g^{-1}(A)} f(g(\omega))\, dP(\omega)\, d\mu(g) = \int_G \int_A f(\omega)\, dP^g(\omega)\, d\mu(g) = \\ &= \int_G \int_A f\, dP\, d\mu = \int_A f\, dP \end{aligned}$$

and this finishes the proof. ■

The preceding theorem gives an effective way to calculate conditional expectations with respect to the σ–field $\mathfrak{U}_G$ and a G–invariant probability measure. Another immediate and remarkable consequence of this theorem and the proposition 2 is the following.

Corollary:

Let $\mathcal{P}$ a set of G–invariant probability measures on $(\Omega, \mathfrak{U})$. Then $\mathfrak{U}_G$ is a sufficient σ–field for the (strongly G–invariant) model $(\Omega, \mathfrak{U}, \mathcal{P})$. The quotient map $\rho : (\Omega, \mathfrak{U}) \longrightarrow (\Omega/G, \mathfrak{U}/G)$ is a sufficient statistic for this model.

Proof:

Since for any $A \in \mathfrak{U}$ the indicator I_A of A is a bounded real r.v., h_{I_A} is a version of the conditional probability $P(A|\mathfrak{U}_G)$ for all $P \in \mathcal{P}$. ■

Finally we give an example to illustrate the obtained results.

Example:

Let us choose $(\Omega, \mathfrak{U}) = (\Re^2, \mathcal{R}^2)$ where $\mathcal{R}^2$ denotes the Borel σ–field on $\Re^2$ and $G = \{r_\theta / \theta \in [0, 2\pi[\}$ where $r_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, $(x,y) \in \Re^2$. The σ–field $\mathcal{R}^2_G$ consists in the Borel subsets of $\Re^2$ which are invariant under rotations. The real G–invariant r.v. on $\Re^2$ are the Borel measurable maps $F : \Re^2 \longrightarrow \Re$ being constant on the circumferences centered at the origin (these circumferences are just the orbits). A Borel probability P on $\mathcal{R}^2$ will be G–invariant if and only if $P(r_\theta^{-1}(A)) = P(A)$ for every θ and every Borel set A; thus, if P has a density f with respect to the Lebesgue measure λ^2 on $\Re^2$, P is G–invariant iff $f(r_\theta(x,y)) = f(x,y)$ λ^2 – a.e. for all θ. For a real bounded Borel measurable map $\varphi : \Re^2 \longrightarrow \Re$ and a G–invariant Borel probability P, the map

$$(x,y) \longrightarrow (2\pi)^{-1} \int_0^{2\pi} \varphi(r_\theta(x,y)\, d\theta$$

is a version of the conditional expectation $E_P(\varphi|\mathcal{R}^2_G)$ since the uniform probability on $[0, 2\pi[$ yields in a natural way a right–invariant probability measure on G and the map $(\theta, (x,y)) \longrightarrow r_\theta(x,y)$ is measurable. The quotient space $(\Re^2/G, \mathcal{R}^2/G)$ can be identified with $([0, +\infty[, \mathcal{R}([0, +\infty[))$ identifying the equivalence class $\{r_\theta(x,y)/\theta \in [0, 2\pi[\}$ with $x^2 + y^2$ where $\mathcal{R}([0, +\infty[)$ denotes the Borel σ–field of $[0, +\infty[$ Thus the quotient map not differs essentially from the map

$$\rho(x,y) = x^2 + y^2.$$

Therefore ρ is a sufficient statistics for the nonparametric model

$$(\Re^2, \mathcal{R}^2, \mathcal{P})$$

where $\mathcal{P}$ denotes the family of all G–invariant Borel probabilities on $\mathcal{R}^2$.

3. REFERENCES

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