

A Survey of Methods to Evaluate Quantified Sentences

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Abstract

The evaluation of quantified sentences is used to solve several problems. Most of the methods proposed in the literature are not satisfactory because they do not verify some intuitive properties. In this paper we propose an extension of both possibilistic and probabilistic methods, based on the Sugeno and the Choquet fuzzy integrals respectively, for the evaluation of type II sentences, the most general kind of sentences. These methods verify good properties, and they are shown to be better than existing ones. Some of the properties of the methods allow us to design an efficient algorithm with linear complexity to perform the evaluation under reasonable assumptions.

Keywords: Quantified sentences, linguistic quantifiers, fuzzy integrals.

1 Introduction

Quantified sentences are assertions about the number or percentage of objects that verify a certain property. They can be applied in many fields such as flexible database querying and expert systems among others (see [7] for a brief review). These sentences are classified into two classes, called type I sentences and type II sentences respectively. A type I sentence is a sentence that take the form "Q of X are A", where $X = \{x_1, \dots, x_n\}$ is a finite set, Q is a linguistic quantifier and A is a fuzzy property defined over X. One example of such sentences is "Most of the students are young". In this case, X is a finite set of students, the property A is "young" and Q is the quantifier "Most". A type II sentence can be described in general as "Q of D are A", where D is also a fuzzy property over X. One example is the sentence "Most of the young people are tall". Obviously, type I sentences are a special case of type II sentences where $D = X$.

The evaluation of quantified sentences tries to obtain an accomplishment degree in the real interval $[0, 1]$ for the sentence. In [3] we propose a set of intuitive properties that any evaluation method should verify. According to those properties, we show that some existing evaluation methods are not suitable in some cases. A

fuzzy quantifier defines a fuzzy quantity (absolute quantifier) or a fuzzy percentage (relative quantifier). Absolute quantifiers are defined as fuzzy sets over the nonnegative integers, while relative quantifiers are fuzzy sets over the real interval $[0, 1]$. A quantifier Q is said to be coherent if it is non-decreasing and $Q(0) = 0$ and $Q(1) = 1$. The case of the evaluation with non-coherent quantifiers is very significant for both type I and type II sentences. The only method that works with non-coherent quantifiers is Zadeh's method, but this method has the drawback that it is very strict when evaluating crisp quantifiers such as "exists" and "all". These problems are sufficient to justify the search for new methods with better properties. Nevertheless, there are other properties that are not verified by some of the existing methods.

In our opinion, methods based on the Sugeno and the Choquet fuzzy integrals are rather good, although they work with coherent quantifiers only. The latter is equivalent to the method of Yager based on the OWA operator [8]. They can be interpreted as a possibilistic and a probabilistic approach respectively. In [4] the relation among them is studied, but none of them is preferred. One of our objectives is to generalize these methods so that they could be used with coherent quantifiers. Another objective is to generalize these methods to evaluate type II sentences, and to study the relation among them in this case to find some criteria, if possible, that could help us to choose one of the methods. Finally, we are concerned with the design of efficient algorithms to perform the evaluation.

2 Type I sentences

2.1 Previous methods

An early method to evaluate type I sentences was introduced by Zadeh for relative quantifiers [10] to be

$$Z_Q(A) = Q\left(\frac{P(A)}{|X|}\right)$$

where $P(A)$ is the power of the fuzzy set A , defined as

$$P(A) = \sum_{i=1}^n A(x_i)$$

This method is strict for crisp quantifiers, in the sense defined in [6]. Basically, this means that the evaluation is crisp for crisp quantifiers. One example is the case of the quantifiers "exists" and "all", that can be defined as

$$\exists(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases} \quad \forall(x) = \begin{cases} 1 & x = 1 \\ 0 & x < 1 \end{cases}$$

Then, the evaluation of the sentences is

$$Z_{\exists}(A) = \begin{cases} 1 & A \neq \emptyset \\ 0 & A = \emptyset \end{cases} \quad Z_{\forall}(A) = \begin{cases} 1 & A = X \\ 0 & A \neq X \end{cases}$$

Other existing methods to evaluate type I sentences are based on the Sugeno and the Choquet integrals, see [1]. They are not strict methods, but both of them are restricted by definition to the case of monotonic increasing quantifiers. The Sugeno integral based method is characterized by

$$S_Q(A) = \max_{1 \leq i \leq n} \min(Q(i/n), b_i)$$

where b_i is the i -th greater value of $A(x_i)$, $i \in \{1, \dots, n\}$. The evaluation according to the Choquet integral based method is

$$C_Q(A) = \sum_{i=1}^n b_i \times (Q(i/n) - Q((i-1)/n))$$

Example 2.1 shows that S_Q is not a suitable method when we are dealing with non-coherent quantifiers. Example 2.2 shows that C_Q has the same problem.

