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Idempotent Operators on a Finite Chain

M. Mas[*], J. Torrens[†], T. Calvo[*] and M. Carbonell[*]
Dpt. de Ciències Matemàtiques i Informàtica
Universitat de les Illes Balears
07071 Palma de Mallorca. Spain
{*dmimmg0,dmijts0,dmitcs0,dmimch0*}*@ps.uib.es*

Abstract

This work is devoted to find and study some possible idempotent operators on a finite chain L. Specially, all idempotent operators on L which are associative, commutative and non-decreasing in each place are characterized. By adding one smoothness condition, all these operators reduce to special combinations of Minimum and Maximum.
Key words: t-norms, t-conorms, idempotent operators, labels, expert systems, uncertainty.

1 Introduction

In last years, many authors have studied binary operators defined on a totally ordered finite set L. Usually, the elements of L are viewed as linguistic terms or "labels" that experts use in their reasonings and the operators on L modelize the propagation of uncertainty in expert systems.

Another way to represent the reasoning in expert systems is given by fuzzy set theory where labels are represented by fuzzy sets and the propagation of uncertainty is performed by t-norms, t-conorms and De Morgan triplets.

The second aproximation is the most usual one, whereas the first aproximation has the advantage that the reasoning of experts is realized directly from these labels and so a numerical interpretation of them is not necessary as it is in the case of fuzzy sets. Following this idea the structure of "directed algebra" on a partially ordered set was introduced in [6], being this structure specially interesting because it preserves most of the properties of continuous De Morgan triplets. In fact, it is stated in [6] that the unique directed algebra structures on $[0,1]$ with the usual order are continuous De Morgan triplets.

However, on one hand it is well known that in directed algebras as well as in De Morgan triplets, the only possibility to have idempotent operators is with the

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Minimum and Maximum operators. On the other hand, the idempotent property has been recognized by many authors (see [7], [8], [9]) as one of the most important properties in logical connectives. Thus in this work, we study some possible modifications in the structure of directed algebras which let us to deal with idempotent operators, specially in the case when L is a totally ordered finite set, i.e. a finite chain. In particular, when L is a finite chain we obtain and describe all possible idempotent operators which are associative, commutative and non-decreasing in each place.

2 Preliminaries

Let us begin with some definitions and basic results that will be used along the paper.

Definition 2.1. A *t-norm* is a two-place function $T : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each place and such that 1 is a neutral element.

Definition 2.2. A *t-conorm* is a two-place function $T : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each place and such that 0 is a neutral element.

Definition 2.3. A *strong negation* or simply a *negation* is a decreasing and involutive function $N : [0,1] \to [0,1]$.

Definition 2.4. A triplet (T, S, N) is said to be a *De Morgan triplet* when T is a t-norm, S a t-conorm and N a negation such that $S(x, y) = N(T(N(x), N(y)))$ for all $x, y \in [0,1]$.

Definition 2.5. Let $(L, \leq)$ be a partially ordered set with minimum 0 and maximum 1. We will say that $(L, \leq, T, S, N)$ is a *directed algebra* if it verifies:

a) T and S are binary operators on L, associative, commutative and such that $T(1,1) = 1$ and $S(0,0) = 0$.

b) $N : L \longrightarrow L$ is an involution satisfying $N(T(x,y)) = S(N(x), N(y))$.

c.1) $x \leq y$ if, and only if, there exists $z \in L$ such that $x = T(y, z)$

c.2) $x \leq y$ if, and only if, there exists $w \in L$ such that $y = S(x, w)$.

In this case we will say that T is an AND operator and S is an OR operator of the directed algebra L. We will also say that N is a negation on L and that S and T are $N-$dual operators.

In [6], all possible directed algebra structures on a finite chain L are found. It is proved there that after some elements on L are selected, there is one and only one directed algebra structure on L in such a way that the corresponding AND operator has exactly these selected elements as idempotents. Consequently, the only idempotent AND operator of directed algebra on L is the Minimum and also, there is one and only one directed algebra structure with the corresponding AND operator archimedean.

Definition 2.6. Let F be a binary operator on a finite chain $L = \{0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1\}$. It is said that F verifies *1-smoothness* or that F is a *1-smooth* operator when for all $i, j \geq 1$, if $F(x_i, x_j) = x_k$ then

$$F(x_{i-1}, x_j) = x_l \quad \text{and} \quad F(x_i, x_{j-1}) = x_m$$

where l, m satisfy $k - 1 \leq l, m \leq k$.

