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Topological Automorphism Groups of Chains

S. Ovchinnikov
Mathematics Department
San Francisco State University
San Francisco, CA 94132
sergei@sfsu.edu

Abstract

It is shown that any set–open topology on the automorphism group $A(X)$ of a chain X coincides with the pointwise topology and that $A(X)$ is a topological group with respect to this topology. Topological properties of connectedness and compactness in $A(X)$ are investigated. In particular, it is shown that the automorphism group of a doubly homogeneous chain is generated by any neighborhood of the identity element.

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1 Introduction

Let X be a chain and $A(X)$ denotes the automorphism group of X, i.e., the set of all order–preserving bijections from X onto X. $A(X)$ is also a lattice–ordered group with respect to meet and join operations defined, respectively by

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \quad \text{and} \quad (f \vee g)(x) = \max\{f(x), g(x)\}$$

for all $f, g \in A(X)$ and $x \in X$ [5].

The usual interval topology on X makes it completely normal Hausdorff space [1]. Thus $A(X)$ is a function space. In Section 2 we show that the pointwise topology is the smallest admissible topology on $A(X)$ and compatible with the group structure on $A(X)$. We also prove that this topology coincides with any set–open topology and, consequently, with the compact–open topology. The pointwise topology is also compatible with the lattice structure on $A(X)$. Thus $A(X)$ is a *topological lattice–ordered group*.

Homogeneity properties of X affect topological properties of $A(X)$. In Section 3 we establish some geometric and topological properties of doubly homogeneous chains that will be used in the succeeding sections.

A connected topological group is generated by any neighborhood of the identity element [2]. This fact motivates our studies of connectedness properties of $A(X)$

in Section 4. We show that $A(X)$ is connected (resp. totally disconnected) if and only if X is connected (resp. totally disconnected). We also prove that the automorphism group of a doubly homogeneous chain is generated by any neighborhood of the identity.

In Section 5, we establish three criteria for a subset to be compact in $A(X)$.

Finally, in Section 6, we use the techniques developed in this paper to develop an approach to robustness of aggregation procedures.

2 Topologies on automorphism groups

Let X be a chain and $A(X)$ its automorphism group. Since X is also a topological space with the usual interval topology, $A(X)$ is a function space.

For each pairs of sets $A \subseteq X$ and $B \subseteq X$, (A, B) denotes the set of all functions $f \in A(X)$ such that $f(A) \subseteq B$.

Definition 2.1. *The pointwise topology (p–topology) $\mathfrak{T}_p$ on $A(X)$ is that having as subbasis all sets $(\{x\}, V)$ where $x \in X$ and V belongs to the subbasis of the interval topology on X consisting of open rays.*

Definition 2.2. *A topology $\mathfrak{T}$ on $A(X)$ is admissible (or jointly continuous [6]) if the evaluation mapping $\mathcal{E} : A(X) \times X \to X$ defined by*

$$\mathcal{E}(f, x) = f(x)$$

for all $f \in A(X)$ and $x \in X$ is continuous.

Theorem 2.1. *$\mathfrak{T}_p$ is the smallest admissible topology on $A(X)$.*

Proof. (a) First we show that $\mathfrak{T}_p$ is admissible. It suffices to show that the inverse image W of an open ray in X is an open set in $A(X) \times X$. Consider an open ray $(a, \to)$ (the case of open rays in the form $(\leftarrow, a)$ is treated similarly). Then

$$W = \mathcal{E}^{-1}((a, \to)) = \{ (f, x) : f(x) > a \}$$

Let $(f_0, x_0) \in W$. Then $f_0(x_0) > a$ or, equivalently, $f_0^{-1}(a) < x_0$. Let us consider two cases.

(i) There exists b such that $f_0^{-1}(a) < b < x_0$ or, equivalently, $a < f_0(b) < x_0$. Then

$$x_0 \in V = (b, \to) \quad \text{and} \quad f_0 \in U = \{ f : f(b) > a \}.$$

Clearly, $U \times V$ is an open neighborhood of (f_0, x_0). Suppose $(f, x) \in U \times V$. Then $f(x) > f(b) > a$ implying $U \times V \subseteq W$. Hence W is an open set.

