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AN EIGENVECTOR PATTERN ARISING IN NON LINEAR REGRESSION

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Let $A = (a_{ij})$ an $n \times n$ matrix defined by $a_{ij} = a_{ji} = i$, $i = 1, \cdots, n$. This paper gives some elementary properties of A and other related matrices. The eigenstructure of A is conjectured: given an eigenvector v of A the remaining eigenvectors are obtained by permuting up to sign the components of v. This problem arises in a distance based method applied to non linear regression.

Keywords: distance analysis, non linear regression, permutation matrices, permuting eigenvectors.

1. INTRODUCTION

In regressing a dependent variable Y on several explanatory variables $x_1, \ldots, x_p$ by using a distance-based model, as proposed by Cuadras (1989) and Cuadras and Arenas (1990), the diagonalization of a special patterned matrix arises. This method seems to be useful for mixed explanatory variables and can also be applied to perform a non-linear regression

$$(1) \qquad y_i = f(x_{i1}, \ldots, x_{ip}; \theta) + e_i \qquad i = 1, \ldots, n,$$

where the explanatory variables are continuous and the square distance

$$d_{ij}^2 = |x_{i1} - x_{j1}| + \cdots + |x_{ip} - x_{jp}|$$

between two observations $(x_{i1}, \ldots, x_{ip})$, $(x_{j1}, \ldots, x_{jp})$ is adopted.

The metric scaling on the $n \times n$ distance matrix $D = (d_{ij})$, where n is the number of observations (see Mardia *et al*, 1979), provides several principal

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coordinates defining linear, quadratic, cubic ... dimensions, and the vector of observations of Y is projected on them. This distance-based method has an interesting advantage: it is not necessary to specify the non-linear function f. In some applications f is unknown, so this approach could be useful.

As an illustration, table 1 reports the data observed in an experiment using newborn turkeys, relating the average body weight to two sources A and B of methionine as dietary supplement. The non-linear model considered is

$$y_i = \theta_1 + \theta_2 \left[1 - exp(\theta_3 x_{i1} + \theta_4 x_{i2})\right] + e_i \tag{2}$$

where y_i is the weight and x_{i1} and x_{i2} are the dose of methionine from sources A and B respectively (Weisberg, 1985, page 262). The left column in table 1 contains the observed data, the fourth column gives predictions from model (2) and the right column gives the predictions obtained by using the distance-based model mentioned above, ***without knowing*** the non-linear regression model.

TABLE I

Observed			Predicted	
	A	B	Model(2)	Distance-based model
y_i	x_{i1}	x_{i2}	$\hat{y}_i$	$\hat{y}_i$
672	0.04	0	671	672.5
709	0.10	0	708.6	707.5
729	0.16	0	736.4	730.5
778	0.28	0	772	777.4
797	0.44	0	795.7	797.2
680	0	0.04	678.5	679.5
721	0	0.10	721.4	722.5
750	0	0.16	751.4	748.5
790	0	0.28	785.3	790.6
799	0	0.44	804.2	798.8

The performance of this distance-based method has been confirmed in other examples. However, in studying the Euclidean coordinates related to distance (1), the following matrix pattern is found

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$$

i.e., $A = (a_{ij})$ is defined by

$$a_{ij} = a_{ji} = i \qquad i = 1, \dots, n.$$

The aim of this paper is to give some elementary properties of A and to enunciate an interesting conjecture.

2. SOME PROPERTIES

It is obvious that

$$A = MM' = N^2$$

where

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

Therefore A is positive definite matrix and both A and M have the same eigenvectors.

Some properties of certain patterned matrices C, which include M and A, are given by Graybill (1983, pages 186-187). However "the characteristic equation and roots of matrix C are difficult to evaluate in general" (page 206). Consequently, the eigenvectors of C also seem to be difficult to obtain algebraically.

Instead of A, let us consider the matrix B obtained by permuting rows and columns in A,

$$B = \begin{pmatrix} n & n-1 & \dots & 2 & 1 \\ n-1 & n-1 & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

i.e.,

$$b_{ij} = b_{ji} = n + 1 - i \qquad i = 1, \dots, n.$$

This special matrix appears as an example in Gregory and Karney (1969).

A and B are related by

$$B = APA$$

where P is the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Thus A and B have the same eigenvalues. If $\boldsymbol{u} = (u_1, \dots, u_n)'$ is an eigenvector of A, $A\boldsymbol{u} = \lambda \boldsymbol{u}$, then, since $P^2 = I$,

$$PAPP\boldsymbol{u} = PA\boldsymbol{u} = \lambda P\boldsymbol{u},$$

hence $P\boldsymbol{u} = (u_n, \dots, u_1)'$ is an eigenvector of B with the same eigenvalue λ.

