ON THE POSSIBILITIES OF FUZZIFICATION OF THE SOLUTION IN FUZZY COOPERATIVE GAMES*

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Some possibilities of fuzzification of the von Neumann – Morgenstern solution of cooperative games with transferable utility (TU games) are briefly investigated. The fuzzification based on the transformation of individual fuzzy TU game into a fuzzy class of (deterministic) TU games with their own specific solutions is discussed.

1 Introduction

The von Neumann – Morgenstern solution of cooperative games (see [5, 1]) belongs to basic and classical approaches to the formulation of solution concept. Its typical feature is a relatively small attention paid to individual coalitions and their forming, and a concentration on the achievability and stability of the pay-off vectors, called imputations.

The fuzzification of TU game, considered in this paper, means a fuzzification of numerical values of the expected pay-offs of coalitions, i. e. of the values of the characteristic function. Such fuzzification and its properties are formulated in several papers and summarized in [4].

In the deterministic case, the definition of the (von Neumann – Morgenstern) solution uses a significant criterion for the characterization of relation between pay-off vectors, called relation of domination. It could be expected that the definition of fuzzy solution of the fuzzy game will be derived by means of some kind of fuzzification of this relation. Unfortunately, even the properties of deterministic domination are rather poor, and its fuzzification appears very problematic, as shown in [2]. It means that it is useful to find another approach to the problem. The perspective method can be found in [3], where a transformation of individual fuzzy game into a fuzzy class of deterministic TU games is suggested. This method is used below to define also fuzzy solution as a fuzzy class of deterministic solutions of those games. The details of this procedure, as well as the elementary properties of its outcome are the main topic of the following sections.

The main methodological conclusion is that even this procedure, in spite of its advantages, offers only very limited degree of fuzzification of solution which appears to be essentially deterministic in its structure.

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2 Deterministic TU Game and its Solution

In the whole paper we denote by I the (non-empty and finite) set of players. Each non-empty subset K of I is called coalition.

The cooperative game with transferable utility is a pair (I, v), for the simplification we often written only v, where v is a mapping connecting every coalition K with a real number $v(K)$. The mapping is called characteristic function of the game (I, v). Without any significant loss of generality (compare, e. g., with [1]) we suppose that

(1) $$v(\{i\}) = 0 \quad \text{for all players } i \in I,$$

and that v is superadditive, i. e.,

(2) $$v(K \cup L) \geq v(K) + v(L) \quad \text{for disjoint } K,\ L.$$

Every real valued vector $\boldsymbol{x} = (x_i)_{i \in I} \in R^I$ such that

(3) $$\sum_{i \in K} x_i \leq v(K) \quad \text{for some } K \subset I,$$

is called an imputation. If the condition (3) is fulfilled, we also say that $\boldsymbol{x}$ is accessible for coalition K.

If $\boldsymbol{x} = (x_i)_{i \in I}$, $\boldsymbol{y} = (y_i)_{i \in I}$ are two vectors from R^I then we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ via a coalition K and write

(4) $$\boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}$$

iff

(5) $$\boldsymbol{x} \text{ is accessible for } K,$$

(6) $$x_i > y_i \text{ for all } i \in K.$$

Further, we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ and write

(7) $$\boldsymbol{x} \operatorname{Dom} \boldsymbol{y},$$

iff $\boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}$ for some coalition $K \subset I$.

Finally, (von Neumann – Morgenstern) solution of the game (I, v) is called a set $S \subset R^I$ such that

(8) if $\boldsymbol{x}$, $\boldsymbol{y} \in S$ then neither $\boldsymbol{x} \operatorname{Dom} \boldsymbol{y}$ nor $\boldsymbol{y} \operatorname{Dom} \boldsymbol{x}$,

(9) if $\boldsymbol{z} \in R^I$ is an imputation and $\boldsymbol{z} \notin S$ then there exists at least one $\boldsymbol{x} \in S$ such that $\boldsymbol{x} \operatorname{Dom} \boldsymbol{z}$.

Note, that the domination relation is defined only for imputations, i. e. vectors accessible for some coalition. Any extension of this relation on other vectors has no sense.

3 Fuzzification of a Game

Let only briefly remember a concept which is used in this section. Fuzzy quantity we call any fuzzy subset a of the real line R with membership function $\mu_a : R \to [0, 1]$ fulfilling

(10) there exist $x_1,\ x_2,\ x_3 \in R,\ x_1 < x_2 < x_3$, such that $\mu_a(x_2) = 1,\ \mu_a(x) = 0$ for $x \notin [x_1, x_3]$.

Individual Fuzzy Game

As we have already mentioned in the Introduction, by fuzzification of a coalitional game we understand fuzzification of the expected pay-offs.

In accordance with [4] we substitute crisp values $v(K)$ by fuzzy quantities $w(K)$ with membership functions $\mu_K : R \to [0,1]$. With respect to (1) we suppose that for every coalition K

(11) $$\mu_{\{i\}}(0) = 1,\ \mu_{\{i\}}(x) = 0 \text{ for } x \neq 0, \text{ if } K \text{ is a one player coalition.}$$

Moreover, we suppose that

(12) $$\mu_K(x) = 0 \text{ for } x < 0 \text{ for every } K,$$

which assumption rather reflects the original assumption that $v(K) \geq v(\{i\}) = 0$.

