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Parametric Families of Fuzzy Consequence Operators

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Abstract

In a previous paper ([6]) we explored the notion of coherent fuzzy consequence operator. Since we did not know of any example in the literature of non-coherent fuzzy consequence operator, we also showed several families of such operators. It is well-known that the operator induced by a fuzzy preorder through Zadeh's compositional rule is always a coherent fuzzy consequence operator. It is also known that the relation induced by a fuzzy consequence operator is a fuzzy preorder if such operator is coherent ([5]). The aim of this paper is to show a parametric family of non-coherent fuzzy consequence operators which induce a preorder and also a family of non-coherent fuzzy consequence operators which do not induce a preorder. These families of operators can be implemented through very simple algorithms.

1 Introduction

The use of Consequence Operators in classical logic is well-known. These operators are introduced by A. Tarski in 1930 ([13]).

The Fuzzy Preorder concept and the Fuzzy Consequence Operator concept (FCO for short) are essentials on fuzzy logic. These notions have been defined as a natural generalization.

Given a non-empty universal set X which will represent a set of propositions, a fuzzy binary relation R on X (fuzzy subset of $X \times X$) is called a fuzzy preorder if it verifies :

(R1) $R(x,x) = 1 \ \forall \ x \in X$ (reflexivity)

(R2) $R(x,z) \geq min\big(R(x,y), R(y,z)\big) \ \forall \ x,y,z \in X$ (transitivity).

Notice that we consider the minimum as t-norm and the *min*-fuzzy preorders will be called fuzzy preorder for short.

Γ', Γ_r and Γ will represent the class of fuzzy relations on X, the family of fuzzy reflexive relations and the subfamily of fuzzy preorders, respectively.

Fixed a closed lattice L which will be the range of the memberships of the fuzzy subsets of X, J. Pavelka introduces in 1979 the concept of FCO on X in fuzzy logic

([11]), extending the concept of consequence operator in Tarski's sense in a natural way. A function $C : L^X \longrightarrow L^X$ is a FCO on X if it verifies :
(C1) $\mu \subset C(\mu)$ for all fuzzy subset $\mu \in L^X$ (inclusion)
(C2) Si $\mu_1 \subset \mu_2 \Longrightarrow C(\mu_1) \subset C(\mu_2)$ for all $\mu_1, \mu_2 \in L^X$ (monotony)
(C3) $C(C(\mu)) \subset C(\mu)$ for all $\mu \in L^X$ (idempotence)

Notice that, under the inclusion axiom, (C3) may be so written equivalently as (C3') $C(C(\mu)) = C(\mu)\ \forall\ \mu \in L^X$.

Under the logical point of view, the previous axioms could be interpreted in the following way:

If X denotes a set of propositions, μ is a (fuzzy) subset of axioms and $C(\mu)$ the theory generated by μ then
(C1) shows that the axioms are included into the theory;
(C2) shows that if the set of axioms increase, the generated theory also increase;
(C3') shows that the theory is stationary in the first step, this is, if the theory is considered itself as a new set of axioms to generate a new theory, it does not increase.

These operators, sometimes called closure operators, are studied in a general context ([1]), ([2]), ([3]), ([15]). During the last decade, these operators have been also studied in the context of fuzzy logic, taking the chain $L = [0, 1]$ as a special case ([4]), ([5]), ([6]), ([7]), ([8]), ([9]), ([10]), ([12]), ([14]). In this paper we will consider $L = [0, 1]$.

Recall some notions, notations and results about fuzzy consequence operators.

If R is a fuzzy preorder on X then the operator C_R from $[0, 1]^X$ into $[0, 1]^X$ given by the Zadeh's compositional rule: $C_R(\mu) = \mu \circ R$, is a fuzzy consequence operator (induced by R), where

$$\mu \circ R(x) = \sup_{w \in X} \big\{ min\big(\mu(w), R(w, x)\big)\big\} \qquad (1)$$

Moreover, E. Trillas and J.L. Castro added the coherence axiom to the FCO concept ([5]). A FCO C is called coherent if

$$min\big(\mu(a), C(\varphi_a)(x)\big) \leq C(\mu)(x) \forall\ \mu \in [0, 1]^X\ \ \forall\ (a, x) \in X \times X \qquad (2)$$

where $\varphi_a(t) = \varphi_{\{a\}}(t) = \begin{cases} 1 & t = a \\ 0 & t \neq a \end{cases}$ is the crisp membership of the singleton $\{a\}$.