(ii) Suppose that x_0 covers $f_0^{-1}(a)$. Then $f_0(x_0)$ covers a. We define

$$V = (f_0^{-1}(a), \to) \quad \text{and} \quad U = \{f : f(x_0) > a\}.$$

Then $U \times V$ is an open neighborhood of (f_0, x_0) and

$$V = [x_0, \to) \quad \text{and} \quad U = \{f : f(x_0) \geq f_0(x_0)\}.$$

Suppose $(f, x) \in U \times V$. Then

$$f(x) \geq f(x_0) \geq f_0(x_0) > a.$$

Thus $U \times V \subseteq W$ implying that W is an open set.

(b) We prove now that $\mathfrak{T}_p$ is the smallest admissible topology on $A(X)$. Let $\mathfrak{T}$ be an admissible topology. For any given $x \in X$, the mapping $\mathcal{E}_x : A(X) \to X$ defined by $f \mapsto f(x)$ is continuous, since $\mathcal{E}$ is a continuous mapping. Let U be an open set in X. Then

$$\mathcal{E}_x^{-1}(U) = \{ f : f(x) \in U \}.$$

These sets are open in $\mathfrak{T}$ and form a subbasis for $\mathfrak{T}_p$. Thus $\mathfrak{T}_p \subseteq \mathfrak{T}$.

□

Since any topology containing an admissible topology is admissible, we can reformulate the previous theorem as follows.

Theorem 2.2. *A topology $\mathfrak{T}$ on $A(X)$ is admissible if and only if $\mathfrak{T} \supseteq \mathfrak{T}_p$.*

We now prove that group operations are continuous in p–topology on $A(X)$.

Theorem 2.3. *$A(X)$ endowed with the p–topology is a topological group.*

Proof. First we prove that $f \mapsto f^{-1}$ is a continuous mapping of $A(X)$ onto itself. Let

$$(\{a\}, (b, \to)) = \{f : f(a) > b)$$

be an element of subbasis for $\mathfrak{T}_p$. The inverse image of this set is given by

$$\{ f : f^{-1}(a) > b \} = \{ f : f(b) < a \} = (\{b\}, (\leftarrow, a))$$

which is an element of the same subbasis. Similarly, the inverse image of $(\{a\}, (\leftarrow, b))$ is $(\{b\}, (a, \to))$. Thus $f \mapsto f^{-1}$ is continuous.

Now we prove that the binary group operation in $A(X)$ is continuous. Let $W = (\{a\}, (b, \to))$ be an element of the subbasis in $A(X)$ (the case of elements in the form $(\{a\}, (\leftarrow, b))$ is treated similarly) and $h_0 = f_0 g_0$ be an element of W. Then $f_0(g_0)(a)) > b$. To prove continuity of the composition operation in $A(X)$ it suffices to find open neighborhoods U and V of f_0 and g_0, respectively, such that for any $f \in U$ and $g \in V$, $fg \in W$. Consider two cases.

(i) There is $c \in X$ such that

$$b < c < f_0(g_0(a)).$$

Then

$$f_0^{-1}(b) < f_0(^{-1}(c) < g_0(a).$$

Let $V = (\{a\}, (d, \to))$ and $U = (\{d\}, (b, \to))$, where $d = f_0^{-1}(c)$. We have $f_0 \in U$, since $f_0(d) = c > b$, and $g_0 \in V$, since $g_0(a) > d$. Thus U and V are neighborhoods of f_0 and g_0, respectively. Let $f \in U$ and $\in V$. Then

$$f(g(a)) > f(d) > b$$

implying $fg \in W$.

(ii) Suppose now that $f_0(g_0))$ covers b. Then $g_0(a)$ covers $c = f_0^{-1}(b)$. We define $V = (\{a\}, (c, \rightarrow))$ and $U = (\{g_0(a)\}, (b, \rightarrow))$. Then $g_0 \in V$, since $g_0(a) > f_0^{-1}(b) = c$, and $f_0 \in U$, since $f_0(g_0)) > b$. Since $f_0(g_0(a))$ covers b and $g_0(a)$ covers $f_0^{-1}(b)$, we have

$$V = (\{a\}, [\,g_0(a), \rightarrow)) \quad \text{and} \quad U = (\{g_0(a)\}, [\,f_0(g_0(a)), \rightarrow)).$$

Let f and g be any elements of U and V, respectively. Then

$$f(g(a)) \geq f(g_0(a)) \geq f_0(g_0(a)) > b$$

implying $fg \in W$.

