Fuzzy Markov Chains : Uncertain Probabilities

James J. Buckley *
University of Alabama at Birmingham
Department of Mathematics
Birmingham, Alabama, 35294, USA
email: buckley@math.uab.edu

Esfandiar Eslami
Shahid Bahonar University of Kerman
Department of Mathematics
Kerman, Iran
email: eslami@arg3.uk.ac.ir

Abstract

We consider finite Markov chains where there are uncertainties in some of the transition probabilities. These uncertainties are modeled by fuzzy numbers. Using a restricted fuzzy matrix multiplication we investigate the properties of regular, and absorbing, fuzzy Markov chains and show that the basic properties of these classical Markov chains generalize to fuzzy Markov chains.

Keywords: Markov chains, uncertain probabilities.

1 Introduction

This paper continues our research into fuzzy Markov chains. In [3] we employed possibility distributions in finite Markov chains. The rows in a transition matrix were possibility distributions, instead of discrete probability distributions. Using possibilities we went on to look at regular, and absorbing, Markov chains and Markov decision processes.

There have been a few other papers published on fuzzy Markov chains ([1],[2],[4],[6],[7],[8],[10]). In [7] the elements in the transition matrix are fuzzy probabilities, which are fuzzy subsets of $[0, 1]$, and we will do the same but under restricted fuzzy matrix multiplication (see below). That is, in [7] the authors use

*Corresponding Author

the extension principle to find powers of the fuzzy transition matrix which is different from our method which we call restricted fuzzy matrix multiplication. The paper [10] is more abstract and about a Markov fuzzy process with a transition possibility measure in an abstract state space. The paper [1] is like [3], containing good results about convergence of powers of the transition matrix, but does not use possibility distributions. In [6] the authors use Dempster–Shafer type mass functions to construct transition probabilities for set–valued Markov chains in which the sets are subsets of the original state space. The authors in [2] were the first to consider stochastic systems in a fuzzy environment. By a "fuzzy environment" they mean the system has fuzzy goals and fuzzy constraints. Their transition matrix uses probabilities and they employed dynamic programming to obtain an optimal solution. This work was continued in [4] showing how fuzzy dynamic programming can be used to solve these types of problems. Fuzzy Markov decision problems were addressed in [8]. In this paper both the state and action are fuzzy, the transition of states is defined using a fuzzy relation, and the discounted total reward is described as a fuzzy number in a closed bounded interval. Our paper is quite different from all of these other papers involving fuzzy Markov chains.

Let us now review some of the basic results from classical finite Markov chains ([5],[9]). A finite Markov chain has a finite number of possible states (outcomes) S_1, $S_2, \ldots$, S_r at each step $n = 1, 2, 3 \ldots$, in the process. Let

$$p_{ij} = Prob(S_j \ at \ step \ n+1 | S_i \ at \ step \ n), \tag{1}$$

$1 \leq i, j \leq r$, $n = 1, 2, \ldots$. The p_{ij} are the transition probabilities which do not depend on n. The transition matrix $P = (p_{ij})$ is a $r \times r$ matrix of the transition probabilities. An important property of P is that the row sums are equal to one and each $p_{ij} \geq 0$. Let $p_{ij}^{(n)}$ be the probability of starting in state S_i and ending up in S_j after n steps. Define P^n to be the product of P n–times and it is well known that $P^n = (p_{ij}^{(n)})$ for all n. If $p^{(0)} = (p_1^{(0)}, \ldots, p_r^{(0)})$, where $p_i^{(0)}$ = the probability of initially being in state S_i, and $p^{(n)} = (p_1^{(n)}, \cdots, p_r^{(n)})$, where $p_i^{(n)}$ = the probability of being in state S_i after n steps, we know that $p^{(n)} = p^{(0)} P^n$.

Before going on to uncertain probabilities let us define the notation we will be using in this paper. We place a "bar" over a symbol to denote a fuzzy set. All our fuzzy sets will be fuzzy subsets of the real numbers ($\mathbf{R}$). So, $\overline{p}$, $\overline{A}$, $\overline{\beta}$, ... all represent fuzzy subsets of the real numbers. If $\overline{A}$ is a fuzzy set, then $\overline{A}(x) \in [0, 1]$ is the membership function for $\overline{A}$ evaluated at $x \in \mathbf{R}$. An α-cut of $\overline{A}$, written $\overline{A}[\alpha]$ is defined as $\{x | \overline{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. $\overline{A}[0]$, the support of $\overline{A}$, is separately defined as the closure of the union of all the $\overline{A}[\alpha]$, $0 < \alpha \leq 1$.

