A NEW PROOF OF THE
MILLIKEN-AKDENIZ THEOREM

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A simple proof is given for a theorem by Milliken and Akdeniz (1977) about the difference of the Moore-Penrose inverses of two positive semi-definite matrices.

Keywords: Moore-Penrose inverse, positive semi-definite matrix (difference)

Let us denote $A^+, B^+$ the Moore-Penrose inverses of the matrices $A, B$.

THEOREM (Milliken and Akdeniz)

Let $A, B$ and $B - A$ be positive semi-definite matrices. Necessary and sufficient for also $A^+ - B^+$ to be positive semi-definite is: $r(A) = r(B)$.

Proof (sufficiency)

Let

\[ A = SMS' \text{ and } B = T\Lambda T' \]

be the spectral decompositions of $A$ and $B$, where $M$ and $\Lambda$ are full rank diagonal matrices, $S'S = I$ and $T'T = I$.

It follows from the third assumption that $M(S) \subset M(T)$, where $M(S)$ is the column space of $S$.

The equalities of ranks implies
\[ S = TR \] with \ orthogonal R.

Hence

\[ B - A = TAT' - SMS' = TA^{\frac{1}{2}} \left( I - \Lambda^{-\frac{1}{2}} RM R'A^{-\frac{1}{2}} \right) \Lambda^{\frac{1}{2}} T' \]

and

\[ A^+ - B^+ = SM^{-1} S' - TA^{-1} T' = TA^{-\frac{1}{2}} \left( \Lambda^{\frac{1}{2}} RM^{-1} R'A^{\frac{1}{2}} - I \right) \Lambda^{-\frac{1}{2}} T' \]

The matrix \( I - \Lambda^{-\frac{1}{2}} RM R'A^{-\frac{1}{2}} \) is positive semi-definite by assumption, hence \( \Lambda^{\frac{1}{2}} RM^{-1} R'A^{\frac{1}{2}} - I \) is also positive semi-definite (Bear in mind that \( R' = R^{-1} \)).

(necessity)

In addition to \( M(S) \subset M(T) \), we have \( M(T) \subset M(S) \).

Hence

\[ M(S) = M(T) \]

This yields the desired rank equality.

2. BIBLIOGRAPHY