□

We proved that the p–topology is the smallest admissible topology on $A(X)$ and that $A(X)$ is a topological group with respect to $\mathfrak{T}_p$. Our next theorem shows that some other topologies that play an important role in the theory of transformation groups coincide with the p–topology in the case of automorphism groups. First we introduce the following definition.

Definition 2.3. *Let $\mathcal{S}$ be the set of all subsets A of X satisfying the following conditions:*

(a) If $\inf A$ *exists, then it belongs to A.*

(b) If $\sup A$ *exists, then it belongs to A.*

The $\mathcal{S}$–topology $\mathfrak{T}_{\mathcal{S}}$ on $A(X)$ is defined by its subbasis which consists of sets (A, U) where $A \in \mathcal{S}$ and U is an open ray in X.

Clearly, $\mathcal{S}$ contains all closed subsets of X. Thus $\mathfrak{T}_{\mathcal{S}}$ contains the largest set–open topology [8] and, consequently, compact–open and pointwise topologies.

The following theorem shows that the $\mathcal{S}$–topology coincides with the pointwise topology in the case of automorphism groups.

Theorem 2.4. $\mathfrak{T}_{\mathcal{S}} = \mathfrak{T}_p$ *on $A(X)$.*

Proof. Let $W = (A, (a, \rightarrow))$ be a nonempty element of the subbasis for $\mathfrak{T}_{\mathcal{S}}$ (the case of elements in the form $(A, (\leftarrow, a))$ is treated similarly). If $b = \inf A$ exists, then $b \in A$ and

$$(A, (a, \rightarrow)) = (\{b\}, (a, \rightarrow)),$$

since all functions in $A(X)$ are strictly increasing. Suppose $\inf A$ does not exist. Since $(A, (a, \rightarrow))$ is not empty, there is $f \in A(X)$ such that $f(x) > a$ for all $x \in X$, or, equivalently, $f^{-1}(a) < x$ for all $x \in X$. Thus the set L of all lower bounds of A is not empty. Since $\inf A$ does not exists, for any $x \in L$ there is $y \in L$ such that $y > x$. Then $x \in (\leftarrow, y) \subseteq L$ implying that L is open. Consider $V = (\{a\}, L)$. We have

$$f \in V \Leftrightarrow f(a) \in L \Leftrightarrow f(a) < x \ (\forall x \in A)$$
$$\Leftrightarrow f^{-1}(x) > a \ (\forall x \in A) \Leftrightarrow f^{-1} \in W.$$

Thus $W = V^{-1}$. Since V is open in the p–topology, we conclude that W is also open in this topology.

□

Let X be a chain and X' be the set of all *inner* elements of X, i.e.,

$$X' = \{x \in X : u < x < v, \text{ for some } u, v \in X\}.$$

Thus $a \in X \backslash X'$ if and only if a is a maximal or minimal element in X. If f is an automorphism of X, then its restriction on X' is an automorphism of the subchain X', since maximal and minimal elements are fixed points of automorphisms. Moreover, any automorphism of the subchain X' can be uniquely extended to an automorphism of X. Clearly, $A(X)$ and $A(X')$ are isomorphic algebraic groups. They are also isomorphic as topological groups endowed with the p–topology. Indeed, any element of the subbasis of $\mathfrak{T}_p$ in $A(X)$ in the form $(\{a\}, U)$, where a is a maximal or minimal element, is either empty or coincides with $A(X)$.

In what follows, we consider only chains without minimal and maximal elements and assume that automorphism groups are endowed with the pointwise topology.

3 Properties of chains

The purpose of this section is to introduce some basic geometric and topological properties of chains.

Let (a, b) be an open interval in X. Consider all automorphisms of X that coincide with the identity automorphism on the complement of (a, b) in X. These automorphisms form a subgroup of $A(X)$ which is isomorphic to $A((a, b))$. We shall often identify $A((a, b))$ with its isomorphic image in $A(X)$.

First we introduce notions of transitivity and homogeneity [5].

