A Fuzzy and Intuitionistic Fuzzy Account of the Liar Paradox

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Abstract

The Liar paradox, or the sentence

What I am now saying is false.

and its various guises have been attracting the attention of logicians and linguists since ancient times. A commonly accepted treatment of the Liar paradox [7, 8] is by means of Situation semantics, a powerful approach to natural language analysis. It is based on the machinery of non-well-founded sets developed in [1]. In this paper we show how to generalize these results including elements of fuzzy and intuitionistic fuzzy logic [3, 4]. Basing on the results, a way is proposed towards solving the problem of modelling the two levels of Situation theory – infons and propositions – with a single one retaining the specific features of the two-levels logics.

1 Models and propositions

Theorem. Let λ be a Liar sentence $\neg True(\texttt{this},\texttt{h})$, where

Following [7], we prefer not to deal with the actual 'liar' sentence cited above but rather with a slightly revised version of it:

The proposition expressed by this sentence is false.

It is however a purely technical device and not a matter of principle, as it is easier to talk about truth values of propositions arising from the logical analysis of natural language utterances than of utterances themselves.

Any proposition p expressed by the liar sentence must satisfy

$$p \leftrightarrow \neg p \tag{1}$$

If there is a classical first-order logic proposition satisfying (1) then it is necessarily neither true nor false, since assuming it is the one we obtain that it is the other, too. But this contradicts a fundamental property of the classical logic of being able to qualify any given proposition as either true or false.

Barwise and Moss [8] have shown the conditions necessary for the existence of a model of (1). First, some necessary definitions from [8] (or parts of them appropriate for our needs).

Given disjoint sets Rel of relation symbols, $Const$ of constants, and Var of variables, a *partial model* M is a tuple

$$\langle D_M, L_M, Ext_M, Anti_M, d_M, c_M \rangle$$

where

- D_M is a non-empty set called the *domain* of M;
- $L_M \subset Rel \cup Const$ is called the *language* of M;
- Ext_M and $Anti_M$ are functions with domain $L_M \cap Rel$ such that for each n-ary symbol $R \in L_M$, $Ext_M(R)$ and $Anti_M(R)$ are disjoint n-ary relations on D_M, called the *extension* and *anti-extension* of R in M, respectively.
- a function $d_M : L_M \cap Const \to D_M$; if $d_M(c) = b$ then b is called the *denotation* of c in M;
- a function $c_M : Var \to D_M$, called the *context* of M.

We will call *denotation* of t $d_M(t)$ if t is a constant, or $c_M(t)$ if t is a variable.

Sentences are built up in L_M using the usual connectives $\neg, \vee$, and $\wedge$. We say that a sentence φ of L_M is *defined* in M if every constant and variable of φ has a denotation in M; the set of sentences defined in M will be denoted by $Def(M)$.

A crucial feature of the above definition is the presence of the concepts of extension and anti-extension of a relation. There may be relations R such that $Ext_M(R) \cup Anti_M(R)$ may not exhaust the whole domain; thus we obtain a way to develop partial models.

For any model M and sentence φ defined in M, we define $M \models \varphi$ and $M \models^- \varphi$ as follows:

i) (atomic sentences) if $\varphi = R(b_1 \ldots, b_n)$ then:

$$\begin{array}{lll} M \models \varphi & \text{iff} & \langle b_1, \ldots, b_n \rangle \in Ext_M(R) \\ M \models^- \varphi & \text{iff} & \langle b_1, \ldots, b_n \rangle \in Anti_M(R) \end{array}$$

ii) (compound sentences)

$$\begin{array}{lll} M \models \neg\varphi & \text{iff} & M \models^- \varphi \\ M \models^- \neg\varphi & \text{iff} & M \models \varphi \\ M \models \varphi_1 \wedge \varphi_2 & \text{iff} & M \models \varphi_1 \text{ and } M \models \varphi_2 \\ M \models^- \varphi_1 \wedge \varphi_2 & \text{iff} & M \models^- \varphi_1 \text{ or } M \models^- \varphi_2 \end{array}$$

We assume that the language L contains a truth predicate $T(x, y)$: the sentence denoted by x is true in the model denoted by y (so, it will be possible for domains to include models, among other things). When a model M is fixed, $True(x,y)$ will mean that $\langle x,y\rangle \in Ext_M(T)$ and $False(x,y)$ will mean that $\langle x,y\rangle \in Anti_M(T)$.

A model is *truth-correct* if the following conditions are satisfied:

(T1) If $True(\varphi, N)$, then N is a model, $\varphi \in Def(N)$ and $N \models \varphi$.

