Solving a Possibilistic Linear Program Through Compromise Programming

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Abstract

In this paper we propose a method to solve a linear programming problem involving fuzzy parameters whose possibility distributions are given by fuzzy numbers. To address the above problem we have used a preference relationship of fuzzy numbers that leads us to a solving method that produces the so-called α-degrees feasible solution. It must be pointed out that the final solution of the problem depends critically on this degree of feasibility, which is in conflict with the optimal value of the objective function. Then DM faces a bi-objective problem that we will solve through a Compromise Programming approach, whose solution lets the Decision-Maker express his own preferences about feasibility versus optimality. Our proposed method will be illustrated by a numerical example.

Keywords: Linear Programming, Fuzzy Number, Compromise Programming, Possibility Distribution, Expected interval, Expected value. Fuzzy numbers ranking.


1 Introduction

Linear programming (LP) is the optimisation technique most frequently applied in real-world problems and therefore it is very important to introduce in the approach new aspects that allow the model to fit into the real world as much as possible.

Any linear programming model representing real-world situations involves a lot of parameters whose values are assigned by the experts, and in the conventional approach, they are required to fix an exact value to the aforementioned parameters. However, experts or the Decision-Maker (DM) frequently do not know precisely the value of those parameters. If exact values are suggested, those are only statistical inference from the past data and their stability is doubtful so the problem’s parameters are usually defined by the Decision-Maker in an uncertain way or by
means of language statement Parameters. Therefore, it is useful to consider the knowledge of the experts about the parameters as fuzzy data.

This paper considers LP problems whose parameters are fuzzy numbers but the decision variables are crisp. In this kind of problems there are two key questions: how to handle the relationship between the fuzzy left and the crisp right hand side of the constraints, and how to find the optimal value for the fuzzy objective function. The answer is related to the problem of ranking fuzzy numbers.

A variety of methods for comparing or ranking fuzzy numbers has been reported in literature (Wang and Kerre [12]) and ranking methods do not always agree with each other. Different properties have been applied to justify ranking methods, such as: distinguishability (Bortolan and Degani [2]), rationality (Nakamura [8]), fuzzy or linguistic presentation (Delgado, Verdegay and Vila [4] and Tong and Bonisone [11]), and robustness (Yuan [14]).

In this paper we use a method (Jimnez [7]) that verifies all the above properties, and that, besides, is computationally efficient to solve an LP problem, because it preserves its linearity.

Looking at the property of representing the preference relationship in linguistic or fuzzy terms, ranking methods can be classified in two approaches. One of them associates, by means of different functions, each fuzzy number to a single point of the real line and then a total crisp order relationship between fuzzy numbers is established.

The other approach ranks fuzzy numbers by means of a fuzzy relationship. It allows decision makers to present their preferences in a gradual way, which in an LP program permits us to handle with different degrees of satisfaction of constraints and, with regard to objective value, it allows us to look for a non dominated satisfying solution.

Obviously if the Decision-Maker establishes a high degree of satisfaction of constraints for a solution, the feasible solutions set becomes less and, consequently, the objective optimal value is worse. So, the DM has to find a balanced solution between two objectives in conflict: to improve the objective function value and to improve the degree of satisfaction of constraints.

In order to address the above problem it has been suggested formulating an auxiliary bi-objective program that we will solve through Compromise Programming. The paper concludes with a numerical example to clarify the proposed methodological approach.

2 Possibilistic linear program

In a previous paper [1] we proposed to solve the following possibilistic linear program:

\[
\text{Minimize} \quad \tilde{z} - \tilde{c} \cdot \tilde{x} \\
\text{subject to} \quad x \in \mathcal{X}(\tilde{A}, \tilde{b}) = \left\{ x \in \mathbb{R}^n \mid \tilde{A} \cdot x \geq \tilde{b}, x \geq 0 \right\}
\]  

(1)

where \( \tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n) \), \( \tilde{A} = [\tilde{a}_{ij}]_{m \times n} \), \( \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \), represent fuzzy parameters whose possibility distributions are given by fuzzy numbers.
Our approach in the previous paper allowed us to obtain a fuzzy optimal solution in the objective space for problem (1). In the present work we propose obtaining a non-fuzzy optimal solution in the decision variables space to the same problem. To do that the following questions [9] arise:

1) How can we define the feasibility of a decision vector \( x \), when the parameters of constraints are fuzzy numbers?