Example 2.1 Let $X = \{x_1, x_2, x_3\}$ and $A = \{1/x_1 + 1/x_2\}$ and let $Q(0) = 0.5$, $Q(1/3) = 1$, $Q(2/3) = 0.5$ and $Q(1) = 0$. The quantifier Q can be interpreted as "approximately 1/3". Then the evaluation of "Q of X are A" by means of Sugeno's fuzzy integral is

$$S_Q(A) = \max\{\min(1, 1), \min(0.5, 1), \min(0, 0)\} = 1$$

But A is crisp, $|A| = 2$ and $|X| = 3$, so the expected result of the evaluation is $Q(2/3) = 0.5$.

Example 2.2 Let X and Q be as in example 2.1, and let $A = \{1/x_1\}$. Then $|A| = 1$, so the expected result is $Q(1/3) = 1$. Instead, $C_Q(A) = 0.5$.

2.2 Our methods

In [4] we introduced an extension of the method C_Q , called GD , to be

$$GD_Q(A) = \sum_{i=0}^n (b_i - b_{i+1}) \times Q(i/n)$$

In the same paper we show that if Q is coherent, then $GD_Q(A) = C_Q(A)$. This extension can be used with non-monotonic quantifiers and it has good properties.

The results obtained for the examples 2.1 and 2.2 by using GD are the expected results (since A is crisp, the evaluation must be $Q(|A|/|X|)$). In general, it is shown in [6] that if A is crisp then $GD_Q(A) = Q(|A|/|X|)$. Moreover, the evaluation method is not strict, i.e., given a quantifier Q we can find a fuzzy set A such that $GD_Q(A) \in]0, 1[$. This property holds even if Q is crisp.

In [5] we introduced an extension called ZS of the Sugeno integral based method, to be

$$ZS_Q(A) = \max_{\alpha \in M(A)} \min\left(\alpha, Q\left(\frac{|A_\alpha|}{|X|}\right)\right)$$

where

$$M(A) = \{\alpha \in [0, 1] \mid \exists x_i \in X : A(x_i) = \alpha\} \cup \{1\}$$

We show in [5] that if Q is coherent, then $ZS_Q(A) = S_Q(A)$. We also show that this method yields the expected result when A is crisp. Also, the method is not strict.

3 Type II sentences

3.1 Previous methods

Zadeh's method was introduced in [10] to be

$$Z_Q(A/D) = Q\left(\frac{P(A \cap D)}{P(D)}\right)$$

This is also an strict method in the case that Q is crisp. For example, in the case of the quantifiers "exists" and "all" the evaluations are

$$Z_{\forall}(A/D) = \begin{cases} 1 & D \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

and

$$Z_{\exists}(A/D) = \begin{cases} 0 & A \cap D = \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Yager introduced another method in [9] to be

$$Y_Q(A/D) = \sum_{i=1}^n w_i c_i$$

where c_i is the i -th largest value of membership to the fuzzy set $D' \cup A$, where D' is the standard complement of D

$$w_i = Q(S_i) - Q(S_{i-1}) \quad i \in \{1, \dots, n\}$$

$$S_i = \frac{1}{d} \sum_{j=1}^i e_j \quad \text{and} \quad d = \sum_{k=1}^n e_k$$

e_k being the k -th smallest value of membership to D , and $S_0 = 0$.

This method has several drawbacks, mainly because it is defined only for coherent quantifiers. The following examples show two additional cases of its misbehavior.

Example 3.1 *Let us consider $A = \{1/x_1 + 0/x_2\}$ and let $D = \{0/x_1 + 0.1/x_2\}$. Also let $Q = \exists$. Then $A \cap D = \emptyset$, so clearly the percentage of objects of D that pertain to A is 0, and hence the evaluation of the sentence "Q of D are A" should be $\exists(0) = 0$ (see [6]), but Yager's method yields $Y_{\exists}(A/D) = 0.9$ instead.*

Example 3.2 Let us consider $A = \{1/x_1, 0.9/x_2\}$ and $D = \{1/x_1, 0.5/x_2\}$. Also let $Q(x) = x$. Clearly $D \subseteq A$ and D is normalized, so the percentage of objects of D that pertain to A is 1, and hence the evaluation should be $Q(1) = 1$ (see [6] again). But instead of the expected result, Yager's method yields $Y_Q(A/D) = 0.93$. The same problem arises if $Q = \forall$. The expected result is $\forall(1) = 1$, but the method yields $Y_{\forall}(A/D) = 0.9$.