Definition 3.1. *Let X be a chain and $A(X)$ its automorphism group. We say that $A(X)$ is transitive if for any $x, y \in X$ there is $f \in A(X)$ such that $f(x) = y$. If $A(X)$ is transitive, then X is said to be homogeneous.*

Definition 3.2. *If for any $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ in X there is $f \in A(X)$ such that $f(x_i) = y_i$ for all i, $1 \leq i \leq n$, we say that $A(X)$ is n–transitive and X is n–homogeneous.*

Obviously, n–homogeneity (n–transitivity) implies m–homogeneity (m–transitivity) for $m \leq n$. The converse, in general, is not true.

Double homogeneous chains are considered to be "well-behaved" in the theory of chains [5, Chapter 2]. The following theorem gives some equivalent descriptions of these chains.

Theorem 3.1. *Let X be a chain. The following conditions are equivalent.*

(1) X is doubly homogeneous;

(2) X is n–homogeneous for all $n \geq 2$;

(3) any open interval in X is homogeneous;

(4) any open interval in X is n–homogeneous for all $n \geq 2$.

Proof. (1) $\Leftrightarrow$ (2) by [5, Lemma 1.10.1].

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4). Let (a, b) be an open interval in X. Consider two sequences

$$a < x_1 < \ldots < x_n < b$$

and

$$a < y_1 < \ldots < y_n < b,$$

where $n \geq 1$. Since X is $(n+2)$–homogeneous, there is $f \in A(X)$ such that $f(a) = a, f(b) = b,$ and $f(x_i) = y_i$ for all $i,\ 1 \leq i \leq n$. The restriction f^* of f on (a, b) is an automorphism of (a, b) satisfying $f^*(x_i) = y_i$ for all $i,\ 1 \leq i \leq n$.

(3) $\Rightarrow$ (1). We do not distinguish automorphisms of open intervals and their trivial extensions to X. Let $x < y$ and $u < v$ be elements of X. Consider an open interval (a, b) containing these elements. Since (a, b) is homogeneous, there is an automorphism f such that $f(y) = u$. Since $u < v$, we have $f^{-1}(u) < f^{-1}(v) = y$. Thus $f^{-1}(u) \in (a, y)$. Since $u < v$, we have $f^{-1}(u) < f^{-1}(v) = y$. Thus $f^{-1}(u) \in (a, y)$. Since (a, y) is homogeneous and $x \in (a, y)$, there is an automorphism g of (a, y) such that $g(x) = f^{-1}(u)$. Consider automorphism $h = fg$. We have $h(x) = f(g(x)) = u$ and $h(y) = f(g(y)) = f(y) = v$.

(4) $\Rightarrow$ (3). Trivial.

□

Corollary 3.1. *If X is doubly homogeneous, it does not have gaps.*

Proof. Suppose $(a, b) = \emptyset$ for some $a < b$. Since X has no minimal element, there are elements c and d such that $d < c < a < b$. Then (d, b) is an open interval containing at least two elements. Since (a, b) is empty, a is the maximal element in (d, b) and therefore a fixed point of any automorphism of (d, b). Thus (d, b) is not homogeneous, which contradicts condition (3) of the previous theorem.

□

We conclude that all open intervals with distinct end points in a doubly homogeneous chain are nonempty. In fact, each of these intervals contains infinitely many elements.

Now we consider connectedness and compactness in X.

Lemma 3.1. *Let X be a disconnected doubly homogeneous chain. For any two elements $u < v$ in X, there are disjoint open sets U and V such that $u \in U, v \in V$ and $U \cup V = X$.*

Proof. Since X is disconnected, there are two nonempty disjoint open sets A and B such that $A \cup B = X$. Let a and b be some elements in A and B, respectively. We may assume that $a < b$. Since X is doubly homogeneous, there is $f \in A(X)$ such that $f(a) = u$ and $f(b) = v$. Then $U = f(A)$ and $V = f(B)$ satisfy conditions of the lemma.

□

Theorem 3.2. *(cf. [9]) A doubly homogeneous chain X is either a connected or totally disconnected topological space.*

Proof. Suppose X is not connected and let K be a component containing at least two elements $u < v$. Let U and V be sets from 3.1. Then $K = (K \cap U) \cup (K \cap V)$ and $(K \cap U) \cap (K \cap V) = \emptyset$, a contradiction. Thus each component contains has exactly one element.

□

Any topologically connected chain X is locally compact. The only connected subsets of X are the intervals. Closed bounded intervals in X are compact. (See [2, IV, 2, Exercise 7]).