A fuzzy number $\overline{N}$ is a fuzzy subset of the real numbers satisfying two basic properties: (1) $\overline{N}(x) = 1$ for some $x \in \mathbf{R}$ (normalized); and (2) $\overline{N}[\alpha]$ is a closed and bounded interval for $0 \leq \alpha \leq 1$. We will be using a special type of fuzzy number $\overline{M}$ called a triangular fuzzy number. $\overline{M}$ is defined by three numbers $a_1 < a_2 < a_3$ where: (1) $\overline{M}(x) = 1$ at $x = a_2$; (2) the graph of $y = \overline{M}(x)$ on $[a_1, a_2]$ is a straight line from $(a_1, 0)$ to $(a_2, 1)$ and also on $[a_2, a_3]$ the graph is a straight line from $(a_2, 1)$ to $(a_3, 0)$; and (3) $\overline{M}(x) = 0$ for $x \leq a_1$ or $x \geq a_3$. We write $\overline{M} = (a_1/a_2/a_3)$ for triangular fuzzy number $\overline{M}$. For any fuzzy number $\overline{N}$

we have $\overline{N}[\alpha] = [n_1(\alpha), n_2(\alpha)]$, $0 \leq \alpha \leq 1$, which describes the closed, bounded, intervals as functions of α. Define $\overline{N} > x$ (or $\geq x$) if $n_1(0) > x$ ($n_1(0) \geq x$) for any real number x. Similarly we define $\overline{N} < x$ and $\overline{N} \leq x$.

In the transition matrix $P = (p_{ij})$ all the p_{ij} must be known exactly. Many times these values are estimated or they are provided by "experts". We now assume that some of the p_{ij} are uncertain and we will model this uncertainty using fuzzy numbers. So, for each p_{ij} we substitute $\overline{p}_{ij}$ and define the fuzzy transition matrix $\overline{P} = (\overline{p}_{ij})$. Not all the p_{ij} need to be fuzzy, some can be crisp (a real number). If a p_{ij} is crisp we will still write it as $\overline{p}_{ij}$. If a $p_{ij} = 0$ or $p_{ij} = 1$, then we assume that there is no uncertainty in this value. If $0 < p_{ij} < 1$ and there is uncertainty in its value, then we assume that $0 < \overline{p}_{ij} < 1$ also.

The uncertainty is in some of the p_{ij} values but not in the fact that the rows in the transition matrix are discrete probability distributions. So we now put the following restriction on the $\overline{p}_{ij}$: there are $p_{ij} \in \overline{p}_{ij}[1]$ so that $P = (p_{ij})$ is the transition matrix for a finite Markov chain (the row sums equal one). This restriction on the p_{ij} is basic to the rest of the paper.

Now we need to define restricted fuzzy matrix multiplication since we will need to compute $\overline{P}^n$ for $n = 1, 2, 3, ...$ But first we require some definitions. Let

$$S = \{x = (x_1, ..., x_r) | x_i \geq 0, \sum_{i=1}^{r} x_i = 1\}, \tag{2}$$

and then define

$$Dom_i[\alpha] = (\prod_{j=1}^{r} \overline{p}_{ij}[\alpha]) \bigcap S, \tag{3}$$

for $0 \leq \alpha \leq 1$ and $1 \leq i \leq r$. Then (Dom for "domain")

$$Dom[\alpha] = \prod_{i=1}^{r} Dom_i[\alpha]. \tag{4}$$

Next set $\overline{P}^n = (\overline{p}_{ij}^{(n)})$ where we will define $\overline{p}_{ij}^{(n)}$ and show that they are fuzzy numbers.