The pair (I, w), or briefly w, is called fuzzy coalitional game.

If there exists a deterministic game (I, v) such that

$$\mu_K(v(K)) = 1 \quad \text{for any } K,$$

then we consider (I, w) for fuzzy extension of (I, v) or, vice-versa, we consider (I, v) for deterministic reduction of (I, w).

It is natural to expect that the solution of fuzzy game will be a fuzzy set of imputations. Its construction demands some kind of fuzziness of the relation of domination. This idea leads to some paradox phenomena following especially from the essential determinism and sophisticated structure of the solution concept, as well as from the very poor properties of the domination and its fuzzification (as shown in [2]). The technical problems and contradictions which we meet when trying to fuzzify the solution of a fuzzy game of the considered type provokes a question if there is another, more transparent, approach to the problem which could better respect the nature of the solution in fuzzy game. We can see that such approach exists.

3.1 General Fuzzy Class of Games

However the fuzzification of a game by means of the substitution of real numbers $v(K)$ by fuzzy quantities $w(K)$ is natural, its processing means some serious problems. They are mostly caused by our intention to handle consequently only fuzzy quantities, fuzzy relations (e. g. ordering), fuzzy mappings, etc., derived from the values μ_K by classical procedures. This method is not only very complicated from formal point of view, but moreover, some of the achieved results do not correspond with analogous properties of deterministic games or with our intuitive expectations, as shown and discussed in [4].

It is desirable to find another approach to the fuzzy game (I, w) and its processing which simplifies the formalism and makes the whole procedure more lucid. Such approach was suggested in [3].

Let us denote by $\mathcal{V}$ the set of all deterministic games (I, v). Let (I, w) be a fuzzy coalitional game with membership functions μ_K of $w(K)$, $K \subset I$.

Then we define a fuzzy class of (deterministic) games $\mathcal{W}$ with membership function $\pi : \mathcal{V} \to [0,1]$, where

(13) $$\pi(v) = \min\left[\mu_K(v(K)) : K \subset I\right].$$

for any $v \in \mathcal{V}$.

Remark 1. It is obvious that if w is a fuzzy extension of some $v_0 \in \mathcal{V}$ then (13) turns into

$$\pi(v) = \min\left[\mu_K(v_0(K)) : K \subset I\right].$$

By means of (13) we have generated a fuzzy subclass $\mathcal{W}$ of $\mathcal{V}$ reflecting the structure of the individual fuzzy game w.

The formal properties of such transformation including the conditions of its uniqueness are summarized in [3]. It is shown there that this method significantly simplifies the processing of fuzzy game w, as well as the formalism connected with related concepts like superadditivity, convexity, core of the game and Shapley value. Here,we try to use it also for the definition of (fuzzy) solution of w.

4 Fuzzy Solution of the Generated Class of Games

The previous concepts can be used for the definition of the following fuzzy solution concept.

Let us denote by $\mathcal{P}(R^I)$ the power set

$$\mathcal{P}(R^I) = \{M : M \subset R^I\}. \tag{14}$$

Then the fuzzy solution of w is defined as a fuzzy subset of $\mathcal{P}(R^I)$ T with membership function $\tau : \mathcal{P}(R^I) \to [0,1]$ where for any $S \in \mathcal{P}(R^I)$

$$\tau(S) = \max\left[\pi(v) : S \text{ is solution of } v,\ v \in \mathcal{V}\right]. \tag{15}$$

Remark 2. Evidently, if $v \in \mathcal{V}$, $\pi(v) = 1$ and v has a solution then

$$\max\left(\tau(S) : S \in \mathcal{P}(R^I)\right) = 1.$$

Lemma 1. Let $w \in \mathcal{W}$ and $w' \in \mathcal{W}$ be fuzzy coalitional games with membership functions π, π', respectively, and let for every $v \in \mathcal{V}$, $\pi(v) \geq \pi'(v)$. Let τ, τ' be membership functions derived from π, π' by means of (15). Then for every $S \in \mathcal{P}(R^I)$

$$\tau(S) \geq \tau'(S).$$

Proof. The statement immediately follows from (15).

5 Conclusive Remark

If we analyze the concepts of the previous section, we see that even in this case the fuzzy solution displays essential features of determinism. Under the concept of fuzzy solution we intuitively imagine something like a fuzzy subset of R^I whose elements have their origin in particular solutions S_v of the games v in the fuzzy class $\mathcal{W}$, and where the values of membership function are defined by means of values $\pi(v)$. Unfortunately, such fuzzy set hardly respects conditions (8) and (9). Moreover, it appears that there is no rational way how to formulate these conditions for fuzzy set of vectors from R^I. This difficulty is also connected with the problematic possibility of fuzzification of the relation of domination.

Summarizing the above conclusions, as well as the conclusions of subsection 3.1, we can say that the concept of (von Neumann – Morgenstern) solution is essentially deterministic, based on deterministic methodological roots. Its fuzzification will be always limited and it keeps some important deterministic features.

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