4 Connectedness in $A(X)$

Automorphism groups $A(\mathbb{R})$ and $A(\mathbb{Q})$ provide typical examples for the results of this section. Namely, $A(\mathbb{R})$ is a connected topological group, whereas $A(\mathbb{Q})$ is a totally disconnected topological group. In the rest of this section, X is a doubly homogeneous chain.

First, we consider the case of totally disconnected X.

Theorem 4.1. *If X is totally disconnected, so is $A(X)$.*

Proof. Let K be a component of $A(X)$ containing at least two elements, say, f and g. Then there is $a \in X$ such that $f(a) \neq g(a)$. By Lemma 3.1, there are two nonempty open sets U and V which form a partition of X and such that $f(a) \in U$ and $g(a) \in V$. Consider open sets $U' = (\{a\}, U)$ and $V' = (\{a\}, V)$. Clearly, they form a partition of $A(X)$. Then $K = (K \cap U') \cup (K \cap V')$ which contradicts the definition of a component. Thus each component has exactly one element.

□

According to Theorem 3.2, the remaining possibility is that of connected X. Let $B(X)$ be the set of all automorphisms of X with bounded support. (Recall that the support of $f \in A(X)$ is the set of all $x \in X$ such that $f(x) \neq x$.) We shall need the following lemma.

Lemma 4.1. *$B(X)$ is dense in $A(X)$.*

Proof. Let $U = (x_1, \dots, x_n; U_1, \dots, U_n)$ be a nonempty element of the basis of $A(X)$ and $h \in U$. We may assume that $x_1 < \cdots < x_n$. Let $y_i = h(x_i), 1 \leq i \leq n$. Let $a, b \in X$ be two elements such that

$$a < x_1 < \cdots < x_n < b$$

and

$$a < y_1 < \cdots < y_n < b$$

There is $f \in A(X)$ such that $f(a) = a, f(b) = b$, and $f(x_i) = y_i, 1 \leq i \leq n$. We define $f^*(x) = f(x)$ for $a \leq x \leq b$, and $f^*(x) = x$, otherwise. Clearly, $f^* \in U$ and $f^* \in B(X)$.

□

Theorem 4.2. *If X is connected, so is $A(X)$.*

Proof. Let K be the component of the identity element in $A(X)$. The K is a closed normal subgroup of $A(X)$ ([2, III, 2.2]). By [5, Theorem 2G], $B(X) \subseteq K$. By Lemma 4.1, $K = A(X)$.

□

The following theorem summarizes the above results.

Theorem 4.3. *$A(X)$ is connected (respectively, totally disconnected) if and only X is connected (respectively, totally disconnected).*

It is known [2, III, 2.2] that a connected topological group is generated by any neighborhood of the identity element. In what follows, we show that this true for any $A(X)$ provided X is a doubly homogeneous chain.

We shall use special open neighborhoods of the identity element in $A(X)$. These are neighborhoods in the form

$$U = (x_1, \dots, x_n; J_1, \dots, J_n) = \{f \in A(X) : f(x_i) \in J_i,\ 1 \le i \le n\}$$

where $x_1 < \cdots < x_n$ and $x_i \in J_i = (a_i, b_i),\ 1 \le i \le n$. In addition, we assume that intervals J_i do not overlap. Clearly, any neighborhood of the identity element contains an open set in this form.

To prove that $A(X)$ is generated by any neighborhood of the identity element we only need to show that any neighborhood in the form $(x_1, \dots, x_n; J_1, \dots, J_n)$ generates $A(X)$. First, we prove two lemmas.

Lemma 4.2. *Let $U = (x; J)$ where $x \in J = (a, b)$. Then U generates $A(X)$.*

Proof. Let h be any automorphism of X. It suffices to show that $h = fg$ for some $f, g \in U$. Let $y = h^{-1}(x)$. We may assume that $y \neq x$. (Otherwise $h \in U$.) If $y < x$, then there is $z \in J$ such that $y < x < z$. If $y > x$, there is $z \in J$ such that $z < x < y$. In both cases, by double homogeneity of X, there is an automorphism g such that $g(y) = x$ and $g(x) = z$. Thus $g \in U$. Let $f = gh^{-1}$. Then $f(x) = h(g^{-1}(x)) = h(y) = x$. Thus $f \in U$ and $h = fg$.