(T2) If N is a model, $\varphi \in Def(N)$ and $False(\varphi, N)$, then $N \models \varphi$.

We are now in a position to quote the result of Barwise and Moss:

Theorem. Let λ be a Liar sentence $\neg True(\texttt{this},\texttt{h})$. If M is a truth-correct model then at least one of the following conditions must fail:

1. `this` denotes λ in M.
2. `h` denotes M in M.
3. $M \models \lambda \vee \neg\lambda$.

The question we are interested in here is, can a truth-correct model support to some degrees of truth and falsity the Liar sentence λ?

To study the problem, we introduce in the above definitions fuzzy logic, i. e., taking truth values from the set $[0,1]$ instead of $\{0,1\}$.

Given disjoint sets *Rel* of relation symbols, *Const* of constants, and *Var* of variables, a *fuzzy model* M is a tuple

$$\langle D_M, L_M, Ext^f_M, d_M, c_M\rangle$$

where Ext^f_M is a fuzzy relation on D_M i. e., $\forall R \in Rel : Ext^f_M(R) : D_M \to [0,1]$, and all the other components are as above, where Ext^f stands for *fuzzy extension.*

We need no more the separate notions of extension and anti-extension of a relation, since the fuzzy relation above combines their features. The concept of anti-extension was useful when we wanted to leave some relations partially defined on the domain. Now this role is played by the fuzzy relation — we may model the extension by assigning 1 to Ext^f_M for the corresponding elements of the domain, anti-extension by assigning 0, and the complement of their union by assigning some third value, say $\frac{1}{2}$.

For any models M and sentence φ defined in M, we define $M \models_\mu \varphi$, φ is true in M to the degree of μ, as follows:

i) (atomic sentences) if $\varphi = R(b_1 \ldots, b_n)$ then:

$$M \models_\mu \varphi \text{ iff } Ext_M(R)(\langle b_1, \ldots, b_n\rangle) = \mu.$$

ii) (compound sentences)

$$\begin{array}{lcr} M \models_\mu \neg\varphi & \text{iff} & M \models_{1-\mu} \varphi \\ M \models_\mu \varphi_1 \wedge \varphi_2 & \text{iff} & M \models_{\mu_1} \varphi_1 \text{ and } M \models_{\mu_2} \varphi_2, \\ & & \mu = min(\mu_1, \mu_2). \end{array}$$

Fuzzy degree of truth:

(FT0) $True(\varphi, N, \mu)$ iff N is a model, $\varphi \in Def(N), N \models_\mu \varphi$.

Let us consider what happens with the analysis of the paradox in this case. Assume that for a given model M and a Liar sentence λ $M \models_\mu \lambda$. This leads us to conclude that $M \models_\mu \neg\lambda$ and following the definition of evaluation we have $M \models_{1-\mu} \lambda$. Thus we arrived at the result $\mu = 1 - \mu$, $\mu = \frac{1}{2}$ which comes to show that on the fuzzy account of the Liar sentence no paradox arises – it just turns out to be false to exactly the same degree as it is true.

Taking the valuation function range to be the [0,1]×[0,1] we can build *intuitionistic fuzzy logic* [3, 4, 6, 15]. An intuitionistic fuzzy degree is not just a number in the interval $[0, 1]$ but a pair $\langle\mu, \nu\rangle$ where $\mu > 0, \nu > 0, \mu + \nu \leq 1$. As in the fuzzy case, μ is called the *degree of truth*, ν the *degree of falsity*. The difference is that now their sum does not necessarily amount to 1. The diference $1 - \mu - \nu$ is regarded as the *degree of uncertainty* of the considered object.

Various versions of IF logic and IF relations have been proposed [2, 4, 5, 9, 10, 11, 12, 13, 17, 18]. To finish this section, we perform the above analysis for the case of intuitionistic fuzzy logic. The only differences are:

- the definition of an intuitionistic fuzzy extension Ext^{IF}_M — an intuitionistic fuzzy relation on D_M: $\forall R \in Rel : Ext^{IF}_M(R) : D_M \to [0, 1]^2$.
- For any model M and sentence φ defined in M, we define $M \models_{\langle\mu,\nu\rangle} \varphi$ (the degree of truth of φ in M is μ and the degree of falsity of φ in M is ν) as follows:

  i) (atomic sentences) if $\varphi = R(b_1 \ldots, b_n)$ then:

  $$M \models_{\langle\mu,\nu\rangle} \varphi \text{ iff } Ext_M(R)(\langle b_1, \ldots, b_n\rangle) = \langle\mu, \nu\rangle.$$

  ii) (compound sentences)