Consider a crisp transition matrix P and $P^n = (p_{ij}^{(n)})$. We know that

$$p_{ij}^{(n)} = f_{ij}^{(n)}(p_{11}, ..., p_{rr}), \tag{5}$$

for some function $f_{ij}^{(n)}$. Equation (5) just says that the elements in P^n are some function of the elements in P. Now consider $f_{ij}^{(n)}$ a function of $p = (p_{11}, ..., p_{rr}) \in Dom[\alpha]$. Look at the range of $f_{ij}^{(n)}$ on $Dom[\alpha]$. Let

$$\Gamma_{ij}^{(n)}[\alpha] = f_{ij}^{(n)}(Dom[\alpha]). \tag{6}$$

That is, $\Gamma_{ij}^{(n)}[\alpha]$ is the set of all values of $f_{ij}^{(n)}$ for $(p_{11}, ..., p_{rr}) \in Dom[\alpha]$. Now $f_{ij}^{(n)}$ is continuous and $Dom[\alpha]$ is connected, closed and bounded (compact), which implies that $\Gamma_{ij}^{(n)}[\alpha]$ is a closed and bounded interval for all α, i, j and n. We set

$$\overline{p}_{ij}^{(n)}[\alpha] = \Gamma_{ij}^{(n)}[\alpha], \tag{7}$$

giving the α-cuts of the $\overline{p}_{ij}^{(n)}$ in $\overline{P}^n$. The resulting $\overline{p}_{ij}^{(n)}$ is a fuzzy number because its α-cuts are closed, bounded, intervals and surely they are normalized.

First, by restricting the $p_{ij} \in \overline{p}_{ij}[\alpha]$ to be in $Dom[\alpha]$, we get $P = (p_{ij})$ a crisp transition matrix for a finite Markov chain. Then an α-cut of $\overline{P}^n$ is the set of all P^n for $(p_{11}, ..., p_{rr}) \in Dom[\alpha]$. This is restricted fuzzy matrix multiplication because the uncertainties are in some of the p_{ij} values and not in the fact that each row in P must be a discrete probability distribution.

To compute the $\Gamma_{ij}^{(n)}[\alpha]$ all we need to find are the end points of the intervals. So we need to solve

$$p_{ij1}^{(n)}(\alpha) = min\{f_{ij}^{(n)}(p) | p \in Dom[\alpha]\}, \tag{8}$$

and

$$p_{ij2}^{(n)}(\alpha) = max\{f_{ij}^{(n)}(p) | p \in Dom[\alpha]\}, \tag{9}$$

where $\overline{p}_{ij}^{(n)}[\alpha] = [p_{ij1}^{(n)}(\alpha), p_{ij2}^{(n)}(\alpha)]$, all α.

In some simple cases, as shown in the examples in the next two sections, we can solve equations (8) and (9) for the α-cuts of the $\overline{p}_{ij}^{(n)}$. In general, one would need to employ a directed search algorithm (genetic, evolutionary) to estimate the solutions to equations (8) and (9).

It may appear that we are doing interval arithmetic to find the α-cuts in $\overline{P}^n$. Let us show that this is not the case. Let $\overline{P}[\alpha] = (\overline{p}_{ij}[\alpha])$ and $(\overline{P}[\alpha])^2 = (w_{ij}[\alpha])$ where

$$w_{ij}[\alpha] = \sum_{k=1}^{r} \overline{p}_{ik}[\alpha]\overline{p}_{kj}[\alpha], \tag{10}$$

all i, j and $\alpha \in [0, 1]$. Equation (10) is evaluated using interval arithmetic between all intervals. However, our restricted fuzzy matrix multiplication does now produce equation (10) for the α-cuts of $\overline{P}^2$. The following example, continued into the next section, shows the difference between the two methods and why using equation (10) is not useful in the study of finite fuzzy Markov chains.

Example 1

Let

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \tag{11}$$

for a crisp transition matrix and then let $\overline{p}_{11} = (0.6/0.7/0.8)$, $\overline{p}_{12} = (0.2/0.3/0.4)$, $\overline{p}_{21} = (0.3/0.4/0.5)$ and $\overline{p}_{22} = (0.5/0.6/0.7)$. Use interval arithmetic, equation (10), to compute $\overline{P}^n$. $\overline{P}[\alpha] = (\overline{p}_{ij}[\alpha])$ where $\overline{p}_{11}[\alpha] = [0.6 + 0.1\alpha, 0.8 - 0.1\alpha]$, $\overline{p}_{12}[\alpha] = [0.2 + 0.1\alpha, 0.4 - 0.1\alpha]$, $\overline{p}_{21}[\alpha] = [0.3 + 0.1\alpha, 0.5 - 0.1\alpha]$ and $\overline{p}_{22} = [0.5 +$