□

For a given $U = (x_1, \dots, x_n; J_1, \dots, J_n)$ and $k \le n$, we define

$$U_k = (x_1, \dots, x_k; J_1, \dots, J_k).$$

Lemma 4.3. *For any $h \in U_k,\ k < n$, there are $f, g \in U_{k+1}$ such that $h = fg$.*

Proof. Let $y = h^{-1}(x_{k+1})$. We may assume that $y \neq x_{k+1}$. Since $x_k \in J_k$ and $J_k \cap J_{k+1} = \emptyset$, we have $h(x_k) < x_{k+1}$ or, equivalently, $x_k < y$. If $y < x_{k+1}$, there is $z \in J_{k+1}$ such that $y < x_{k+1} < z$. Thus we have

$$x_1 < \cdots < x_k < y < x_{k+1} < z.$$

If $y > x_{k+1}$, there is $z \in J_{k+1}$ such that $z < x_{k+1} < y$. Note that $z > x_k$, since $J_k \cap J_{k+1} = \emptyset$. Thus we have

$$x_1 < \cdots < x_k < z < x_{k+1} < y.$$

In both cases, by $(k+2)$–homogeneity of X, there exists an automorphism g such that $g(x_i) = x_i$, $(1 \leq i \leq k)$, $g(y) = x_{k+1}$, and $g(x_{k+1}) = z$. Clearly, $g \in U_{k+1}$. Let $f = hg^{-1}$. Then $f(x_i) = h(g^{-1}(x_i)) = h(x_i) \in J_i$ for all $i \leq k$, and $f(x_{k+1}) = h(g^{-1}(x_{k+1})) = h(y) = x_{k+1} \in J_{k+1}$. Thus $f \in U_{k+1}$. Clearly, $h = fg$.

□

Let $U = U_n = (x_1, \ldots, x_n; J_1, \ldots, J_n)$. Consider a nested family of neighborhoods of the identity

$$A(X) \supseteq U_1 \supseteq \cdots \supseteq U_k \supset \cdots \supseteq U_n$$

where $U_k = (x_1, \ldots, x_k; J_1, \ldots, J_k)$, $1 \leq k \leq n$. Obvious inductive argument using lemmas 3 and 4 shows that any element in $A(X)$ is a composition of at most 2^n elements in U. Therefore $A(X)$ is generated by U. We proved the following theorem.

Theorem 4.4. *If X is a doubly homogeneous chain, then the automorphism group $A(X)$ is generated by any neighborhood of its identity element.*

Consider chain $X = \mathbb{Z}$. The automorphism group of this chain is a discrete group isomorphic to $\mathbb{Z}$. Clearly, $U = \{0\}$ does not generate $A(\mathbb{Z})$. Hence the double homogeneity property is an essential condition in Theorem 10.

5 Compact sets in $A(X)$

In this section we introduce three compactness criteria for subsets of $A(X)$. The first two criteria are obtained by embedding $A(X)$ into standard function spaces. The third criterion employs the lattice structure in $A(X)$. In what follows, we do not assume that X is a doubly homogeneous chain unless it is otherwise specified.

First we note that $A(X)$ is a Hausdorff subspace of X^X in p –topology. Thus we have the following theorem [6, 7.1].

Theorem 5.1. *A subset F of $A(X)$ is compact if and only if*
(a) F is closed in X^X, and
(b) for each $x \in X$ the set $\mathcal{E}(F, \{x\})$ has a compact closure.

The second criterion is a form of the Arzela–Ascoli Theorem. It is established under assumption that X is a k–space [6]. This includes, in particular, all doubly homogeneous topologically connected chains and all chains that are first countable topological spaces.

Suppose X is a k–space. Let $C(X)$ be a space of all continuous functions from X into X endowed with the compact–open topology. Then $A(X)$ is a subspace of $C(X)$ (Theorem 2.4). Thus we have the following version of Arzela–Ascoli Theorem [6, Theorem 7.21] and [8, Theorem 3.2.6].

Theorem 5.2. *If X is a k–space, then a subset F of $A(X)$ is compact if and only if*

(a) F is closed in $C(X)$,
(b) the closure of $\mathcal{E}(F,\{x\})$ is compact for each $x \in X$, and
(c) F is evenly continuous.

The third criterion does not use any assumption about X or embeddings of $A(X)$ into a larger space.