3.2 Our methods

The previous evaluation methods for type II sentences are less satisfactory than the existing methods for type I sentences. We have generalized our methods GD and ZS to the case of type II sentences, in order to obtain methods with better properties.

First we introduced in [5] the method ZS for type II sentences to be

$$ZS_Q(A/D) = \max_{\alpha \in M(A/D)} \min \left(\alpha, Q \left(\frac{|(A \cap D)_{\alpha}|}{|D_{\alpha}|} \right) \right)$$

where $M(A/D) = M(A \cap D) \cup M(D)$. We assume D is a normal fuzzy set. Otherwise, we normalize D in the usual way and we apply the same normalization factor to the fuzzy set $A \cap D$ (so, at least conceptually, we are not changing the relative cardinality of D with respect to A).

This method verifies several good properties, as we showed in [6].

- If A and D are crisp, then the expected result is obtained, i.e.

$$ZS_Q(A/D) = Q \left(\frac{|A \cap D|}{|D|} \right)$$

- If $D \subseteq A$ then $ZS_Q(A/D) = Q(1)$.
- If $A \cap D = \emptyset$ then $ZS_Q(A/D) = Q(0)$. Because of this, the method ZS obtain the expected values for the examples 3.1 and 3.2.
- The method yields intuitively good results when dealing with non-coherent quantifiers.
- The method is not strict.
- Its computational complexity is $O(n \log n)$, and the evaluation is faster and easier than the evaluation of Y_Q , because Yager's method needs to obtain the values S_i and w_i previously.

Obviously, when $D = X$ a type II sentence becomes a type I sentence. We showed in [5] that $ZS_Q(A/X) = ZS_Q(A)$ in that case.

We have also obtained a generalization of the probabilistic method GD to type II sentences. In [6] we introduced $GD_Q(A/D)$ to be

$$GD_Q(A/D) = \sum_{\alpha_i \in M(A/D)} (\alpha_i - \alpha_{i+1}) \times Q \left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|} \right)$$

where we write $M(A/D) = \{\alpha_i\}$ with $1 = \alpha_1 < \alpha_2 < \dots < \alpha_m < \alpha_{m+1} = 0$. This method verifies the same properties as method ZS , and hence it yields the expected results for the examples 3.1 and 3.2. Also if $D = X$ then $GD_Q(A/X) = GD_Q(A)$. Its computational complexity is $O(n \log n)$.

We have generalized the methods S_Q and C_Q to the case of any quantifier, either coherent or not, and to the case of type II sentences. In [2], Bosc and Lietard compare both methods and they conclude that the maximum difference between the results is 0.25. We have also shown that in the case of coherent quantifiers, these methods verify the properties we require of any good method. So both S_Q and C_Q are acceptable. Nevertheless, there are more deep differences between their generalizations, GD_Q and ZS_Q , at least in the case of type II sentences. The following example illustrates our claim.

Example 3.3 *Let us consider $A = \{1/x_1 + 0.01/x_2\}$ and $D = \{1/x_1 + 0.99/x_2\}$. Let $Q = \forall$. It is obvious that x_2 is very close to be in D , but it is very far from being in A , so intuitively the percentage of elements of D that pertain to A should not be 1. We obtain a value coherent with that intuition from method GD , since $GD_{\forall}(A/D) = 0.02$. But with ZS we obtain $ZS_{\forall}(A/D) = 1$. As we can see, the difference in the case of type II sentences can be very high. In fact, we can make this difference as close to 1 as desired by decreasing $A(x_2)$ and increasing $D(x_2)$ without reaching 0 and 1 respectively.*

Example 3.3 motivates us to prefer GD to ZS . We think that ZS generates a counterintuitive evaluation in this example because the functions "max" and "min" ignore most of the information contained in A and D , while method GD takes into account all the information (i.e. the information given by all the α -cuts of both A and D). This case arises in the case of S_Q and C_Q as well, but the consequences are smoothed because the information about the referential X is fully used by both methods.