Remark: Note that from the above definition all 1-smooth operators are non-decreasing in each place.

3 From directed algebras to idempotent algebras

As we have already mentioned, the definition of directed algebra given in [6] (see definiton 2.5) is characterized by the fact that the only structures of this kind on $[0, 1]$ with the usual order are continuous De Morgan triplets. But the same authors in their work suggest that other possible definitions can be considered by replacing some of the required properties by another ones. This is precisely the idea of this paper, to study alternately definitions which allow us to obtain idempotent operators. Initially, we will work in the general case when $(L, \leq)$ is simply a partially ordered set with minimum 0 and maximum 1. But in many cases we will restrict our study to the particular case when L is a finite chain; this is the case in which we are specially interested.

Let us begin by noting that we obtain an equivalent definition of directed algebra by changing condition (c.2) by the decreasing condition of N.

Proposition 3.1. $(L, \leq, T, S, N)$ is a directed algebra on L if, and only if, N is decreasing and conditions a), b), c.1) are satisfied.

Proof: To prove the direct implication, let $(L, \leq, T, S, N)$ be a directed algebra. Suppose that $x \leq y$, there exists $z \in L$ such that $x = T(y, z)$ but then $N(x) = N(T(y, z)) = S(N(y), N(z))$ and thus by condition c.2) we have $N(x) \geq N(y)$, i.e. N is decreasing.

Reciprocally, if N is decreasing we have,

$$x \leq y \iff N(x) \geq N(y) \quad \iff \text{there exists } z \in L \text{ such that } N(y) = T(N(x), z)$$

but this is satisfied if and only if there exists $N(z) \in L$ such that $y = N(T(N(x), z))$ $= S(x, N(z))$ which proves condition c.2). ■

In what follows we will use this fact without further comments and our modifications will be made from this equivalent definition. It is well known that associative and idempotent properties rarely coincide, thus our first change is in this way.

Proposition 3.2. If we substitute associative property of T and S by the distributive property of T with respect to S we obtain:

i) S is distributive with respect to T.

ii) T and S are non-decreasing in each place.

iii) T and S are idempotents.

Proof: The proof of the first part follows by duality since given $x, y, z \in L$, there exist $a, b, c \in L$ such that $x = N(a), y = N(b), z = N(c)$ and then:

$$S(x, T(y, z)) = S(N(a), T(N(b), N(c))) = N(T(a, S(b, c))) =$$

$$N(S(T(a, b), T(a, c))) = T(S(N(a), N(b)), S(N(a), N(c))) = T(S(x, y), S(x, z)).$$

Let us prove ii) only for the operator T since the proof for S is similar. Let $x, x' \in L$ such that $x \leq x'$. By condition c.2), there exists $z \in L$ such that $x' = S(x, z)$ and then:

$$T(x', y) = T(S(x, z), y) = S(T(x, y), T(z, y))$$

which prove that $T(x, y) \leq T(x', y)$ again by applying condition c.2).

Let us prove iii) again only for T. Given any $x \in L$, since $0 \leq x$ there must exist $y, z \in L$ such that

$$x = S(y, 0) \quad \text{and} \quad 0 = T(z, x)$$

by ii) we have $T(0, 0) = 0$ and also

$$x = S(y, T(0, 0)) = T(S(y, 0), S(y, 0)) = T(x, x).$$

■

However, this modification leads again to Minimum and Maximum operators in the case that L is a finite chain as it is proved in the following

Proposition 3.3. With the same conditions of the above proposition, it is satisfied:

$$x \leq y \iff T(x, y) = x \iff S(x, y) = y$$

and consequently, when L is totally ordered T is the Minimum and S is the Maximum.

Proof: We prove first that $T(x,1) = x$ for all $x \in L$. By condition c.1) we have $T(x,1) \leq x$ and by the above proposition $T(x,1) \geq T(x,x) = x$. Now, for all $x \leq y$,

$$x = T(x,x) \leq T(x,y) \leq T(x,1) = x$$

The converse is true by condition c.1) and it follows the equivalence for T. In a similar way we obtain the equivalence for S. ■

From now on, we will reduce to the case when L is a finite chain of the form $L = \{0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1\}$. Our following modification is to introduce directly the idempotent property instead of associativity.

