

Truth and Utility in Fuzzy Logic

A. Sobrino

Dept. Logic and Moral Philosophy. Univ. Santiago de Compostela
Campus Universitario, s. n. 15706 Santiago de Compostela, Spain
lfgalex@usc.es

Abstract

The notion of degree of truth used in fuzzy logic can be interpreted in terms of partial truth or in terms of utility. By investigating how these interpretations fit in with Tarski's definition of truth, this paper explores some of their implications and their consequences for the foundations and credibility of fuzzy logic.

Keywords: Vagueness, Truth, Utility, Fuzzy logic.

1 Introduction

The notion of truth is central to classical bivalent logic, which is commonly described as the study of the propagation of truth by valid deductive processes. Fuzzy logic is often described as a formalism for reasoning in situations to which uncertainty is inherent; it associates both premises and conclusions with *degrees of truth* - a concept as fundamental to fuzzy logic as truth is to classical logic - and provides a calculus allowing reasonable evaluation of the degree of certainty with which conclusions can be reached from vague premises and imprecise logical relationships.

It is well known that degrees of truth can be interpreted in several different ways. One of the most usual views sees them as quantifying truth itself, so that a degree of truth less than 100% indicates a *partial truth*; another sees them as quantifying *useful truth* [3]. Interpreted as measures of partial truth, they express approximations to a whole truth that is unattainable because of lack of factual or inferential information; for example, a doctor attending a road accident victim without diagnostic equipment must draw inferences on the basis of imperfect knowledge of the victim's condition. Interpreted as measures of "useful" truth, they express an agent's subjective confidence in the truth of the proposition in question; for example, in distributing their study time, students take into account the degrees of confidence they have in their current ability to pass examinations in the various subjects. Either way, as expressions of partial truth or useful truth, degrees of truth are relative to an objective truth that is unattainable because of current lack of information.

In classical logic, the notion of objective truth is usually related to that of eternal truth, or at least to the notion of a truth with a certain temporal stability. In particular, the verificational criteria of physics and mathematics endow their propositions with a certain stability or immutability. In fuzzy logic the concept of imprecise truths is usually related either to the need to carry out reasoning processes in situations in which time is short and precise information concerning premises and implications cannot be waited for, to situations in which precise information would be so voluminous as to make its processing impractical, or to situations in which there are no available mechanisms for processing precise information in a way that guarantees conclusions of greater reliability than those that can be drawn on the basis of imprecise information.

Classical logic, as the study of the propagation of truth by valid deductive processes, rests on an ideal criterion of truth. Rigorous definitions of truth have been proposed by Tarski (primarily with formal languages in mind) and, in view of the problems arising when Tarski's definition is applied to natural languages, by several other authors. With fuzzy logic in mind, Goguen [6] has proposed a "social" definition of truth with features inherited from speech act theory [1]; a social concept of truth concerns not true sentences but true utterances emitted by particular agents with particular goals in particular situations. A problem with such a concept is that the lack of replicability of the circumstances attending an utterance throw doubts on the objectivity of truth assignments that are made to depend on these circumstances.

Certainly, fuzzy logic appears to require some kind of social concept of truth, since it attempts to model real-life reasoning constrained by limited time and information processing power. At the same time, however, it claims to be objective (even though fuzzy propositions may be, in some context-dependent sense, of limited rather than universal scope), and hence to require a definition of truth that is objectively applicable in a way that seems beyond the reach of social concepts of truth. Fuzzy logic would thus seem to need a concept of graded truth that is nevertheless susceptible to a definition of the kind that Tarski proposed for the truth of classical logic.

In this paper I use Tarski's truth criterion to examine certain limitations and implications of the notions of truth and utility used in fuzzy logic. In Section 2 below I analyse the concepts of ideal truth (2.1), social truth (2.2) and multiple-valued truth (2.3), in Section 3 I consider truth and utility in a fuzzy context, and in Section 4 I sketch some closing reflexions.

2 The notion of truth

2.1 Ideal truth

Classical logic is concerned with the formal reconstruction of the consequences of scientific language - paradigmatically, the languages of physics and mathematics, fields in which research is commonly said to aspire to objective truth. This intended objectivity leads to the requirement that any definition of truth be i) of

general applicability, and ii) ontologically and epistemologically neutral. It is usually considered that both these requisites are satisfied by the theory of truth put forward by Tarski [12] in 1936, which proposes conditions that should be met by any adequate truth theory and presents a definition of truth for formal languages that is said to comply with these conditions.