2) How can we define the optimality for an objective function with fuzzy parameters?

Several authors have studied this problem. Some of them by considering not only fuzzy parameters but also the fuzzy goals of the decision maker, i.e. Tanaka and Asay [10], Sakawa [9]. Others by using some relationship of comparison of fuzzy numbers, i.e., Cadenas and Verdegay [3], Sakawa [9]. Our approach belongs to the second group. We use a fuzzy relationship to compare fuzzy numbers, developed by Jimenez [7], that verify good properties such: distinguishability, rationality, fuzzy presentation and robustness. This method is computationally efficient to solve an LP problem, because it preserves the linearity of the problem.

Before presenting the preference relationship of fuzzy numbers that we will use, we should recall the notions of the expected interval and the expected value of a fuzzy number. Heilpern [5] defines the expected interval of a fuzzy number \( \tilde{a} \), which will be noted \( EI(\tilde{a}) \), as:

\[
EI(\tilde{a}) = [E^a_1, E^a_2] = \left[ \frac{1}{h_0} \int_0^1 f_1^{-1}(h) dh, \frac{1}{h_0} \int_0^1 g_2^{-1}(h) dh \right]
\]

(2)

\[\text{where } f_1(h) \text{ and } g_2(h) \text{ are the left and right hand side of the membership function of the fuzzy number } \tilde{a}. \]

And the expected value of a fuzzy number \( \tilde{a} \), which will be noted \( EV(\tilde{a}) \), as:

\[
EV(\tilde{a}) = \frac{E^a_1 + E^a_2}{2}
\]

(3)

If \( \tilde{a} = (a_1, a_2, a_3, a_4) \) is a trapezoidal fuzzy number, its expected interval and its expected value can be calculated very easily:

\[
EI(\tilde{a}) = \left[ \frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2} \right], \quad EV(\tilde{a}) = \frac{a_1 + a_2 + a_3 + a_4}{4}
\]

(4)

**Definition 1.** For any pair of fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \) we define the relationship of fuzzy preference \( M(\tilde{a}, \tilde{b}) \), through the following membership function [7]:

\[
\mu_M(\tilde{a}, \tilde{b}) = \begin{cases} 
0 & \text{si } E_2^b - E_1^a < 0 \\
\frac{E_2^b - E_1^a}{E_2^b - E_1^a - (E_2^a - E_1^a)} & \text{si } 0 \in [E_1^a - E_2^b, E_2^b - E_1^a] \\
1 & \text{si } E_2^b - E_1^a > 0
\end{cases}
\]

(5)

where \( \mu_M(\tilde{a}, \tilde{b}) \) is the degree of preference of \( \tilde{a} \) over \( \tilde{b} \).

If \( \mu_M(\tilde{a}, \tilde{b}) \geq \alpha \), we denote it by \( \tilde{a} \geq_{\alpha} \tilde{b} \) and we say that \( \tilde{a} \) is greater than \( \tilde{b} \) at least in a degree \( \alpha \). When \( \mu_M(\tilde{a}, \tilde{b}) = \frac{1}{\alpha} \), we say that \( \tilde{a} \) and \( \tilde{b} \) are indifferent.
Definition 2. A decision vector \( x \in \mathbb{R}^n \), is said to be \( \alpha \)-feasible to problem (1) if:

\[
\min_{j=1,\ldots,m} \{ \mu_M (a_j x, b_j) \} - \alpha
\]

that is to say if

\[
\tilde{a}_j x \geq \tilde{a}_j b_j, \ \forall j = 1, \ldots, m
\]

Therefore after (5) we write the following expression:

\[
[(1 - \alpha)E_2^a + \alpha E_1^a] x \geq \alpha E_2^b + (1 - \alpha)E_1^b
\]

The set of all \( \alpha \)-feasible decision vectors is denoted by \( X(\alpha) \). From definition 2 the following property immediately follows:

\[
\alpha_1 < \alpha_2 \Rightarrow X(\alpha_1) \supset X(\alpha_2)
\]

By means of definition 2 we have answered the first question: how to define feasibility when the constraints involve fuzzy numbers.