Proposition 3.4. If we substitute associativity by the idempotent property of T and S in the definition of directed algebra, we obtain:

- i) $T(x_i, 1) = x_i$ for all $i = 0, \ldots, n+1$.
- ii) T is the Minimum and S is the Maximum.

Proof: i) Let us consider the function $T_1 : L \to L$ defined by $T_1(x_i) = T(1, x_i)$. Since any $x_i \in L$ verifies $x_i \leq 1$, condition c.1) proves the surjectivity of T_1 and consequently it is bijective. Since it is also increasing it must be the identity and so $T(x_i, 1) = x_i$ for all $i = 0, \ldots, n+1$.

To prove ii) we use induction on n. When $n = 0$, $L = \{0, 1\}$ and the only operator T on L is the Minimum and the only operator S is the Maximum. Now suppose that the asertion is true for $n-1$ and we prove it for n, i.e. when $L = \{0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1\}$. Let $\overline{L} = L - \{1\}$ and $\overline{T} : \overline{L} \times \overline{L} \to \overline{L}$ the corresponding restriction of T. Thus, it is clear that $\overline{T}$ is an idempotent and commutative operator. Moreover, when $x_i < x_j$ there exists $x_k \in L$ such that $T(x_k, x_j) = x_i$. By applying i), $x_k \neq 1$ and so $x_k \in \overline{L}$ and $x_i = \overline{T}(x_k, x_j)$. We have proved that $\overline{T}$ is an operator on $\overline{L}$ with the same properties that T and by induction it must be the Minimum. Now it is clear by i) that T is also the Minimum operator and by duality S is the Maximum. ■

From the above results, it seems to be clear that condition c.1) is the most restrictive one. Initially introduced to give "some idea of " continuity, it is the only condition that relates the ordering of L with operators T and S. Thus, our final modification, which in this case leads us to idempotent operators, is to remove condition c.1) but introducing monotonicity of T and S with respect to the ordering of L.

Definition 3.5. It is said that $(L, \leq, T, S, N)$ is an *idempotent algebra* on L when

- T and S are associative, commutative, increasing in each place and idempotent binary operators on L.
- N is a decreasing involution and $S(x,y) = N(T(N(x), N(y)))$ for all $x, y \in L$.

In what follows, we will study and classify all idempotent algebra structures on L and we begin with some easy properties. For instance, it is easy to prove (see [2] or [6]) that there is only one negation $N : L \to L$ which is given by

$$N(x_i) = x_{n+1-i} \quad \text{for all} \quad i = 0, \ldots, n+1.$$

Consequently, any idempotent algebra structure (T, S, N) on L is completely determined by the operator T since then S follows from duality. Thus, from now on, we will work only with these operators T of idempotent algebras.

Remark: Note that in idempotent algebras, contrariously to the case of directed algebras, operators T and S are totally symmetric. In fact, in directed algebras it is deduced immediately from condition c) that

$$T(0, x) = 0, \ T(1, x) = x \quad \text{and} \quad S(0, x) = x, \ S(1, x) = 1$$

and so T is viewed as an AND operator and S as an OR operator. However, in idempotent algebras T and S simply satisfy

$$0 \le T(0, x), S(0, x) \le x \le T(1, x), S(1, x) \le 1$$

and thus the properties of T and S are exactly the same.

Thus, let T (or S) be an operator of idempotent algebra. Note first that T is between Minimum and Maximum since for all x_i, x_j with $x_i \le x_j$

$$x_i = T(x_i, x_i) \le T(x_i, x_j) \le T(x_j, x_j) = x_j.$$

Definition 3.6. Let T be an operator of idempotent algebra on L. We will say that an element $x_i \in L$ is *fixed* by 1 when $T(x_i, 1) = x_i$.

Note that if T is an operator of idempotent algebra on L, then 1 is always fixed by 1, but there can be more elements on L with this property. Thus, let us denote by F the set of elements on L that are fixed by 1, that is:

$$F = \{x_i \in L \mid \ T(x_i, 1) = x_i\}$$

and we will see that the structure of T depends on the set F.