In fact, C_R is a coherent FCO. It is also known that if C is a coherent FCO then the relation R_C given by $R_C(x, y) = C(\varphi_x)(y)$ is a fuzzy preorder (induced by C). Finally, for any relation R on X R_{C_R} is exactly the relation R ([5]).

Since any crisp consequence operator is coherent, the equivalence between classical consequence operators and classical preorders is held. This is not true in the fuzzy case.

In Section 2 we will show a parametric family of non-coherent fuzzy consequence operators which induce a preorder.

In Section 3 we will show a parametric family of non-coherent fuzzy consequence operators which do not induce a preorder.

The following definitions about the partial verification of the coherence axiom will be used in Section 2 and Section 3.

Let C be an FCO on X. We will say that C is coherent for the fuzzy set μ with respect to the ordered pair $(a,b) \in X \times X$ if $min\big(\mu(a), C(\varphi_a)(b)\big) \leq C(\mu)(b)$. We will say that C is coherent for the fuzzy set μ with respect to the element a if $min\big(\mu(a), C(\varphi_a)(x)\big) \leq C(\mu)(x)$ for all $x \in X$, that is, if C is coherent for the fuzzy set μ with respect to the ordered pair (a,x) for all $x \in X$. We will say that C is coherent for the fuzzy set μ if C is coherent for the fuzzy set μ with respect to every element a in X. We will say that C is coherent with respect to the ordered pair (a,b) if C is coherent for μ with respect to (a,b) for all fuzzy set μ. We will say that C is coherent with respect to the element a if C is coherent for μ with respect to a for all fuzzy set μ. Remark that a FCO C is coherent if and only if C is coherent for all fuzzy set μ with respect to every pair (x,y).

Finally notice that if the cardinal of the universe X is one, that is, $X = \{x\}$, then every FCO C on X is a coherent operator. In fact, it is enough to use the inclusion axiom of the operator C to obtain $min\big(\mu(x), C(\varphi_x)(x)\big) = min\big(\mu(x), 1\big) = \mu(x) \leq C(\mu)(x)$.

2 A Parametric Family of non-coherent Fuzzy Consequence Operators that induce preorder

For every universe X with card $X \geq 2$ we will show an uniparametric family of non-coherent fuzzy consequence operators that induce a preorder.

$C_\gamma^{a\,b}$ **Family**

For every pairwise different elements a, b in X and for every real number such that $0 \leq \gamma < \frac{1}{2}$, we define the operator from $[0,1]^X$ into $[0,1]^X$ given by $C_\gamma^{a\,b}(\mu) = \begin{cases} \mu' & \text{if } \mu \in E \\ \mu_\gamma & \text{if } \mu \notin E \end{cases}$ where μ' and μ_γ are the following fuzzy subsets:

$$\mu'(x) = \begin{cases} \mu(x) & \text{if } x \neq b \\ max\big(\mu(x)\,,\, \frac{1}{2}\big) & \text{if } x = b \end{cases}$$

$$\mu_\gamma(x) = \begin{cases} \mu(x) & \text{if } x \neq b \\ max\big(\mu(x)\,,\, \gamma\big) & \text{if } x = b \end{cases}$$

and E represents the family of fuzzy subsets $E = \Big\{\mu \in [0,1]^X \,/\, \mu(a) > \frac{1}{2}\Big\}$.

Notice that the family E has the following stability property: if $\mu \in E$ and $\mu \subset \nu$ then $\nu \in E$. This property will be essential in order to prove the monotony axiom of the following family of operators.

In the notation of previous family, the superindexes are used to show the dependence of the operator to the elements of X and the subindex are reserved to show the dependence of the operator to the parameters. We will agree the same for the following operators.

Remark that we put C as $C_{\gamma}^{a\,b}$ in order to abridge the following proofs.

Theorem 1. *$C_{\gamma}^{a\,b}$ is a non-coherent fuzzy consequence operator.*

Proof. Check that $C_{\gamma}^{a\,b}$ verifies the axioms of fuzzy consequence operator:

(C1) Since $\mu' \supset \mu$ and $\mu_{\gamma} \supset \mu$, $\mu \subset C(\mu)\ \forall\ \mu \in [0,1]^X$.

(C2) Given μ_1 , $\mu_2 \in [0,1]^X$ such that $\mu_1 \subset \mu_2$, see that $C(\mu_1) \subset C(\mu_2)$:

(i) If $\mu_1 \in E$. From stability property of E $\mu_2 \in E$ and $C(\mu_1) = \mu_1' \subset \mu_2' = C(\mu_2)$.