$0.1\alpha, 0.7-0.1\alpha]$. Since all the intervals are non-negative we may find the end points of the intervals in $\overline{P}^n$ using

$$P_1(\alpha) = \begin{pmatrix} 0.6+0.1\alpha & 0.2+0.1\alpha \\ 0.3+0.1\alpha & 0.5+0.1\alpha \end{pmatrix}, \tag{12}$$

and

$$P_2(\alpha) = \begin{pmatrix} 0.8-0.1\alpha & 0.4-0.1\alpha \\ 0.5-0.1\alpha & 0.7-0.1\alpha \end{pmatrix}. \tag{13}$$

Let $\overline{P}^n = (w_{ij}^{(n)}(\alpha))$ where $w_{ij}^{(n)}[\alpha] = [w_{ij1}^{(n)}(\alpha), w_{ij2}^{(n)}(\alpha)]$. Then $P_1^n(\alpha) = (w_{ij1}^{(n)}(\alpha))$ and $P_2^n(\alpha) = (w_{ij2}^{(n)}(\alpha))$. But $w_{ij1}^{(n)}(\alpha) \to 0$ and $w_{ij2}^{(n)}(\alpha) \to \infty$ as $n \to \infty$ for $0 \le \alpha < 1$. In other words

$$w_{ij}^{(n)}(x) \to \begin{cases} 0, & x \le 0, \\ 1, & x > 1. \end{cases} \tag{14}$$

This is not a satisfactory result. We obtain better results using restricted fuzzy matrix multiplication in Example 2 in the next section.

2 Regular Markov Chains

We first review some of the basic properties of regular Markov chains. We say that the Markov chain is regular if $P^k > 0$ for some k, which is $p_{ij}^{(k)} > 0$ for all i, j. This means that it is possible to go from any state S_i to any state S_j in k steps. A property of regular Markov chains is that powers of P converge, or $\lim_{n\to\infty} P^n = \Pi$, where the rows of Π are identical. Let w be the unique left eigenvalue of P corresponding to eigenvalue one, so that $w_i > 0$ all i and $\sum_{i=1}^r w_i = 1$. That is $wP = w$ for $1 \times r$ vector w. Each row in Π is equal to w and $p^{(n)} \to p^{(0)}\Pi = w$. After a long time, thinking that each step being a time interval, the probability of being in state S_i is w_i, $1 \le i \le r$, independent of the initial conditions $p^{(0)}$. In a regular Markov chain the process goes on forever jumping from state to state, to state, ...

If P is a regular (crisp) Markov chain, then consider $\overline{P} = (\overline{p}_{ij})$ where $\overline{p}_{ij}$ gives the uncertainty (if any) in p_{ij}. If $(p_{11}, ..., p_{rr}) \in Dom[\alpha]$, then $P = (p_{ij})$ is also a regular Markov chain.

Let $\overline{P}^n \to \overline{\Pi}$ where each row in $\overline{\Pi}$ is $\overline{\pi} = (\overline{\pi}_1, ..., \overline{\pi}_n)$. Also let $\overline{\pi}_j[\alpha] = [\pi_{j1}(\alpha), \pi_{j2}(\alpha)]$, $1 \le j \le n$. We now show how to compute the α-cuts of the $\overline{\pi}_j$.

For each $(p_{11}, ..., p_{rr}) \in Dom[\alpha]$ set $P = (p_{ij})$ and we get $P^n \to \Pi$. Let $\Gamma(\alpha) = \{w | w \textit{ a row in } \Pi, (p_{11}, ..., p_{rr}) \in Dom[\alpha]\}$. $\Gamma(\alpha)$ consists of all vectors w, which are the rows in Π, for all $(p_{11}, ..., p_{rr}) \in Dom[\alpha]$. Then

$$\pi_{j1}(\alpha) = min\{w_j | w \in \Gamma(\alpha)\}, \tag{15}$$

and

$$\pi_{j2}(\alpha) = max\{w_j | w \in \Gamma(\alpha)\}. \tag{16}$$

In equations (15) and (16) w_j is the j^{th} component in the vector w.