Lemma 3.7. Let T be an operator of idempotent algebra on L. If $F = L$, T is the Minimum. On the other hand, if $x_i \in L$ is not fixed by 1 and $T(x_i, 1) = x_k$ with $k > i$, then $x_k \in F$ and

$$T(x_r, x_s) = x_k \quad \text{whenever} \quad i \le min(r, s) \le k \le max(r, s)$$

Proof: Let us suppose that allthe elements of L are fixed by 1. Then if $x_i \le x_j$, we have

$$x_i = T(x_i, x_i) \le T(x_i, x_j) \le T(x_i, 1) = x_i$$

and consequently T is the Minimum. On the other hand, let $T(x_i, 1) = x_k$ with $k > i$. Then

$$T(x_k, 1) = T(T(x_i, 1), 1) = T(x_i, 1) = x_k$$

and so $x_k \in F$. Moreover,

$$T(x_i, x_k) = T(x_i, T(x_i, 1)) = T(T(x_i, x_i), 1) = T(x_i, 1) = x_k$$

and by monotonicity, $T(x_s, x_k) = x_k$ for all x_s with $x_i \leq x_s \leq 1$. On the other hand, when $i \leq r \leq k \leq s$,

$$x_k = T(x_i, x_k) \leq T(x_r, x_s) \leq T(x_k, x_s) = x_k.$$

■

Definition 3.8. Let T and T_0 be operators of idempotent algebra on L and $L_0 = [x_1, 1]$ respectively. We will say that T is a $(Min, T_0)-$*sum* if

$$T(x_i, x_j) = \begin{cases} T_0(x_i, x_j) & \text{when } i, j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

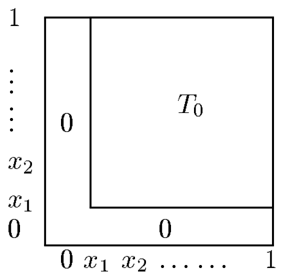
Definition 3.9. Let T and T_0 be operators of idempotent algebra on L and $L_0 = [0, x_n]$ respectively. We will say that T is a $(T_0, Max)-$*sum* if

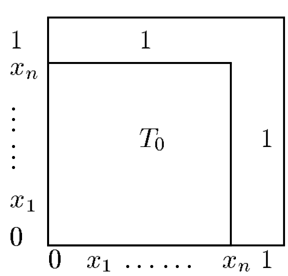
$$T(x_i, x_j) = \begin{cases} T_0(x_i, x_j) & \text{when } i, j \leq n \\ 1 & \text{otherwise} \end{cases}$$

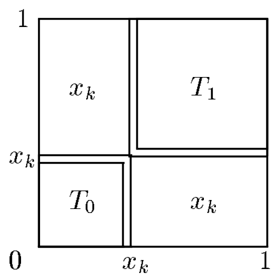
Definition 3.10. Let T be an operator of idempotent algebra on L and $x_k \in L$. Let T_0, T_1 be operators of idempotent algebra on $L_0 = [0, x_{k-1}]$ and $L_1 = [x_{k+1}, 1]$ respectively. We will say that T is a $(T_0, T_1)-$*sum* if

$$T(x_i, x_j) = \begin{cases} T_0(x_i, x_j) & \text{when } x_i, x_j \in L_0 \\ T_1(x_i, x_j) & \text{when } x_i, x_j \in L_1 \\ x_k & \text{when } Min(i, j) \leq k \leq Max(i, j) \end{cases}$$

The following figures show the structure of the operators defined above.







Lemma 3.11. Let T be an operator of idempotent algebra on L and let x_k be the first element of L fixed by 1. If $0 < x_k < 1$ then T is a $(T_0, T_1)-$ sum.

Proof: Consider T_0 and T_1 the restrictions of T to $L_0^2 = [0, x_{k-1}]^2$ and $L_1^2 = [x_{k+1}, 1]^2$ respectively. It is obvious that T_0 and T_1 are operators of idempotent algebra on L_0 and L_1. On the other hand, as x_k is the first element of L which is fixed by 1, we have by the second part of lemma 3.7 $T(0, 1) = x_k$ and also

$$T(x_i, x_j) = x_k \quad \text{whenever} \quad min(i, j) \leq k \leq max(i, j)$$

which ends the proof. ∎

Theorem 3.12. A function $T : L \times L \to L$ is an operator of idempotent algebra on L if, and only if, T is a $(Min, T_0)-$sum, or a $(T_0, Max)-$sum or a $(T_0, T_1)-$sum.