(ii) If $\mu_1 \notin E$ then $C(\mu_1) = \mu_{1_{\gamma}}$ and

(iia) If $\mu_2 \notin E$, $C(\mu_2) = \mu_{2_{\gamma}}$. Thus $\mu_{1_{\gamma}} \subset \mu_{2_{\gamma}}$ (iib) If $\mu_2 \in E$, $C(\mu_2) = \mu_2'$. Since $\mu_1 \subset \mu_2$, $\mu_{1_{\gamma}}(b) = max\big(\mu_1(b),\gamma\big) \le max\big(\mu_2(b),\frac{1}{2}\big) = \mu_2'(b)$. Moreover $\mu_{1_{\gamma}}(x) = \mu_1(x) \le \mu_2(x) = \mu_2'(x)$ for all $x \neq b$, hence $\mu_{1_{\gamma}} \subset \mu_2'$.

Therefore $C(\mu_1) \subset C(\mu_2)\ \forall\ \mu_1$, μ_2 such that $\mu_1 \subset \mu_2$.

(C3) (i) If $\mu \in E$, $C\big(C(\mu)\big) = C(\mu')$. From stability property of E, $\mu' \in E$ and $C(\mu') = \mu''$. Since the maximum of real numbers is an idempotent operation, $\mu' = \mu''$. Hence $C\big(C(\mu)\big) = C(\mu)$.

(ii) If $\mu \notin E$, clearly $\mu_{\gamma} \notin E$ and $C\big(C(\mu)\big) = C(\mu_{\gamma}) = \mu_{\gamma_{\gamma}} = \mu_{\gamma} = C(\mu)$. Therefore, $C\big(C(\mu)\big) = C(\mu)\ \ \forall\ \mu \in [0,1]^X$.

Now see that $C_{\gamma}^{a\,b}$ is not coherent for every $\mu \in [0,1]^X$ with respect to (a,b) if $max\big(\mu(b),\gamma\big) < \mu(a) \le \frac{1}{2}$. Under such condition $\mu \notin E$ and $min\big(\mu(a), C\big(\varphi_a\big)(b)\big) = min\big(\mu(a),\varphi_a'(b)\big) = min\big(\mu(a),\frac{1}{2}\big) = \mu(a) > max\big(\mu(b),\gamma\big) = \mu_{\gamma}(b) = C(\mu)(b)$. $\square$

Therefore the operator $C_{\gamma}^{a\,b}$ is not coherent. However it verifies the inequality (2) in a lot of cases. In fact the following theorem holds.

Theorem 2. *Given two pairwise different elements a, b in X and $0 \le \gamma < \frac{1}{2}$ then*
[1] $C_{\gamma}^{a\,b}$ is coherent with respect to (x,y) for all $(x,y) \neq (a,b)$.
[2] $C_{\gamma}^{a\,b}$ is coherent for μ with respect to (a,b) if and only if $\mu \notin F$, where F represents the following set: $F = \Big\{\mu \in [0,1]^X / max\big(\mu(b),\gamma\big) < \mu(a) \le \frac{1}{2}\Big\}$.

Proof. If $x = y$, $C_{\gamma}^{a\,b}$ is coherent with respect (x,y) as we have shown at the end of Section 1. Thus in this proof we suppose that $x \neq y$

Prove [1]. For every μ we obtain,

(i) If $x \neq a$ and $y \neq b$,

$$min\big(\mu(x), C\big(\varphi_x\big)(y)\big) = min\big(\mu(x),\big(\varphi_x\big)_{\gamma}(y)\big) =$$

$$= min\big(\mu(x),\varphi_x(y)\big) = min\big(\mu(x),0\big) = 0 \le C\big(\mu\big)(y)$$

(ii) If $x = a$ and $y \neq b$,

$$min\big(\mu(a), C\big(\varphi_a\big)(y)\big) = min\big(\mu(a),\varphi_a'(y)\big) =$$

$$= min\big(\mu(a), \varphi_a(y)\big) = min\big(\mu(a), 0\big) = 0 \leq C\big(\mu\big)(y)$$