  $$\begin{array}{lll} M \models_{\langle\mu,\nu\rangle} \neg\varphi & \text{iff} & M \models_{\langle\nu,\mu\rangle} \varphi \\ M \models_{\langle\mu,\nu\rangle} \varphi_1 \wedge \varphi_2 & \text{iff} & \exists\mu_1, \mu_2, \nu_1, \nu_2 \text{ such that } \mu_1 + \nu_1 \leq 1, \mu_2 + \nu_2 \leq 1 : \\ & & M \models_{\langle\mu_1,\nu_1\rangle} \varphi_1 \text{ and } M \models_{\langle\mu_1,\nu_1\rangle} \varphi_2, \\ & & \mu = min(\mu_1, \mu_2), \nu = max(\nu_1, \nu_2). \end{array}$$

Assume that for a given model M and a Liar sentence λ: $M \models_{\langle\mu,\nu\rangle} \lambda$. This leads us to conclude that $M \models_{\langle\mu,\nu\rangle} \neg\lambda$ and following the definition of evaluation we have $M \models_{\langle\nu,\mu\rangle} \lambda$. Thus we arrived at the result $\mu = \nu$: an even more interesting result which shows that the intuitionistic fuzzy treatment of the Liar sentence is more powerful than the pure fuzzy one — not only the paradox disappears, as in the fuzzy case, and its truth degree turns out again to be equal to its falsity degree, but now we have the opportunity of modelling *various* Liars of differing degrees of belief:

This sentence is to some extent false.

This sentence can be modelled by

$$\lambda = \neg True(\texttt{this}, \texttt{h}),$$

$$M \models_{\mu,\mu} \lambda, \mu < \frac{1}{2}.$$

2 Situation semantics and the Liar

One of Situation semantics' main assumptions is that simple assertive statements do not yield any truth value; rather, they give rise to objects of a special kind called *infons* [14, 16]. Infons are intended to carry the information conveyed by the statement rather than to make some declaration about its being true or false. If there is a *situation*[1] that 'supports' the factuality of an infon, we may speak of a *proposition* (having an ordinary truth value) stating that a given infon is a fact in a given situation:

$$p = (s \models \sigma)$$

For example, the infon $\langle loves, john, mary \rangle$, despite its similarity to a Prolog database fact, does not state in terms of truth or falsity anything about John and Mary's affair; it is just an item of information which may be true in some situations and false in other — either because things are not just the way they are described or simply because there are no John or Mary at all in some situation. Thus an infon needs a situation to start having a truth value.

Failing to support an infon does not mean that some form of negation of the infon is supported. *Partiality* is another important aspect of situations and infons. Unlike infons, propositions respect the principle of 'excluded middle', in that any situation either supports a given infon or not – therefore, a proposition is either false or true. Thus two different logics are employed, one classical, at propositions' level, the other, usually containing some 'undefined' truth value, at the level of infons.

This view is contrasted to earlier approaches where the semantical analysis of a statement usually lead to a proposition having a predicate derived from the verb phrase with statement's subject and objects as arguments.

Liar paradox is one of the points where the advantages of this approach emerge clearly. Instead of taking a proposition expressing the liar sentence and then trying to calculate its truth value, situation-semantic analysis first produces the infon carried by the sentence

$$\langle \text{False}, p \rangle$$

and then considers, for any situation s, the proposition p[2]:

$$p = (s \models \langle \text{False}, p \rangle)$$

Now the assumption that p is false does not lead us into a paradox, as the failure of s to support $\langle \text{False}, p \rangle$ is perfectly consistent with that assumption. This is, of course, due to the law of excluded middle being relinquished at the level of infons, and so no contradiction with the theorem from the beginning of the previous section arises.

[1]Situations, infons, propositions, and the 'supports' relation are crucial notions of Situation theory, the underlying theory for situation semantics. It does not seem possible to give a definition here even of the notions which have an exact one; still, as an example, we may consider a simple version of Situation theory where situations are just collections of infons, and a situation s supports an infon σ, $s \models \sigma$, just in case $\sigma \in s$.

[2]The existence of this proposition is guaranteed by a corollary of the Anti-Foundation Axiom [1,2]

3 Degrees of Truth and the Situation-Semantic Account of the Liar Paradox

Even some of the earliest versions of Situation semantics introduced the notion of *polarity*, a characteristic of any infon that tells one which reading of a given infon should be given: direct or opposite. For example, *'John loves Mary'* is represented by the infon $\langle loves, john, mary; 1\rangle$ while *'John does not love Mary'* is represented by the infon $\langle loves, john, mary; 0\rangle$.