4 Algorithm

Our objective is to design an efficient algorithm to perform the evaluation by using GD . The complexity of GD is $O(n \log n)$, because we must arrange $M(A/D)$. But the following result allow us to design an $O(n)$ algorithm:

Proposition 4.1 *Let $M^+(A/D) = M(A/D) \cup \{\alpha\}$ such that $\alpha \notin M(A/D)$. It is always possible to write $M^+(A/D) = \{\beta_i\}$ with $1 = \beta_1 > \beta_2 > \dots > \beta_m > \beta_{m+1} > \beta_{m+2} = 0$ and $m = |M(A/D)|$. Then*

$$GD_Q(A/D) = \sum_{\beta_i \in M^+(A/D)} (\beta_i - \beta_{i+1}) \times Q \left(\frac{|(A \cap D)_{\beta_i}|}{|D_{\beta_i}|} \right)$$

Proof. Since for any $\alpha \notin M(A/D)$, $\alpha_{i_0} \in M(A/D)$ exists such that $\alpha_{i_0} > \alpha > \alpha_{i_0+1}$, and $A_{\alpha_{i_0}} = A_\alpha$ and $D_{\alpha_{i_0}} = D_\alpha$. Then

$$Q\left(\frac{|(A \cap D)_{\alpha_{i_0}}|}{|D_{\alpha_{i_0}}|}\right) = Q\left(\frac{|(A \cap D)_\alpha|}{|D_\alpha|}\right)$$

and

$$\begin{aligned} & \sum_{\beta_i \in M^+(A/D)} (\beta_i - \beta_{i+1}) \times Q\left(\frac{|(A \cap D)_{\beta_i}|}{|D_{\beta_i}|}\right) = \\ & = GD_Q(A/D) - \left((\alpha_{i_0} - \alpha_{i_0+1}) \times Q\left(\frac{|(A \cap D)_{\alpha_{i_0}}|}{|D_{\alpha_{i_0}}|}\right) \right) + \\ & + \left((\alpha_{i_0} - \alpha) \times Q\left(\frac{|(A \cap D)_{\alpha_{i_0}}|}{|D_{\alpha_{i_0}}|}\right) \right) + \left((\alpha - \alpha_{i_0+1}) \times Q\left(\frac{|(A \cap D)_\alpha|}{|D_\alpha|}\right) \right) = \\ & = GD_Q(A/D) - \left((\alpha_{i_0} - \alpha_{i_0+1}) \times Q\left(\frac{|(A \cap D)_{\alpha_{i_0}}|}{|D_{\alpha_{i_0}}|}\right) \right) + \\ & + \left(((\alpha_{i_0} - \alpha) + (\alpha - \alpha_{i_0+1})) \times Q\left(\frac{|(A \cap D)_{\alpha_{i_0}}|}{|D_{\alpha_{i_0}}|}\right) \right) = GD_Q(A/D) \end{aligned}$$

Hence, we can evaluate GD by using any set of α -cuts containing $M(A/D)$. The results from the following proposition avoid normalizing D and $A \cap D$ when D is not normalized.

Proposition 4.2 *Let $nf(D) = \max_{x_i \in X} D(x_i)$. Let*

$$M(A/D)_* = (M(D) \cup M(A \cap D)) - \{\alpha \in M(D) \mid |D_\alpha| = 0\}$$

assuming that we have not applied any normalization factor to D yet. Then

$$GD_Q(A/D) = \frac{1}{nf(D)} \sum_{\alpha_i \in M(A/D)_*} (\alpha_i - \alpha_{i+1}) \times Q\left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|}\right)$$

Proof. When we normalize D and $A \cap D$ before calculating $M(A/D)$, we are in fact normalizing the values in $M(A/D)_*$. Then it is easy to show that, after the normalization,

$$M(A/D) = \left\{ \frac{\alpha}{nf(D)} \mid \alpha \in M(A/D)_* \right\}$$

Also

$$Q\left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|}\right) = Q\left(\frac{|(A \cap D)_{\frac{\alpha_i}{nf(D)}}|}{|D_{\frac{\alpha_i}{nf(D)}}|}\right)$$

since the relative cardinality between A and D does not change when we normalize both $A \cap D$ and D . Hence

$$\begin{aligned}
& \frac{1}{nf(D)} \sum_{\alpha_i \in M(A/D)_*} (\alpha_i - \alpha_{i+1}) \times Q \left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|} \right) = \\
& = \sum_{\alpha_i \in M(A/D)_*} \left(\frac{\alpha_i}{nf(D)} - \frac{\alpha_{i+1}}{nf(D)} \right) \times Q \left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|} \right) = \\
& = \sum_{\alpha_j \in M(A/D)_*} \left(\frac{\alpha_j}{nf(D)} - \frac{\alpha_{j+1}}{nf(D)} \right) \times Q \left(\frac{|(A \cap D)_{\frac{\alpha_j}{nf(D)}}|}{|D_{\frac{\alpha_j}{nf(D)}}|} \right) = \\
& = \sum_{\alpha_j \in M(A/D)} (\alpha_j - \alpha_{j+1}) \times Q \left(\frac{|(A \cap D)_{\alpha_j}|}{|D_{\alpha_j}|} \right) = GD_Q(A/D)
\end{aligned}$$