Since \( \alpha \) indicates the feasibility degree of each constraint, \( 1 - \alpha \) can be interpreted as the degree of unfeasibility, i.e., \( 1 - \alpha \) gives us a measure of risk of violation of such constraint.

To answer the second question: how to define the optimality for an objective function with fuzzy parameters, we consider the following problem:

Minimize \( \tilde{z} - \tilde{c} x \)

subject to: \( x \in X(A, b) - \{ x \in \mathbb{R}^n / Ax \geq b, x \geq 0 \} \)

(10)

where the fuzzy parameters are only in the objective function. According to the ranking method given in definition 1, we can say that:

Definition 3. \( x^\circ \in X(A, b) \) is said to be an acceptable optimal solution to problem (10) if it is verified that:

\[
\tilde{c} x \geq 1/2 \tilde{c} x^\circ, \ \forall x \in X(A, b)
\]

(11)

From definition 1 the above expression can be written as follows:

\[
\frac{E_3^a x - E_1^a x}{E_2^a x - E_1^a x + E_3^a x - E_1^a x} \geq \frac{1}{2}
\]

(12)

so that

\[
\frac{E_1^a x + E_2^a x}{2} \geq \frac{E_1^a x + E_2^a x}{2}
\]

(13)

Therefore, from equation (3) and definition 3 the following proposition can be stated:

Proposition 1. \( x^\circ \in \mathbb{R}^n \) is an acceptable optimal solution to problem (10) if, and only if, it is an optimal solution to the following crisp linear program:

Minimize \( EV(\tilde{c})x \)

subject to: \( x \in X(A, b) \)

(14)
where \( EV(\tilde{c}) - (EV(\tilde{c}_1), EV(\tilde{c}_2), \ldots, EV(\tilde{c}_n)) \).

According to this result we may establish the following definition related to the initial fuzzy model (1).

**Definition 4.** \( x^0 \in \mathbb{R}^n \) is said to be an \( \alpha \)-optimal solution to problem (1) if it is an optimal solution to the following problem:

\[
\begin{align*}
\text{Minimize} & \quad EV(\tilde{c})x \\
\text{subject to} & \quad x \in \mathcal{X}(\alpha)
\end{align*}
\]  

(15)

Through this model we have substituted the initial fuzzy program (1) for a crisp \( \alpha \)-parametric linear program.

From (9) the obtaining of a better value to the optimal objective function implies a lesser degree of feasibility of the constraints. Then the Decision-Maker runs into two conflicting objectives: to improve the objective function value and to improve the degree of satisfaction of constraints. To obtain a balanced solution to this problem with two objectives in conflict, we have drawn up an auxiliary biobjective program, as shown in the following section.

### 3 A biobjective problem: the pay off matrix and the ideal point

According to the above considerations we propose raising the problem (1) again through the following non-linear biobjective problem:

\[
\begin{align*}
\text{Minimize} & \quad \tilde{z}(x) - \tilde{c}x \\
\text{Maximize} & \quad \alpha \quad \text{subject to } \quad \left[ (1-\alpha)E_{2}^{b_j} + \alpha F_{1}^{b_j} \right] x \geq \alpha E_{2}^{b_j} + (1-\alpha)E_{1}^{b_j}, \forall j \\
x & \geq 0, \quad 0 \leq \alpha \leq 1
\end{align*}
\]  

(16)

The set of feasible solutions of model (16) is denoted by \( \mathcal{X} \).

**Definition 5.** \( (x^*, \alpha^*) \in \mathcal{X} \) is said to be a Pareto optimal solution to (16) if there does not exist another \( (x, \alpha) \in \mathcal{X} \) such that

\[
\tilde{c}x^* \geq_{1/2} \tilde{c}x
\]  

(17)

and

\[
\alpha^* \geq \alpha
\]  

(18)

where at least one of these inequalities is strict.