(iii) If $x \neq a$ and $y = b$,
If $x = b$ then $x = y = b$ and $C_\gamma^{a\,b}$ is coherent for μ with respect to (x, y).
Suppose that $x \neq b$. Then $x \neq a$ and $y = b$:

$$min\big(\mu(x), C\big(\varphi_x\big)(b)\big) = min\big(\mu(x), \big(\varphi_x\big)_\gamma(b)\big) = min\big(\mu(x), \gamma\big) \leq \gamma$$

Now, if $\mu \in E$,

$$C\big(\mu\big)(b) = \mu'(b) = max\big(\mu(b), \frac{1}{2}\big) \geq \frac{1}{2} > \gamma$$

and if $\mu \notin E$,

$$C\big(\mu\big)(b) = \mu_\gamma(b) = max\big(\mu(b), \gamma\big) \geq \gamma$$

Prove [2]. It is proved that if $\mu \in F$ then $C_\gamma^{a\,b}$ is not coherent for μ with respect to (a, b).

Conversely, suppose that $\mu \notin F$ and see that $C_\gamma^{a\,b}$ is coherent for μ with respect to (a, b).

Notice that if $\mu \in E$,

$$min\big(\mu(a), C\big(\varphi_a\big)(b)\big) = min\big(\mu(a), \varphi'_a(b)\big) = min\big(\mu(a), \frac{1}{2}\big) =$$

$$= \frac{1}{2} \leq max\big(\mu(b), \frac{1}{2}\big) = \mu'(b) = C\big(\mu\big)(b)$$

and $C_\gamma^{a\,b}$ is coherent for μ with respect to (a, b).

Now, if $\mu \notin F$ then $\mu(a) > \frac{1}{2}$ or $max\big(\mu(b), \gamma\big) \geq \mu(a)$.

(i) If $\mu(a) > \frac{1}{2}$ then $\mu \in E$ and it is proved.

(ii) If $max\big(\mu(b), \gamma\big) \geq \mu(a)$ and $\mu \in E$, it is proved.
(iii) If $max\big(\mu(b), \gamma\big) \geq \mu(a)$ and $\mu \notin E$:

$$min\big(\mu(a), C\big(\varphi_a\big)(b)\big) = min\big(\mu(a), \varphi'_a(b)\big) = min\big(\mu(a), \frac{1}{2}\big) \leq$$

$$\leq \mu(a) \leq max\big(\mu(b), \gamma\big) = \mu_\gamma(b) = C\big(\mu\big)(b) \qquad \square$$

The following theorem shows that the relation $R_{C_\gamma^{a\,b}}$ induced by the operator $C_\gamma^{a\,b}$ is a fuzzy preorder on X. Consequently, the coherence axiom on a FCO C is not a necessary condition in order that the relation induced by C to be a preorder.

Theorem 3. *Given two different elements a, b in X, if $0 \leq \gamma < \frac{1}{2}$ then the relation $R_{C_\gamma^{a\,b}}$ induced by the operator $C_\gamma^{a\,b}$ is a fuzzy preorder.*

Proof. From the inclusion axiom of the operator C, it is clear that

$$R_C(x, x) = C\big(\varphi_x\big)(x) \geq \varphi_x(x) = 1$$

and R_C is reflexive relation.
Now, prove that

$$R_C(x,z) \geq min\big(R_C(x,y), R_C(y,z)\big) \;\forall\; x,y,z \in X$$

that is

$$C\big(\varphi_x\big)(z) \geq min\big(C\big(\varphi_x\big)(y), C\big(\varphi_y\big)(z)\big) \tag{3}$$

If $x = y$ or $x = z$ or $y = z$ the previous condition holds. Thus, suppose that x, y, z are pairwise different elements.

Notice that if $y \neq b$ then $C\big(\varphi_x\big)(y) = \varphi_x(y)$. Since $x \neq y$, $C\big(\varphi_x\big)(y) = \varphi_x(y) = 0$. Therefore, the second member of the inequality (3) is equal to 0 and it is held.

Analogously, if $z \neq b$, $C\big(\varphi_y\big)(z) = \varphi_y(z) = 0$ and (3) holds.

Finally, if $y = b = z$ then x, y, z are not pairwise different elements and (3) also holds. □

3 A Parametric Family of non-coherent Fuzzy Consequence Operators that do not induce preorder

For every universe X with card $X \geq 3$ we will show an uniparametric family of fuzzy consequence operators that do not induce a preorder. Consequently they are not coherent operators.

Remark that every FCO C verifies the inclusion axiom then R_C is a reflexive relation. Moreover, the inequality (3) holds if x, y, z are not pairwise different elements. In particular, if card $X \leq 2$ then every FCO induces a fuzzy preorder.