Proof: Clearly, operators of the three defined forms are commutative, increasing in each place and idempotent. Associativity follows easily by distinguishing several cases. We show only one of them as an example. Let T be a $(T_0, T_1)-$sum and let $x_i, x_j, x_s \in L$ such that $s < k < i \leq j$. Then

$$T(T(x_i, x_j), x_k) = T(T_1(x_i, x_j), x_k) = x_k$$

whereas

$$T(x_i, T(x_j, x_k)) = T(x_i, x_k) = x_k.$$

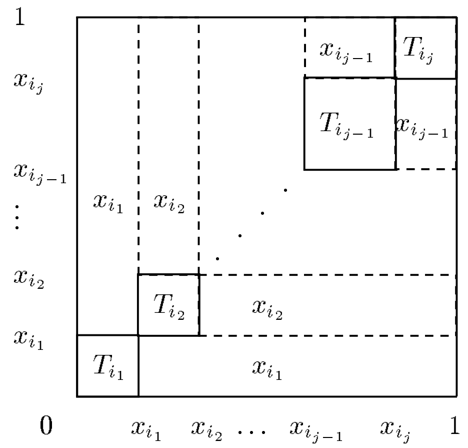
Reciprocally, let T be an operator of idempotent algebra on L and F the set of elements of L fixed by 1.

- If $0 \in F$, then $T(0, x_i) = 0$ for all $x_i \in L$ and by considering T_0 the restriction of T to L_0^2 where $L_0 = [x_1, 1]$ it follows that T is a $(Min, T_0)-$sum.
- If $F = \{1\}$, then $T(x_i, 1) = 1$ for all $x_i \in L$ and by considering T_0 the restriction of T to L_0^2 where $L_0 = [0, x_n]$ it follows that T is a $(T_0, Max)-$sum.
- If $0 \notin F$ and $x_k \neq 1$ is the first element of L fixed by 1, let us consider T_0 and T_1 the restrictions of T to $L_0^2 = [0, x_{k-1}]^2$ and $L_1^2 = [x_{k+1}, 1]^2$ respectively. It follows from lemma 3.7 that T is a $(T_0, T_1)-$sum. ∎

From the above theorem we can classify all operators T of idempotent algebra on L. Given the elements of L fixed by 1, T must be a sum of "suboperators" T_i; each one of them can be classified in the same way from the last theorem. For instance, if the set F is given by

$$F = \{x_{i_1}, \ldots, x_{i_j}\} \quad \text{with} \quad 0 < x_{i_1} < \ldots < x_{i_j} = 1$$

the structure of T is as follows



where each T_{i_l} is an operator T on the corresponding interval and its structure depends similarly on the set of elements fixed by its maximum element.

Examples 3.13. When $n = 0$, $L = \{0, 1\}$ and there are only the operators Minimum and Maximum. When $n = 1$, $L = \{0, x, 1\}$ and there are five possible operators of idempotent algebra which are given by:

1	0	x	1
x	0	x	x
0	0	0	0
	0	x	1

1	0	1	1
x	0	x	1
0	0	0	0
	0	x	1

1	1	1	1
x	x	x	1
0	0	x	1
	0	x	1

1	1	1	1
x	0	x	1
0	0	0	1
	0	x	1

1	x	x	1
x	x	x	x
0	0	x	x
	0	x	1

In fact, we can give the number of operators T of idempotent algebra on L for any natural number n as we prove in the following:

Proposition 3.14. Let d_n be the number of operators T of idempotent algebra on $L = \{0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1\}$. Then $d_0 = 2$, $d_1 = 5$, $d_2 = 2d_0 + 2d_1 = 14$ and for all $n \geq 3$:

$$d_n = 2d_{n-1} + 2d_{n-2} + \sum_{j=2}^{n-1} d_{j-2}d_{n-1-j}.$$

Proof: The above example proves the first two numbers. The value of d_2 can be proved similarly to the general case and we leave if to the reader. To prove the general case, let $n \geq 3$ and let T be any operator of idempotent algebra and x_k the first element of L fixed by 1. Then,

- If $k = 0$, T is a (Min, T_0)−sum and there exists exactly d_{n-1} operators of this kind.
- If $k = 1$ we necessarily have
$$T(0, x_i) = x_1 = T(x_1, x_i) \quad \text{for all } i \geq 1$$
whereas T restricted to $[x_2, 1]^2$ is an operator of idempotent algebra and so there are d_{n-2} operators of this kind.
- Dually we obtain the cases $k = n + 1$, where T is a (T_0, Max)−sum with d_{n-1} operators and $k = n$ with d_{n-2} operators.
- Otherwise, $2 \leq k \leq n - 1$ and T is a (T_0, T_1)−sum where T_0 and T_1 are operators on $L_0 = [0, x_{k-1}]$ and $L_1 = [x_{k+1}, 1]$ respectively. Thus, for each k the number of operators in this case is $d_{k-2}d_{n-k-1}$.