It might be worthwhile to fuzzify infon polarity – an obvious application may be sentences like 'John is a little ill', represented by an infon of the form $\langle ill, john; \mu\rangle$ for a suitable small value of μ or even taken from a discrete set of possible values. Such a fuzzification may be naturally followed by introducing a second degree, of falsity, into the infon polarity

$$\langle r, a, b\rangle_{\langle \mu,\nu\rangle}$$

to represent appropriate undetermined statements.

An interesting case appears when this possibility is applied to the Liar sentence itself. We can imagine *The Unconvinced Liar*:

It is not clear whether this sentence is false.

or *The Fuzzy Truth-Teller*[3]:

The truth degree of this sentence is μ.

We will present a representation of these sentences within the framework that will be developed below.

Observe now that while 'John loves Mary' may well be true in one situation, false in the next, and 'to some extent true' in a third situation, there are statements expressing propositions whose degree of truth or falsity is in a sense inherent, i. e., rather independent from the situation they are put in. Various kinds of Liar and Truth-teller sentences fall into this category. One thing that distinguishes such sentences from our everyday utterances is that very 'independence'.

Introducing intuitionistic fuzzy polarity of the infons makes exactly the step necessary to capture this property. There remains however the question about how should the intuition be reflected that an infon of higher $\mu-$polarity, and hence of a greater gegree of independence, must be supported by more situations than an infon with lower $\mu-$polarity? In particular, should infons with polarity $\langle 1, 0\rangle$, if any, be supported by all situations?

The answer to the above question requires that we distinguish between infon's *polarity*, in the sense of truth or falsity degree, and its *independence degree* in the

[3]The Truth-teller:

What I am now saying is true.

leads to no paradox – its proposition is definitely true. Maybe this is the reason that it is typically regarded as not so interesting as the paradoxical Liar sentence. Both are, however, equally good examples of 'situational independence'.

above sense. The latter notion is captured by now by the sum $\mu - \nu$. The opposite of this sum, $1 - \mu - \nu$ is normally called in the fuzzy literature 'degree of uncertainty'.

So we have now infons of various sorts — absolutely or fuzzy true or false, as well as situation-independent vs. situation-dependent ones. The next section presents a uniform representation of all these varying subjects.

4 Fuzzy Situation Semantics

As discussed above, we take the polarity of an infon to be an ordered pair $\langle \mu, \nu \rangle$ meaning its inherent truth-degree and its inherent falsity degree. To sum up the above discussions, we take it as a principle that inherent degrees set up minimal values which should be respected by all situations ever.

We introduce a second degree pair which belongs to the $\models$ relation. A situation s will now be able to support an infon σ to the degree of μ and not to support it to the degree of ν: $s \models_{\mu,\nu} \sigma_{\mu_\sigma,\nu_\sigma}$ where $\mu + \nu \leq 1$ and $\mu_\sigma + \nu_\sigma \leq 1$. We pose the following requirement on this new relation:

$$s \models_{\mu,\nu} \sigma_{\mu_\sigma,\nu_\sigma} \text{ only if } \mu \geq \mu_\sigma \text{ and } \nu \geq \nu_\sigma.$$

It is now clear how to coin a uniform representation of infons and propositions. Aside from Liars and Truth-teller sentences, another obvious candidate is the class of infons whose relation is '$\models$'. Let us model every proposition $p = (s \models \sigma)$ by the infon $\langle \models, s, \sigma \rangle_{\langle 1,0 \rangle}$. By this we guarantee that any situation s would support $\langle \models, s, \sigma \rangle$ with maximum degree of truth and independence.

The fuzzy truth-teller is modelled by $\sigma = \langle \models, s, \langle True, \sigma \rangle \rangle_{\langle \mu, 0 \rangle}$. Thus different situations will be able to support it with various falsity (and so overall) degrees, from $\langle \mu, 0 \rangle$ to the fully determined $\langle \mu, 1 - \mu \rangle$.

The ordinary convinced Liar will be represented as $\sigma = \langle \models, s, \langle False, \sigma \rangle \rangle_{\langle \frac{1}{2}, \frac{1}{2} \rangle}$. No other options are allowed except $\langle \frac{1}{2}, \frac{1}{2} \rangle$ because the Liar is fully situational-independent.

Any ordinary infon (fully situation-dependent) will have fuzzy truth and falsity degrees $\langle 0, 0 \rangle$ so that the corresponding situations may choose whatever truth and falsity values are appropriate.

The approach presented in this paper, as the author hopes, may turn out useful both in modelling real phenomena and in studying the fundamental parts of the still developing theory of situations.

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