$C^{xyz}_{\alpha\beta\gamma\delta}$ **Family**

Let X be an universe with card $X \geq 3$, for every three pairwise different elements x, y, z consider the following families :

$$E_x = \Big\{\mu \in [0,1]^X / \mu(x) = 1\Big\}$$

$$E_y = \Big\{\mu \in [0,1]^X / \mu(y) = 1\Big\}$$

Notice that both families and their union and their intersection have the same stability property that the set E of previous family, namely if $\mu \in E$ and $\mu \subset \nu$ then $\nu \in E$.

For every real numbers a, b, c in $[0,1]$, we define the following fuzzy subset of X:

$$\mu_{abc}(t) = \begin{cases} max\big(\mu(t)\,,\,a\big) & \text{if } t = x \\ max\big(\mu(t)\,,\,b\big) & \text{if } t = y \\ max\big(\mu(t)\,,\,c\big) & \text{if } t = z \\ \mu(t) & \text{if } t \notin \{x,y,z\} \end{cases}$$

Then we define the operator $C^{xyz}_{\alpha\beta\gamma\delta}$ from $[0,1]^X$ into $[0,1]^X$ given by

$$C^{xyz}_{\alpha\beta\gamma\delta}(\mu) = \begin{cases} \mu_{1\alpha\beta} & \text{if } \ \mu \in E_x \setminus E_y \\ \mu_{\delta 1\gamma} & \text{if } \ \mu \in E_y \setminus E_x \\ \mu_{11\gamma} & \text{if } \ \mu \in E_x \cap E_y \\ \mu & \text{if } \ \mu \notin E_x \cup E_y \end{cases}$$

Theorem 4. *If $1 > \alpha > \beta \geq 0$, $1 \geq \gamma > \beta \geq 0$ and $1 > \delta \geq 0$ then the operator $C^{xyz}_{\alpha\beta\gamma\delta}$ is a fuzzy consequence operator such that $R_{C^{xyz}_{\alpha\beta\gamma\delta}}$ is not a fuzzy preorder on X. In particular $C^{xyz}_{\alpha\beta\gamma\delta}$ is a non-coherent operator.*

Proof. Check that $C \equiv C^{xyz}_{\alpha\beta\gamma\delta}$ verifies the axioms of fuzzy consequence operator:

(C1) If $\mu \in E_x \cup E_y$ it is clear that $\mu \subset \mu_{pqr} = C(\mu)$ for all p, q, r.

If $\mu \notin E_x \cup E_y$ then obviously $\mu \subset \mu = C(\mu)$.

Hence $\mu \subset C(\mu) \ \ \forall \ \mu \in [0,1]^X$.

(C2) Given μ_1 , $\mu_2 \in [0,1]^X$ such that $\mu_1 \subset \mu_2$, see that $C(\mu_1) \subset C(\mu_2)$.

Observe that if $\mu_1 \notin E_x \cup E_y$, $C(\mu_1) = \mu_1 \subset \mu_2 \subset C(\mu_2)$.

(i) If $\mu_2 \notin E_x \cup E_y$, as $E_x \cup E_y$ has the stability property $\mu_1 \notin E_x \cup E_y$ and it is proved.

Thus, in the following cases we can suppose that $\mu_1 \in E_x \cup E_y$:

(ii) If $\mu_2 \in E_x \cap E_y$:

(iia) If $\mu_1 \in E_x \cap E_y$, $C(\mu_1) = \mu_{1_{11\gamma}} \subset \mu_{2_{11\gamma}} = C(\mu_2)$.

(iib) If $\mu_1 \in E_x \setminus E_y$, $C(\mu_1) = \mu_{1_{1\alpha\beta}}$, as $\alpha < 1$ y $\beta < \gamma$ then $C(\mu_1) = \mu_{1_{1\alpha\beta}} \subset \mu_{1_{11\gamma}} \subset \mu_{2_{11\gamma}} = C(\mu_2)$.

(iic) If $\mu_1 \in E_y \setminus E_x$, $C(\mu_1) = \mu_{1_{\delta 1\gamma}}$, as $\delta < 1$ then $C(\mu_1) = \mu_{1_{\delta 1\gamma}} \subset \mu_{1_{11\gamma}} \subset \mu_{2_{11\gamma}} = C(\mu_2)$.