Finally, the given formula follows by joining all the above numbers. ■

The last proposition shows that the number of idempotent operators on L is high. It is mainly due to the fact that without condition c) of directed algebra we lose any idea of "continuity". However, there are some authors (see [2], [3]) which use 1-smoothness condition (definition 2.6) in order to obtain this idea of continuity. If we add this condition to the operators of idempotent algebras we reduce drastically the number of possible operators but we still obtain more than the Minimum and Maximum operators.

Proposition 3.15. Let T a 1-smooth operator of idempotent algebra.

- If $T(1,0) = 0$, T is the Minimum.
- If $T(1,0) = 1$, T is the Maximum.
- If $T(1,0) = x_k$ with $0 \neq x_k \neq 1$, T is given by the following:

$$T(x_i, x_j) = \begin{cases} Max(x_i, x_j) & \text{when } x_i, x_j \leq x_k \\ Min(x_i, x_j) & \text{when } x_i, x_j \geq x_k \\ x_k & \text{otherwise} \end{cases}$$

Proof: Let us see the three cases separately.

i) If $T(1,0) = 0$ let us prove that $F = L$. Suppose on the contrary that there is an element of L which is not fixed by 1 and let x_i be the first element with this condition. Then,

$$T(x_i, 1) > x_i \quad \text{whereas} \quad T(x_{i-1}, 1) = x_{i-1}$$

which is a contradiction with 1-smoothness. Thus, by lemma 3.7 we obtain $T = Min$.

ii) If $T(1,0) = 1$ let us prove that $T(0, x_i) = x_i$ for all $i \in \{0, \ldots, n+1\}$. Suppose on the contrary that there is an element of L that do not satisfy this property and let x_i be the biggest one satisfying $T(0, x_i) < x_i$. Then,

$$T(0, x_i) < x_i \quad \text{whereas} \quad T(0, x_{i+1}) = x_{i+1}$$

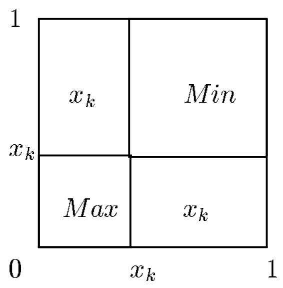
which is again a contradiction. Thus, if $i \leq j$

$$x_j = T(0, x_j) \leq T(x_i, x_j) \leq T(x_i, x_j) = x_j$$

and so $T = Max$.

iii) If $T(1,0) = x_k$ with $0 \neq x_k \neq 1$, then x_k must be the first element of L fixed by 1 and T is a $(T_0, T_1)-$sum with T_0 and T_1 1-smooth operators on $L_0 = [0, x_{k-1}]$ and $L_1 = [x_{k+1}, 1]$ respectively. Moreover, note that $T(0, x_k) = x_k$ and since $T(0,0) = 0$ and T is 1-smooth we necessarily have $T(0, x_i) = x_i$ for all $i \in \{0, \ldots, k\}$. Thus, T_0 is in the case ii) and it must be the Maximum. Similarly, T_1 is in the case i) and it must be the Minimum. ■

The structure of these operators can be viewed in the following figure:



This kind of operators on L are special cases of the so-called $t-$operators which are studied in [5]. It is proved there, the last result through a classification of all $t-$operators on L. The same special combinations of Minimum and Maximum but defined on $[0,1]$ are also studied in [1] and appear also as a particular case of $t-$operators on $[0,1]$ in [4].

As an immediate corollary to Proposition 3.15 we obtain.

Corollary 3.16. There are $n+2$ 1-smooth operators of idempotent algebra on L.

4 Conclusions

The concept of idempotent algebra over a finite chain L is introduced as an structure (T,S,N) where T and S are associative, commutative, increasing in each place and idempotent binary operators on L and N is a decreasing involution on L in such a way that T and S are N-dual. Afterwards, all possible idempotent algebra structures on L are characterized. A recursive formula is given to compute the total number of these structures depending on the number of elements of L. Finally, by adding the hypotesis of 1-smoothness, it is proved that these structures reduce to special combinations of Minimum and Maximum which coincide with the so called idempotent t-operators in [5].

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