	$\alpha = 1$	$\alpha = 0$
$\overline{\pi}_1$	0.2609	[0.1923,0.3443]
$\overline{\pi}_2$	0.5217	[0.4255,0.6176]
$\overline{\pi}_3$	0.2174	[0.1600,0.2857]

Table 1: Alpha-cuts of the Fuzzy Numbers $\overline{\pi}_i$ in Example 3.

Example 2

This continues Example 1. If p is a 2×2 regular Markov chain, then we may find that $w_1 = p_{21}/(p_{21} + p_{12})$, $w_2 = p_{12}/(p_{21} + p_{12})$ where $w = (w_1, w_2)$ is a row in Π. Now we may solve equations (15) and (16) since $\partial w_1/\partial p_{21} > 0$, $\partial w_1/\partial p_{12} < 0$ and $\partial w_2/\partial p_{21} < 0$, $\partial w_2/\partial p_{12} > 0$. If $\overline{p}_{21}[\alpha] = [p_{211}(\alpha), p_{212}(\alpha)]$ and $\overline{p}_{12}[\alpha] = [p_{121}(\alpha), p_{122}(\alpha)]$ we obtain

$$\overline{\pi}_1[\alpha] = [\frac{p_{211}(\alpha)}{p_{211}(\alpha) + p_{122}(\alpha)}, \frac{p_{212}(\alpha)}{p_{212}(\alpha) + p_{121}(\alpha)}], \tag{17}$$

and

$$\overline{\pi}_2[\alpha] = [\frac{p_{121}(\alpha)}{p_{121}(\alpha) + p_{212}(\alpha)}, \frac{p_{122}(\alpha)}{p_{122}(\alpha) + p_{211}(\alpha)}], \tag{18}$$

both triangular fuzzy numbers. We know that, with restricted fuzzy matrix multiplication, $\overline{P}^n \rightarrow \overline{\Pi}$, where each row in $\overline{\Pi}$ is $(\overline{\pi}_1, \overline{\pi}_2)$. We may simplify equations (17) and (18) and get $\overline{\pi}_1 = (\frac{3}{7}/\frac{4}{7}/\frac{5}{7})$ and $\overline{\pi}_2 = (\frac{2}{7}/\frac{3}{7}/\frac{4}{7})$.

Example 3

Let

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.25 & 0.75 & 0 \\ 0 & 0.6 & 0.4 \end{pmatrix} \tag{19}$$

be a transition matrix. $P^2 > 0$ so P is regular. If some p_{ij} are uncertain (substitute $\overline{p}_{ij}$), then another p_{ij} in the same row must also be uncertain since the row sums must equal one. Therefore, let $\overline{p}_{11} = \overline{p}_{13} = (0.4/0.5/0.6)$, $\overline{p}_{21} = (0.2/0.25/0.3)$, $\overline{p}_{22} = (0.7/0.75/0.8)$, $\overline{p}_{32} = (0.5/0.6/0.7)$, and $\overline{p}_{33} = (0.3/0.4/0.5)$. In this example we may also solve equations (15) and (16) because $w_1 = p_{32}p_{21}/S$, $w_2 = (1 - p_{11})p_{32}/S$, $w_3 = (1 - p_{11})p_{21}/S$ where $S = p_{32}p_{21} + (1 - p_{11})p_{32} + (1 - p_{11})p_{21}$. We determine that: (1) $\partial w_1/\partial p_{11} > 0$, $\partial w_1/\partial p_{21} > 0$, $\partial w_1/\partial p_{32} > 0$; (2) $\partial w_2/\partial p_{11} < 0$,$\partial w_2/\partial p_{21} < 0$, $\partial w_2/\partial p_{32} > 0$; and (3) $\partial w_3/\partial p_{11} < 0$, $\partial w_3/\partial p_{21} > 0$, $\partial w_3/\partial p_{32} < 0$. This allows us to find the α-cuts of the $\overline{\pi}_i$, $i = 1, 2, 3$. The $\overline{\pi}_i$ will be triangular shaped fuzzy numbers whose $\alpha = 0$ and $\alpha = 1$ cuts are presented in Table 1. Triangular "shaped" means that the sides of the "triangle" are curves not straight lines. Then $\overline{P}^n \rightarrow \overline{\Pi}$ where the rows of $\overline{\Pi}$ are $(\overline{\pi}_1, \overline{\pi}_2, \overline{\pi}_3)$.