From definitions 4 and 5 we deduce the following proposition:

**Proposition 2.** All Pareto optimal solutions \( (x^*, \alpha^*) \) to problem (16) are \( \alpha^* \)-solutions to problem (1) and reciprocally.

**Proof.** Let \( (x^*, \alpha^*) \) be an \( \alpha^* \)-solution to problem (1) but not a Pareto Optimal solution to problem (16); then there exists another \( (x, \alpha) \) verifying (17) and (18) where at least one of these inequalities is strict.
From (17) and the proposition 1 we have: $EV(\hat{c})x^* \geq EV(\hat{c})x$ and from (18) $x \in \mathcal{X}(\alpha^*)$; then $x^*$ is not an optimal solution to problem (15) and $(x^*, \alpha^*)$ is not an $\alpha^*$-solution to problem (1).

And reciprocally, let $(x^*, \alpha^*)$ be a Pareto Optimal solution to problem (16) but not an $\alpha^*$-optimal solution to problem (1). From definition 4 there is another $x \in \mathcal{X}(\alpha^*)$ such that $EV(\hat{c})x^* \geq EV(\hat{c})x$; from (3) we have that $\hat{c}x^* > \frac{1}{2} \hat{c}x$ and $(x^*, \alpha^*)$ is not a Pareto solution to problem (16). $\Diamond$

Small degrees of satisfaction of constraints can not be accepted by the Decision Maker. If $\alpha_0$ is the minimum degree of feasibility of constraints that the decision maker is willing to admit, the feasibility interval of $\alpha$ in $\alpha_0$ is reduced to $\alpha_0 \leq \alpha \leq 1$. We refer to this new problem as (16-bis):

\textit{Model (16) with: } $\alpha_0 \leq \alpha \leq 1$

To solve the problem (16-bis) through Compromise Programming it is necessary to obtain the pay-off matrix. To do that we optimise each objective separately, calculating the value reached by the other objective on the optimal decision variable. This matrix allows us to quantify the conflict level between the two objectives. In our case, we can obtain the two rows of the matrix starting from the minor and greater feasibility degrees: $\alpha - \alpha_0$ and $\alpha - 1$. According to definition 4, that means solving the following two crisp mono-objective programs:

a) Problem (15) with $\alpha - \alpha_0$, whose optimal solution $x^*(\alpha_0)$ gives the fuzzy number $\hat{z}^*(\alpha_0) - \hat{c}x^*(\alpha_0)$ as possibility distribution of the corresponding optimal objective value.

b) Problem (15) with $\alpha - 1$, whose optimal solution $x^*(1)$ gives the fuzzy number $\hat{z}^*(1) - \hat{c}x^*(1)$ as possibility distribution of the corresponding optimal objective value.

Therefore the pay off matrix is the following

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>Feasibility Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{z}^*(\alpha_0)$</td>
<td>$\alpha_0$</td>
</tr>
<tr>
<td>$\hat{z}^*(1)$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The elements of principal diagonal ($\hat{z}^*(\alpha_0), 1$) form the ideal point, which optimises the two objective functions simultaneously. Obviously the ideal point is unattainable. On the other hand, the secondary diagonal forms the anti-ideal point ($\hat{z}^*(1), \alpha_0$).

4 Compromise solutions

The Compromise Programming approach to solve a multiobjective optimization problem is based on the idea that a rational Decision Maker should choose the
Pareto optimal solution, or an area of Pareto optimal set, that is nearest to the ideal point. The points that verified the previous property are called compromise solutions.

In our case (problem (16-bis)), the points of the objective space have the form \((\tilde{z}(x), \tilde{a})\), with \(\tilde{z}(x) = \tilde{c}x\); that is to say the first component is a fuzzy number. We say that they are fuzzy points.