Tarski requires of a truth theory that it be both “materially adequate” and “formally correct”. A theory is materially adequate when it implies all instances of the T- schema

$$S \text{ is true iff } p \quad (\text{T})$$

where “S” is a name given to some sentence belonging to the language for which truth is being defined, and p is a sentence that asserts what the sentence S asserts; the standard example of an instance of the T-schema is

$$\text{“Snow is white” is true iff snow is white}$$

A theory is formally correct if the language for which truth is being defined and the definition of truth itself satisfy certain conditions (see Tarski [12] for precise details). Tarski’s theory satisfies the requirement of ontological and epistemological neutrality because application of the T-schema does not intrinsically involve the agent’s knowledge or beliefs. Satisfaction of the requirement of generality is less obvious: first, because there is no single definition of truth applicable to all languages, but only particular definitions for particular formal languages or appropriately regularized natural languages; and secondly, because even as regards a single given language, there is no definition of truth in general, only of the truth of particular sentences S.

Let us consider the material adequacy condition in the context of natural language, including the language of empirical science. Note that satisfaction of the T- schema does not by itself constitute a definition of truth. Tarski remarked that if a natural or scientific language had a finite number of sentences, each of which was semantically closed, the definition of truth would be trivial because it would be possible to list all sentences and define as true just those sentences S that translated into true sentences p. However, natural languages have an infinite number of sentences. Tarski met this problem by defining truth in terms of the satisfaction of a sentential function by given objects, and proving that this definition was materially adequate. However, this solution fails to tackle a different kind of problem that arises for natural languages: the existence of vague predicates, which means that in the T-schema S and p may include words of imprecise meaning. This poses the questions of whether the T-schema is formally compatible with the existence of vague predicates, and what the implications are for a theory of truth.

2.2 Social truth

In a social approach to truth, the question of what truth is is replaced by questions about how the word “true” is used, i.e. how it is applied to sentences that are asserted to be true. On this view, assertions about truth are evaluated in the course of social interaction, using ordinary language, rather than in an ideal “objective” system designed for some specific purpose such as the formal reconstruction of science. A social definition of truth does not start by requiring that true sentences

reflect the world, but instead that they describe situations in a way that results in successful communication between language users, so that the language user receiving them understands the user by whom they are emitted. Propositions do not have to *reconstruct* reality; rather, they construct a useful subjective understanding of reality.

In this context, a theory of truth must take into account that the truth-value of a proposition is not independent of the agents using it or the situation in which it is used; that it may change in time; that propositions must be evaluated just as they are in real language, without any alteration of their lexicon; and that it is a fact that most natural language propositions include words of imprecise meaning. In ordinary communication there are no independent truths, but true sentence utterances performed by particular individuals. In fact, in the social context of communities of specialists in particular subject matters, the attribution of truth to a proposition depends more on the linguistic community than on its individual members, since it is generally the community that establishes, by institutional mechanisms, what the correct use of a symbol or string of symbols is and which of its members are competent - and in what degree - to establish truth at some particular time. In everyday language, utterances usually refer to situations that are not replicable, cannot be examined by others, and cannot even be re-examined by the speaker, since there is no certainty that two different experiences are instances of the same situation.

In the absence of replicates, all that can be said is that assertions can be made with different degrees of success in various dimensions, can fit the facts more or less loosely, and in different ways, on different occasions and with different intents and purposes [6]. The universe of truth values cannot be limited to “true” and “false”.

As noted above, fuzzy logic appears to require a social definition of truth. Since it is concerned with reasoning that is imprecise, like everyday reasoning in ordinary language, it ought to take into account the agents involved, time, context, and the existence of vague predicates. However, recognition of the social nature of truth provides no criterion of how to use this concept; rather, it appears to imply that, because of the variety and transience of social phenomena, no such criterion is possible. Thus although it is not immediately clear how the T-schema may be applied to natural or scientific language, neither is it clear that there is no need to apply it. Any scientific field seeks objectivity, including the attempts, within the field of fuzzy logic, to establish a formal or informal theory of vagueness; for anyone wishing to endow the logical analysis of imprecise language with objectivity, the T-schema is an instrument that cannot be overlooked.