Let $\boldsymbol{v}_1 = (v_1, \dots, v_n)'$, the eigenvector of B with the largest eigenvalue. It is satisfied that

(3) $$v_1 > \dots > v_n > 0.$$

Proof

All components of $\boldsymbol{v}_1$ are positive (Penrose's theorem) and $B\boldsymbol{v}_1 = \boldsymbol{w}_1 = (w_1, \dots, w_n)'$ satisfies

$$w_j - w_{j+1} = \sum_{i=1}^{n} v_j \qquad j = 1, \dots, n-1,$$

hence $w_1 > \dots > w_n > 0$. Since $\boldsymbol{w}_1 = \lambda \boldsymbol{v}_1$ inequality (3) holds.

The eigenvalues of B are given by

$$\lambda_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{2n+1}\pi\right)\right]^{-1} \qquad i = 1, \dots, n,$$

(see Frank, 1958), but the eigenvectors, as far as we know, are not well known.

3. THE CONJECTURE

To elucidate the structure of the eigenvectors of B, we firstly consider the case $n = 3$. Then $B = V\Lambda V'$, where

$$V = \begin{pmatrix} .737 & .591 & .328 \\ .591 & -.328 & -.737 \\ .328 & -.737 & .591 \end{pmatrix}$$

Let us denote the columns of V by $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and introduce the matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Note that:

a) V is a symmetric matrix.

b) The second and third columns of V are permutations up to sign of the first one.

c)

$$\boldsymbol{v}_2 = Q\boldsymbol{v}_1$$
$$\boldsymbol{v}_3 = -Q\boldsymbol{v}_2 = -Q^2\boldsymbol{v}_1$$

d) $\quad Q^3 = I,$

where I is the identity matrix.

Therefore the eigenstructure of B depends on $\boldsymbol{v}_1$ and Q.

Generally, suppose that the B matrix has order $n \geq 2$. We say that an $n \times n$ matrix Q is a signed-permutation matrix if each row and each column contains either 1 or -1 and the remaining elements are zero. It is easily proved that Q is non-singular,

$$Q' = Q^{-1}$$
$$Q^{n-1} = Q',$$

and

$$Q^n = I,$$

where I is the $n \times n$ identity matrix.

Conjecture

Let $B = (b_{ij})$ the $n \times n$ matrix defined by

$$b_{ij} = b_{ji} = n + 1 - i \qquad i = 1, \ldots, n,$$

and suppose $n \neq \dot{3} + 1$. Let $\boldsymbol{v}_1$ be the eigenvector with the largest eigenvalue λ_1. Then there exists a signed-permutation matrix Q such that

$$\boldsymbol{v}_1' Q \boldsymbol{v}_1 = 0$$

and

$$v_k = Q^{k-1} v_1 \qquad 2 \le k \le n$$

are the remaining eigenvectors of B. It is even possible to write the spectral decomposition

$$B = V \Lambda V \tag{4}$$

in such a way that V is an orthogonal symmetric matrix whose columns are standardized eigenvectors obtained from permutations up to signs of the components of v_1.

If $n = \dot{3} + 1$ *then V is also symmetric and some columns are obtained similarly, but other columns contain certain elements of* v_1 *and certain zeros.*

As an example, for $n = 6$ we obtain

$$V = \begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 2 & -1 & -4 & -6 & -3 \\ 4 & -1 & -6 & -2 & 3 & 5 \\ 3 & -4 & -2 & 5 & 1 & -6 \\ 2 & -6 & 3 & 1 & -5 & 4 \\ 1 & -3 & 5 & -6 & 4 & -2 \end{pmatrix}$$

where we write ranks instead of numerical values.

Remarks

i) The spectral decomposition (4), where V is a biorthogonal matrix, is also found for a special pattern of correlation matrix with the structure of Latin square (Tiit, 1984, 1986).
ii) A generalization is possible. Given a vector v and a signed-permutation matrix Q such that $v'Q^k v = 0$ $(k = 1, \ldots, n-1)$ and $Q^n = I$, we may consider the class $\mathcal{C}$ of $n \times n$ matrices such that $C \in \mathcal{C}$ iff

$$C = \sum_{i=1}^{n} \mu_i Q^{i-1} v v' Q^{-(i-1)}$$

where $\mu_1, \ldots, \mu_n$ are real values. It is conjectured that both v and Q exists.

Note that $B \in \mathcal{C}$ and the following property holds:

$$C \in \mathcal{C} \qquad \text{iff} \qquad Q^k C Q^{-k} \in \mathcal{C}$$

for any integer k (remember that $Q' = Q^{-1}$).

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