3 Absorbing Markov Chains

First we will discuss the basic results for absorbing Markov chains. We will call a state S_i absorbing if $p_{ii} = 1$, $p_{ij} = 0$ for $i \neq j$. Once in S_i you can never leave. Suppose there are k absorbing states, $1 \leq k < r$, and then we may rename the states (if needed) so that the transition matrix P can be written as

$$P = \begin{pmatrix} I & O \\ R & Q \end{pmatrix}, \tag{20}$$

where I is the $k \times k$ identity, O is the $k \times (r-k)$ zero matrix, R is $(r-k) \times k$ and Q is $(r-k) \times (r-k)$. The Markov chain is called an absorbing Markov chain if it has at least one absorbing state and from every non–absorbing state it is possible to reach some absorbing state in a finite number of steps. Assume the chain is absorbing and then we know that

$$P^n = \begin{pmatrix} I & O \\ SR & Q^n \end{pmatrix}, \tag{21}$$

where $S = I + Q + \cdots + Q^{n-1}$. Then $\lim_{n \to \infty} P^n = \Pi$ where

$$\Pi = \begin{pmatrix} I & O \\ R^* & O \end{pmatrix}, \tag{22}$$

for $R^* = (I - Q)^{-1} R$. Notice the zero columns in Π which implies that the probability that the process will eventually enter an absorbing state is one. The process eventually ends up in an absorbing state.

If $R = (r_{ij})$ and $Q = (q_{ij})$ we now assume that there is uncertainty in some of the r_{ij} and/or the q_{ij} values. We then substitute $\overline{r}_{ij}$ for r_{ij} and $\overline{q}_{ij}$ for q_{ij} and obtain $\overline{P}$ an absorbing fuzzy Markov chain. We now show, under restrictive fuzzy matrix multiplication, that $\overline{P}^n \to \overline{\Pi}$ where

$$\overline{\Pi} = \begin{pmatrix} I & O \\ \overline{R}^* & O \end{pmatrix} \tag{23}$$

with $(r-k) \times k$ matrix $\overline{R}^* = (\overline{r}^*_{ij})$. For any $p \in Dom[\alpha]$, P^n converges to the Π in equation (22) which implies that $\overline{Q}^n \to O$, the (crisp) zero matrix. Also, for any $p \in Dom[\alpha]$, $R^* = (I - Q)^{-1} R = (r^*_{ij})$. Let $\overline{r}^*_{ij}[\alpha] = [r^*_{ij1}(\alpha), r^*_{ij2}(\alpha)]$. It follows that

$$r^*_{ij1}(\alpha) = min\{r^*_{ij} | p \in Dom[\alpha]\}, \tag{24}$$

and

$$r^*_{ij2}(\alpha) = max\{r^*_{ij} | p \in Dom[\alpha]\}. \tag{25}$$

To find the limit of $\overline{P}^n$, as $n \to \infty$, which is $\overline{\Pi}$, all we need to do is solve equations (24) and (25) for the α-cuts of the $\overline{r}^*_{ij}$ in $\overline{R}^*$ in equation (23).

	$\alpha = 1$	$\alpha = 0$
$\overline{r}^*_{11}$	0.3404	[0.1633.0.5682]
$\overline{r}^*_{12}$	0.6596	[0.5306,0.9091]
$\overline{r}^*_{21}$	0.4681	[0.3163,0.6704]
$\overline{r}^*_{22}$	0.5319	[0.3469,0.7727]

Table 2: Alpha-cuts of the Fuzzy Numbers $\overline{r}^*_{ij}$ in Example 5.

Example 4

Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \tag{26}$$

have one absorbing state. Substitute $\overline{p}_{ij}$ for p_{ij} expressing the uncertainty in the p_{ij} values. Then $\overline{P}^n \to \overline{\Pi}$ where $\overline{\Pi}$ is the crisp matrix

$$\overline{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{27}$$

because $\overline{Q}^n \to O$ and the $\overline{r}^*_{i1}$ must equal crisp one because the row sums are one.