Let L be a lattice. The *interval topology* on L is defined by taking the intervals $[p, \rightarrow)$ and $(\leftarrow, q]$ for all $p, q \in L$ as a subbasis of closed sets. The following theorem is found in [1] and [4].

Theorem 5.3. *A lattice is compact in its interval topology if and only if it is complete.*

We shall establish a similar result for sublattices of topological lattice $A(X)$ endowed with the pointwise topology.

First note that sets $(\{a\}, [b, \rightarrow))$, and $(\{a\}, (\leftarrow, b])$ form a subbasis for the p–topology in $A(X)$. We have

$$[f, \rightarrow) = \{g : g \geq f\} = \bigcap_{x \in X} (\{x\}, [f(x), \rightarrow))$$

Thus $[f, \rightarrow)$ is a closed set for any $f \in A(X)$. Similarly, all sets $(\leftarrow, f]$ are closed sets. Thus interval topology is smaller that the p–topology in $A(X)$.

Let L be a sublattice of $A(X)$. The interval topology in L is smaller than the induced topology. Thus, if L is a compact subset of $A(X)$, then it is also compact in the interval topology. By Theorem 12, L is complete. We proved the following lemma.

Lemma 5.1. *Any compact sublattice of $A(X)$ is complete.*

The following lemma asserts that the induced topology coincides with the interval topology provided L is a complete sublattice of $A(X)$.

Lemma 5.2. *Let L be a complete sublattice of $A(X)$. Then the induced topology is smaller than the interval topology in L.*

Proof. Let $(\{a\}, [b, \rightarrow))$ be an element of subbasis of closed sets in $A(X)$. We define

$$K_{a,b} = L \cap (\{a\}, [b, \rightarrow)) = \{f \in L : f(a) \geq b\}.$$

Since L is a complete lattice, $f_{a,b} = \inf K_{a,b}$ exists and $f_{a,b} \in K_{a,b}$. For any $f \in L$ such that $f \geq f_{a,b}$, we have $f \in K_{a,b}$. Thus $K_{a,b} = [f_{a,b}, \rightarrow)$ in L. Similarly, an intersection of $(\{a\}, (\leftarrow, b])$ with L is a closed interval in the form $(\leftarrow, f]$. Therefore any subset of L which is closed in the induced topology is also closed in the interval topology.

□

By Lemma 5.2 and Theorem 5.2, any complete sublattice of $A(X)$ is a compact subset in $A(X)$. We proved the following theorem.

Theorem 5.4. *A sublattice L of $A(X)$ is compact if and only if L is a complete lattice.*

Corollary 5.1. *Let H be a closed subset of $A(X)$. If the lattice $[H]$ generated by H is complete, then H is compact.*

6 Robustness of aggregation procedures

Generally speaking, an aggregation procedure is a function assigning to n–tuples of objects all belonging to a given set X a single object from the same set. Typical examples include $X = \mathbb{R}$ (aggregation of real numbers) and $X = [0,1]^m$ (aggregation of fuzzy sets with a finite universe).

We begin with the case $X = \mathbb{R}$. Thus we assume that the objects are represented by real numbers; in other words, we deal with *measurements* of real objects rather than with objects themselves.

In this setting, an aggregation procedure is just a function

$$y = M(x_1, x_2, \dots, x_n)$$

from $\mathbb{R}^n$ to $\mathbb{R}$. Naturally, some conditions on M should be imposed in order to justify the name “aggregation procedure”. Here, we assume that M is a continuous symmetric function of its arguments satisfying the following condition

$$\min\{x_1, x_2, \dots, x_n\} \le M(x_1, x_2, \dots, x_n) \le \max\{x_1, x_2, \dots, x_n\}$$

for all n–tuples $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Following Cauchy [3], we call these functions *means* [9]. Yager’s OWA operators [12] are common examples of aggregation functions satisfying the above conditions as well as the usual arithmetic mean.

Because numbers $x_1, x_2, \dots, x_n$ represent measurements, we should specify a scale in which these measurements were performed. Here, we are concerned with the case of ordinal scales. Moreover, we probably want the aggregation function M to represent a *meaningful* relation with respect to this scale. The concept of meaningfulness is formalized in the representational measurement theory [7] as the *invariance* property as follows. For any $f \in A(\mathbb{R})$ (the automorphism group of $\mathbb{R}$), the equation

$$f(y) = M(f(x_1), f(x_2), \dots, f(x_n))$$

is equivalent to the equation

$$y = M(x_1, x_2, \dots, x_n).$$

Functions from $A(\mathbb{R})$ are called *admissible* transformations in measurement theory [7].

