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A NEW PROOF OF THE MILLIKEN-AKDENIZ THEOREM

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A simple proof is given for a theorem by Milliken and Akdeniz (1977) about the difference of the Moore-Penrose inverses of two positive semi-definite matrices.

Keywords: Moore-Penrose inverse, positive semi-definite matrix (difference)

Let us denote A^+, B^+ the Moore-Penrose inverses of the matrices A, B.

THEOREM (Milliken and Akdeniz)

Let A, B and $B - A$ be positive semi-definite matrices. Necessary and sufficient for also $A^+ - B^+$ to be positive semi-definite is: $r(A) = r(B)$.

Proof:(sufficiency)

Let

$$A = SMS' \text{ and } B = T\Lambda T'$$

be the spectral descompositions of A and B, where M and Λ are full rank diagonal matrices, $S'S = I$ and $T'T = I$.

It follows from the third assumption that $M(S) \subset M(T)$, where $M(S)$ is the column space of S.

The equalities of ranks implies

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$$S = TR \text{ with orthogonal } R.$$

Hence

$$B - A = T\Lambda T' - SMS' = T\Lambda^{\frac{1}{2}} \left(I - \Lambda^{-\frac{1}{2}} RMR'\Lambda^{-\frac{1}{2}} \right) \Lambda^{\frac{1}{2}} T'$$

and

$$A^{+} - B^{+} = SM^{-1}S' - T\Lambda^{-1}T' = T\Lambda^{-\frac{1}{2}} \left(\Lambda^{\frac{1}{2}} RM^{-1}R'\Lambda^{\frac{1}{2}} - I \right) \Lambda^{-\frac{1}{2}} T'$$

The matrix $I - \Lambda^{-\frac{1}{2}} RMR'\Lambda^{-\frac{1}{2}}$ is positive semi-definite by assumption, hence $\Lambda^{\frac{1}{2}} RM^{-1}R'\Lambda^{\frac{1}{2}} - I$ is also positive semi-definite (Bear in mind that $R' = R^{-1}$).

(necessity)

In addition to $M(S) \subset M(T)$, we have $M(T) \subset M(S)$.

Hence

$$M(S) = M(T)$$

This yields the desired rank equality.

2. BIBLIOGRAPHY

[1] **Milliken, G.A.** and **Akdeniz, F.** (1977). "A theorem on the difference of the generalized inverses of two nonnegative matrices". *Communications in Statistics A, 6, 73-79.*