To find the compromise solutions, it is necessary to define the distance \(d_j\) between each objective function \(f_j\) and its ideal value \(f_j^*\). We work with the mean distance between fuzzy numbers [6]:

\[
d(\tilde{a}, \tilde{b}) = \|EV(\tilde{a}) - EV(\tilde{b})\|
\]

From (19) we can establish the following proposition:

**Proposition 3.** Any compromise solution to the following crisp program:

\[
\begin{align*}
\text{Minimize} & \quad EV[\tilde{z}(x)] - EV(\tilde{c})x \\
\text{Maximize} & \quad \alpha \\
\text{subject to} & \quad \left[(1 - \alpha)E_{2j}^{b_j} + \alpha E_{1j}^{b_j}\right] x \geq \alpha E_{2j}^{b_j} + (1 - \alpha)E_{1j}^{b_j}, \forall j \\
x & \geq 0, \quad \alpha_0 \leq \alpha \leq 1
\end{align*}
\]

is also a compromise solution to fuzzy program (16-bis), and reciprocally.

In our case, we can define the distance between each objective value and the corresponding ideal value as follows:

\[
\begin{align*}
d_1 (\tilde{z}(x), \tilde{z}^*(\alpha_0)) - \|EV(\tilde{z}(x)) - EV(\tilde{z}^*(\alpha_0))\| \\
d_2 (\alpha, 1) - 1 - \alpha
\end{align*}
\]

As the objectives are measured in different units, it is necessary to homogenise the distances:

\[
\begin{align*}
d_1 &= \frac{EV(\tilde{z})x - EV(\tilde{z}^*(\alpha_0))}{EV(\tilde{z}^*(1)) - EV(\tilde{z}^*(\alpha_0))} \\
d_2 &= \frac{1 - \alpha}{1 - \alpha_0}
\end{align*}
\]

To determine all possible compromise solutions, we consider the following distance measures:

\[
L_p = \begin{cases} 
(W_1d_1 + W_2d_2)^{1/p} & \text{for } 1 \leq p < \infty \\
\max(W_1d_1, W_2d_2) & \text{for } p = \infty
\end{cases}
\]

Where \(W_1\) and \(W_2\) are the weight that represent the Decision-Maker’s preferences with regard to the divergence between each objective value and its ideal value.

A compromise solution is the one which minimises \(L_p\). Therefore we will solve the following crisp monobjective program:

\[
\begin{align*}
\text{Minimize} & \quad L_p \\
\text{subject to} & \quad \left[(1 - \alpha)E_{2j}^{b_j} + \alpha E_{1j}^{b_j}\right] x \geq \alpha E_{2j}^{b_j} + (1 - \alpha)E_{1j}^{b_j}, \forall j \\
x & \geq 0, \quad \alpha_0 \leq \alpha \leq 1
\end{align*}
\]
Obviously the solution depends on the chosen metric. By varying $p$ from 1 to \( \infty \), the set of compromise solutions can be determined.

To illustrate the proposed methodology a possibilistic linear program is provided as a numerical example.

\section{Numerical example}

Let be the following a possibilistic linear program:

\[
\begin{align*}
\text{Minimize} & \quad (19, 20, 21)x_1 + (29, 30, 31)x_2 \\
\text{subject to:} & \quad (4.5, 5.5, 5.5)x_1 + (2.5, 3.4, 4)x_2 \geq (194, 300, 206) \\
& \quad (3.4, 5)x_1 + (6.5, 7, 7.5)x_2 \geq (230, 240, 250) \\
& \quad x_1 \geq 0, \ x_2 \geq 0
\end{align*}
\]

For simplicity we have supposed that all imprecise parameters are triangular fuzzy numbers, but any other fuzzy numbers could be implemented.

Let us assume that the minimum degree of satisfaction of constraints that DM is willing to accept is \( \alpha_0 = 0.4 \). According to expression (16-bis) we will solve the problem:

\[
\begin{align*}
\text{Minimize} & \quad (19, 20, 21)x_1 + (29, 30, 31)x_2 \\
\text{Maximize} & \quad \alpha \\
\text{subject to:} & \quad x \in \mathcal{X}(\alpha), \ 0.4 \leq \alpha \leq 1
\end{align*}
\]

To obtain the payoff matrix it is necessary to solve the above problem for \( \alpha = 0.4 \) and for \( \alpha = 1 \); then, according to epigraph 2 we will solve program (15) with \( \alpha = 0.4 \) and for \( \alpha = 1 \):

\[
\begin{array}{|c|c|}
\hline
\text{O.F.} & \text{F.D.} \\
\hline
\text{O.F.} & \tilde{z}^*(0.4) = (1048.2, 1089.1134.8) \quad 0.4 \\
\text{F.D.} & \tilde{z}^*(1) = (1174.8, 1226.1277.2) \quad 1 \\
\hline
\end{array}
\]

The principal diagonal is the ideal point: \((\tilde{z}^*(0.4), 1)\).