(iii) If $\mu_2 \in E_x \setminus E_y$, as E_y has the stability property, $\mu_1 \notin E_y$ and $\mu_1 \in E_x \cup E_y$ then $\mu_1 \in E_x \setminus E_y$ and $C(\mu_1) = \mu_{1_{1\alpha\beta}} \subset \mu_{2_{1\alpha\beta}} = C(\mu_2)$.

(iv) If $\mu_2 \in E_y \setminus E_x$, as E_x has the stability property, $\mu_1 \notin E_x$ and $\mu_1 \in E_x \cup E_y$ then $\mu_1 \in E_y \setminus E_x$ and $C(\mu_1) = \mu_{1_{\delta 1\gamma}} \subset \mu_{2_{\delta 1\gamma}} = C(\mu_2)$

Therefore $C(\mu_1) \subset C(\mu_2) \ \forall \ \mu_1$, μ_2 such that $\mu_1 \subset \mu_2$.

(C3) Let $\mu \in [0,1]^X$ be a fuzzy subset:

If $\mu \notin E_x \cup E_y$ then $C(\mu) = \mu \notin E_x \cup E_y$

If $\mu \in E_x \cap E_y$ then $C(\mu) = \mu_{11\gamma} \in E_x \cap E_y$. As $\alpha < 1$ and $\delta < 1$:

If $\mu \in E_x \setminus E_y$ then $C(\mu) = \mu_{1\alpha\beta} \in E_x \setminus E_y$

If $\mu \in E_y \setminus E_x$ then $C(\mu) = \mu_{\delta 1\gamma} \in E_y \setminus E_x$

Now, in the previous four cases, the idempotence of our operator is obtained from the idempotence of the maximum.

Therfore $C(C(\mu)) = C(\mu) \ \ \forall \ \mu \in [0,1]^X$.

See that R_C is not a transitive fuzzy relation. Observe that

$$C\big(\varphi_x\big)(y) = \big(\varphi_x\big)_{1\alpha\beta}(y) = \alpha > \beta$$

$$C\big(\varphi_y\big)(x) = \big(\varphi_y\big)_{\delta 1\gamma}(z) = \gamma > \beta$$

$$C(\varphi_x)(z) = (\varphi_x)_{1\alpha\beta}(z) = \beta$$

Then $C(\varphi_x)(z) < min(C(\varphi_x)(y)\,,\,C(\varphi_y)(z))$, that is
$R_C(x,z) < min(R_C(x,y)\,,\,R_C(y,z))$. □

In consequence, $C^{xyz}_{\alpha\beta\gamma\delta}$ is a non-coherent operator. Nevertheless, show this explicitly:

If $\mu = C(\varphi_x) = (\varphi_x)_{1\alpha\beta}$, that is:

$$\mu(t) = \begin{cases} 1 & \text{si } \; t = x \\ \alpha & \text{si } \; t = y \\ \beta & \text{si } \; t = z \\ 0 & \text{si } \; t \notin \{x, y, z\} \end{cases}$$

Then

$$min(\mu(y)\,,\,C(\varphi_y)(z)) = min(C(\varphi_x)(y)\,,\,C(\varphi_y)(z))$$

Recall that we have proved that

$$min(C(\varphi_x)(y)\,,\,C(\varphi_y)(z)) > C(\varphi_x)(z)$$

Now by the idempotence of the operator C:

$$min(\mu(y)\,,\,C(\varphi_y)(z)) > C(\varphi_x)(z) =$$

$$= C(C(\varphi_x))(z) = C(\mu)(z)$$

Therefore $C^{xyz}_{\alpha\beta\gamma\delta}$ is not coherent for the fuzzy set μ with respect to the ordered pair (y, z).

Finally, we can observe that the relation induced by the operator $C^{xyz}_{\alpha\beta\gamma\delta}$ verifies the transitivity condition (3) except for the ordered elements x, y, z. However, the coherence condition (2) does not hold a lot of cases.

4 Conclusions

In this paper we have shown that for every universe X with card $X \geq 2$ there exists an infinite number of non-coherent fuzzy consequence operators such that they induce a fuzzy preorder and for every universe X with card $X \geq 3$ there exists an infinite number fuzzy consequence operators such that they do not induce a fuzzy preorder and consequently they are not coherent operators.

In fact, we have shown the previous examples as parametric families of fuzzy consequence operators which can be implemented through very simple algorithms.

In particular we have proved that the coherence axiom on a fuzzy consequence operator C is not a necessary condition in order that the relation induced by C to be a preorder.

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