In practice, algorithms that deal with fuzzy sets usually assume the number of distinct membership degrees to be finite. This is a reasonable assumption that will improve the time expended in the evaluation. However, it is needed that the set of values provided a suitable level set in order to obtain a good representation of any fuzzy set by means of its *alpha*-cuts. In designing our algorithms, we have used a fixed number k of equidistant values as the possible membership degrees to A and D . The possible values will be

$$W = \left\{ \frac{1}{k}, \frac{2}{k}, \dots, \frac{n-1}{k}, 1 \right\}$$

and k should ensure that the set $\{F_\alpha \mid \alpha \in W\}$ is a good representation of any fuzzy set F defined over X . We define a pair of vectors V_D and $V_{A \cap D}$ of size k . $V_D(i)$ and $V_{A \cap D}(i)$ store the cardinality of the sets $\{x_i \in X \mid D(x_i) = i/k\}$ and $\{x_i \in X \mid (A \cap D)(x_i) = i/k\}$ respectively. These vectors can be calculated in time $O(n)$. Then, $GD_Q(A/D)$ is calculated in $O(k) = O(1)$ from the vectors by means of algorithm 1. This last step takes advantage of proposition 4.1 because $M(A/D) \subseteq W$, and hence

$$GD_Q(A/D) = \frac{1}{nf(D)} \sum_{\alpha_i \in W \cap [0, nf(D)]} (\alpha_i - \alpha_{i+1}) \times Q \left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|} \right)$$

Since $\alpha_i = \frac{k+1-i}{k}$ then $\alpha_i - \alpha_{i+1} = 1/k \forall i \in \{1, \dots, k\}$ and hence

$$GD_Q(A/D) = \frac{1}{k \times nf(D)} \sum_{\alpha_i \in W \cap [0, nf(D)]} Q \left(\frac{|(A \cap D)_{\alpha_i}|}{|D_{\alpha_i}|} \right)$$

In summary, the complexity of the evaluation is $O(n)$.

Algorithm 1 Algorithm to obtain $GD_Q(A/D)$ from V_D and $V_{A \cap D}$

1. $j \leftarrow k$
 $GD \leftarrow 0$
 $nf(D)^* \leftarrow k$
 $acum_D \leftarrow 0$
 $acum_{A \cap D} \leftarrow 0$
 2. { Calculate $nf(D)^* = nf(D) \times k$ }
While $(nf(D)^* > 0)$ and $(V_D(nf(D)^*) = 0)$
 - (a) $nf(D)^* \leftarrow nf(D)^* - 1$
 3. If $(nf(D)^* = 0)$ then return("Error"); End
 4. While $j > 0$
 - (a) $acum_{A \cap D} \leftarrow acum_{A \cap D} + V_{A \cap D}(j)$
 - (b) $acum_D \leftarrow acum_D + V_D(j)$
 - (c) If $(j \leq nf(D)^*)$
 - i. $GD \leftarrow GD + Q\left(\frac{acum_{A \cap D}}{acum_D}\right)$
 - (d) $j \leftarrow j - 1$
 5. { Normalization }
 $GD \leftarrow \frac{GD}{nf(D)^*}$
 6. return(GD); End
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5 Conclusions

We have briefly reviewed some previously existing evaluation methods for type I and type II sentences. We have discussed on these methods from the point of view of some intuitive properties that we think that any method should verify. We show by means of some examples that some of the existing methods are not suitable in some cases. An example is the case of sentences with non-coherent quantifiers. Zadeh's method is the only existing method that can deal with such kind of quantifiers, but it has been shown to be very strict for the evaluation of both type I and type II sentences. We have developed generalizations of two good existing methods, based on Sugeno's and Choquet's fuzzy integrals. These generalizations have better properties than previous ones. Our experiments, from which we have selected 3.3 to illustrate our claim, showed that the method GD should be preferred in general because it takes into account all the information involved and (to our experiments), the results are always coherent with our intuition. Finally, we have designed an efficient algorithm that runs in $O(n)$ time, to evaluate GD .

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