The compromise set is obtained solving the following programs with metrics \( L_p \) to the crisp program (24). Let \( W_1 = 0.5 \) and \( W_2 = 0.5 \):

a) with metric \( L_1 \), we have the problem:

\[
\begin{align*}
\text{Minimize} & \quad L_1 - 0.5 \left( \frac{20x_1 + 30x_2 - 1093.3}{1226.1093.3} \right) + 0.5 \left( \frac{x_1 - 1003.3}{1003.3} \right) \\
\text{subject to:} & \quad [(1 - \alpha)5.25 + \alpha 4.75]x_1 + [(1 - \alpha)3.5 + \alpha 2.75]x_2 \geq \\
& \quad \geq \alpha 203 + (1 - \alpha)197 \\
& \quad [(1 - \alpha)4.5 + \alpha 3.5]x_1 + [(1 - \alpha)7.25 + \alpha 6.75]x_2 \geq \\
& \quad \geq \alpha 245 + (1 - \alpha)235 \\
& \quad 0.4 \leq \alpha \leq 1, \ x_1 \geq 0, \ x_2 \geq 0
\end{align*}
\]
The solution is: $\alpha = 0.71$, $x_1^\infty = 29.73$, $x_2^\infty = 18.76$, $\tilde{z}^\infty = (1109.14, 1157.64, 1206.14)$ and the expected value of $\tilde{z}^1$ is 1129.67.

b) with metric $L_\infty$ the problem is:

\[
\begin{align*}
\text{Minimize } & \quad L_\infty - d \\
\text{subject to: } & \quad \text{Model (27) constraints set} \\
& \quad 0.5 \left( \frac{20x_1+30x_2-1063.5}{1226-1063.5} \right) \leq d \\
& \quad 0.5 \left( \frac{1-x_1}{100} \right) \leq d
\end{align*}
\]  

The solution is: $\alpha = 0.71$, $x_1^\infty = 29.73$, $x_2^\infty = 18.76$, $\tilde{z}^\infty = (1109.14, 1157.64, 1206.14)$ and the expected value of $\tilde{z}^\infty$ is 1157.64.

Comparing these solutions with the ideal point, we will observe that the one corresponding to $L_\infty$ is more balanced than the other, and it would be offered to the decision-centre as a solution. If the decision-maker does not consider the above solution acceptable, it may be the starting point of an interactive process.

6 Conclusions

In this paper we have proposed a method to solve a linear programming problem involving fuzzy parameters whose possibility distributions are given by fuzzy numbers. In a previous work, [1], we had solved the same problem obtaining a fuzzy solution in the objective space, but now we have constructed a solving method that produces crisp optimal solutions in decision space.

To construct the above method a preference relationship of fuzzy numbers it has been used. This leads us to the so-called $\alpha$-degree feasible solutions, and then to the acceptable optimal solution.

We have shown that the obtaining of a better value to the optimal objective function implies a lesser degree of feasibility of the constraints, and therefore the Decision-Maker faces a problem involving two conflicting objectives: to improve the objective function value and to improve the degree of satisfaction of constraints. In this problem, the degree of feasibility of constraints becomes a new decision variable.

To obtain a balanced solution to this problem we have solved a Compromise Programming problem, which allows the Decision-Maker to establish diverse requirements about feasibility versus optimality. This scheme requires a certain interaction with the Decision-Maker in order to adjust his structure of preferences; but it is much more flexible than the usual formulations.

To solve the above Compromise Programming program it has been necessary to extend to Possibilistic Programming the main results on classical Compromise Programming.
References