The invariance condition greatly restricts the set of means on $\mathbb{R}$. The following theorem follows from a more general result established in [9].

Theorem. *An invariant mean on $\mathbb{R}$ is an order statistics.*

The set $A(\mathbb{R})$ of all admissible transformations of $\mathbb{R}$ is a huge set. It consists of all strictly increasing functions from $\mathbb{R}$ onto itself. What happens if we consider only "small" admissible transformations which are "near" the identity transformation? In mathematics, the concept of "nearness" is formalized by introducing a topological structure on the set under consideration. Here, we consider $A(\mathbb{R})$ as a topological group so we can use the results of the previous sections. The transformations that are "near" the identity element belong to some neighborhood V of the identity transformation. These observations motivate the following definition (cf. [10, 11]).

Definition 6.1. *A mean M on $\mathbb{R}$ is V–*robust *if*

$$M(f(x_1), f(x_2), \dots, f(x_n)) = f(M(x_1, x_2, \dots, x_n))$$

for all $x_1, x_2, \dots, x_n \in \mathbb{R}$ and all $f \in V$ where V is a neighborhood of the identity in $A(\mathbb{R})$.

Clearly, any invariant mean is V–robust for any neighborhood V of the identity. It turns out that in the case $X = \mathbb{R}$ the converse is also true. Namely we have the following theorem [11].

Theorem 6.1. *Any V–robust mean on $\mathbb{R}$ is invariant.*

Proof. By Theorem 4.4, $A(\mathbb{R})$ is generated by V. In other words, for a given $f \in A(\mathbb{R})$ there are transformations $f_1, f_2, \dots, f_k \in V$ such that

$$f(x) = (f_1 f_2 \cdots f_k)(x)$$

for all $x \in \mathbb{R}$. Since

$$M(f_i(x_1), f_i(x_2), \dots, f_i(x_n)) = f(M(x_1, x_2, \dots, x_n))$$

for all $1 \leq i \leq k$, we have

$$M(f(x_1), f(x_2), \dots, f(x_n)) = f(M(x_1, x_2, \dots, x_n)).$$

□

The situation is quite different in the case $X = [0,1]^m$ if $m > 1$. In this case we are concerned with the aggregation problem for fuzzy sets defined on a finite universe U with $|U| = m$. We denote $x(u)$ the membership function of a fuzzy set $x \in [0,1]^U$. Consider the set of all "pointwise" admissible transformations of X of the form

$$F(x)(u) = f_u(x(u)),$$

where , for each $u \in U$, f_u is an increasing bijection of $[0,1]$ onto itself (an automorphism of $[0,1]$). Thus defined transformations preserve lattice structure on X and form a group $A^o(X)$ which is a proper subgroup of the automorphism group $A(X)$. We have the following theorem.

Theorem 6.2. *Let V be a neighborhood of the identity in $A(X)$ such that $V \subseteq A^o(X)$. Then any V–robust mean is $A^o(X)$–robust.*

Proof. The group $A^o(X)$ is the direct product of m copies of automorphism groups $A([0,1])$, i.e.,

$$A^o(X) = (A([0,1]))^m.$$

Since the group $A([0,1])$ is isomorphic to the group $A(\mathbb{R})$, it is a connected topological group. Thus $A^o(X)$ is also connected and therefore is generated by V. Now we apply the same argument as in the proof of the previous theorem to complete the proof.

□

The following theorem follows immediately from Theorem 6.1 in [9].

Theorem 6.3. *Let*

$$y(u) = M(x_1(u), x_2(u), \dots, x_n(u))$$

be a V–robust mean on $X = [0,1]^U$, where $V \subseteq A^o(X)$. There exists a function $p : U \to \{1, 2, \dots, m\}$ such that, for any given $u \in U$, $y(u)$ is the $p(u)$'s order statistic.

It is proven in [9] that any *invariant* mean on X is a p'th order statistic for some p which does not depend on $u \in U$. Thus, in the case $X = [0,1]^m$, the class of V–robust means is much wider than the class of invariant means.

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