Example 5

We next consider two absorbing states with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.3 \\ 0.4 & 0.4 & 0.2 & 0 \end{pmatrix}. \tag{28}$$

Substitute $\overline{r}_{ij}$ for r_{ij} and $\overline{q}_{ij}$ for q_{ij} where $\overline{r}_{11} = (0.1/0.2/0.3), \overline{r}_{12} = (0.4/0.5/0.6)$, $\overline{r}_{21} = (0.3/0.4/0.5), \overline{r}_{22} = (0.3/0.4/0.5), \overline{q}_{12} = (0.2/0.3/0.4)$ and $\overline{q}_{21} = (0.1/0.2/0.3)$. First we determine the r^*_{ij} values in terms of the r_{ij} and the q_{ij}. The result is $r^*_{11} = (r_{11} + r_{21}q_{12})/T$, $r^*_{12} = (r_{12} + r_{22}q_{12})/T$, $r^*_{21} = (r_{21} + r_{11}q_{21})/T$ and $r^*_{22} = (r_{22} + r_{12}q_{21})/T$ where $T = 1 - q_{21}q_{12}$. We may solve equations (24) and (25) by noting that $\partial r^*_{ij}/\partial r_{ab} > 0$, for $i, j = 1, 2$ and $a, b = 1, 2$ and also $\partial r^*_{ij}/\partial q_{ab} > 0$ for $i, j = 1, 2$ and $ab \in \{12, 21\}$. We then find the α-cuts of all the $\overline{r}^*_{ij}$ whose $\alpha = 1$ and $\alpha = 0$ cuts are shown in Table 2.

4 Summary and Conclusions

In this paper we studied finite Markov chains which have uncertainties in some of the probabilities in the transition matrix. We modeled these uncertainties using fuzzy numbers. One could also model the uncertainties using (crisp) intervals. We used what we called restricted fuzzy multiplication to find powers of the fuzzy transition matrix. Although there may be uncertainty in some of the probability values, there is no uncertainty that each row in the transition matrix must be a discrete probability distribution (the row sums are always one). The row sums always one formed the basis of our restricted matrix multiplication. We then showed that the basic properties of regular, and absorbing, Markov chains carry over to our fuzzy Markov chains.

We presented five elementary examples where we could do the computations by hand, but in general one would need a directed search algorithm (genetic, evolutionary) to determine the α-cuts of the limit of the powers of the fuzzy transition matrix. Also, in the examples we used only triangular fuzzy numbers but the results easily extend to any fuzzy numbers.

Future research will be concerned with extending these ideas to other probability models.

5 References

1. K. E. Avrachenkov and E. Sanchez, Fuzzy Markov Chains: Specificities and Properties, Proc. IPMU, July 3-7, Madrid, Spain, 2000, 1851-1856.

2. R. E. Bellman and L. A. Zadel, Decision-Making in a Fuzzy Environment, Management Science, 17(1970)141-164.

3. J. J. Buckley, T. Feuring and Y. Hayashi, Fuzzy Markov Chains, Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems. Under review.

4. R. E. Bellman and L. A. Zadel, Decision-Making in a Fuzzy Environment, Management Science, 17(1970)141-164.

5. A. O. Esogbue and R. E. Bellman, Fuzzy Dynamic Programming and its Extensions, TIMS Studies in the Management Sciences, 20(1984)147-167.

6. J.G. Kemeny and J.L. Snell, Finite Markov Chains, Van Nostrand, N.Y., 1960.

7. R. M. Kleyle and A. deKorvin, Constructing One-Step and Limiting Fuzzy Transition Probabilities for Finite Markov Chains, J. Intelligent and Fuzzy Systems, 6(1998)223-235.

8. R. Kruse, R. Buck-Emden and R. Cordes, Processor Power Considerations - An Application of Fuzzy Markov Chains, Fuzzy Sets and Systems, 21(1987)289-299.

9. M. Kurano, M. Yasuda, J. Nakagami and Y. Yoshida, Markov-type Fuzzy Decision Processes with a Discounted Reward on a Closed Interval, European J. Oper. Res., 92(1996)649-662.

10. M. Olinick, An Introduction to Mathematical Models in the Social and Life Sciences, Addison-Wesley , Reading, Mass., 1978.

11. Y. Yoshida, Markov Chains with a Transition Possibility Measure and Fuzzy Dynamic Programming, Fuzzy Sets and Systems, 66(1994)39-57.