2.3 Ideal truth and multiple-valued truth

According to Putnam [10], it is possible to formulate a T-schema for languages with vague predicates, but only at a price: the introduction of vagueness in the meta-language. This is illustrated by the following truth table, in which “I” represents any degree of truth between T and F.

S is true	iff (\times)	p
T	T	T
F	T	F
I	T	I

In other words, the logic of the metalanguage is influenced by the logic of the object language: if one wants to preserve the equivalence expressed in the T-schema, then vagueness cannot be confined to the object language and inspected from the vantage point of a crisp metalanguage - the metalanguage, too, must allow vagueness. Furthermore, there must even be metasentences predicating truth that have truth values other than T and F, so that the predicate “true” must necessarily be as imprecise as the vague predicates of the object language: if the T-schema is preserved, to an imprecise S there must correspond an imprecise p, and to the imprecise p an imprecise “S is true”. In short, Putnam’s arguments show that if the metalanguage is to conform to classical logic, then the object language must be regularized so as to eliminate vagueness; while if vague predicates are to be admitted in the object language as susceptible of logical analysis, then classical logic must be replaced by a multiple-valued logic in both the object language and the metalanguage.

Even if the T-schema can, at a price, be generalized from precise to imprecise languages, the above generalization would appear to require that, whatever degree of truth a sentence may have, that degree of truth be unique, distinguishable, and independent of the nature of agents and situations. Is this so in languages with vague predicates? The contrary is suggested by the paradox of Sorites: if someone with just a few hairs on their head, n say, is bald, then someone with $n + 1$ hairs is also bald, and so on; so everyone is bald, no matter how many hairs they have on their head. Here, a patently false conclusion is arrived at through the absence of a holistic view of the problem and the use of reasoning that maintains unaltered, at each step, the strength with which the predicate “bald” is applied. This paradox thus highlights what appears to be an incompatibility between rigorous application of the rules of logic and the common sense evaluation of truth in everyday situations: it is absurd to conclude that even the hairiest of heads is bald, but it is no less absurd to try to identify a definite number of hairs, possession of which constitutes the borderline between baldness and non-baldness.

What the paradox of Sorites really highlights is a difference between the objectives of “logico-mathematical” reasoning and the objectives of common sense reasoning. “Logico-mathematical” reasoning is appropriate and effective in proving propositions with immutable truth values, whereas common sense reasoning concerns the assignment of truth values for specific useful purposes in specific communicative situations.

At this point, let us recall that in fuzzy logic “degree of truth” is usually interpreted either as “partial truth” or as “useful truth”. From the point of view of utility, it will in some cases be irrelevant whether a given sentence is assigned as its truth value 0.860 or 0.861; these two values will be indistinguishable as regards the success of the communication in which they are used and the desirability or otherwise of their consequences. However, interpreted as measures of partial truth,

as correlates of reality (perhaps a vague object), 0.860 is essentially distinct from 0.861; indeed, assignment of the truth value 0.860 could be treated as negating assignment of the value 0.861. Thus for a proposition designating its linguistic correlate ambivalently, the T-schema is not satisfied, at least for one of the values.

Application of the T-schema to sentences that can only be modelled semantically using degrees of truth therefore leads inevitably to undesirable conclusions. If the interpretation of the notion of degree of truth is logical, there ought to be a sharp borderline between a vague predicate and its negation so as to avoid Sorites-type paradoxes; but there is no sharp borderline. If, on the other hand, the interpretation is based on utility, then there are criteria for drawing the borderline, but the borderline drawn is hazy and mutable, depending on contextual and extralinguistic factors. This dilemma between useless objectivity and subjective utility is characteristic of the status of the notion of truth in fuzzy logic. In what follows I discuss some aspects of this.

3 Truth and utility in fuzzy logic

Fuzzy logic is a logic of vague predicates. A frequently used paradigm of a vague predicate is “tall”. It is common to say that this predicate is observational, i.e. it belongs to that class of predicates that for neopositivists are essential components of the propositions with which science is constructed. In spite of this exalted function, we cannot but recognize that vague observational predicates can be used in ways that are contradictory (in the classical sense of contradiction) or at least mutually distinct. If we wanted the concept to have a single valuation, we would have to express it with greater precision.

A common way of sharpening the meaning of a vague predicate is by reference to instruments that amplify or refine observation, affording a more detailed view of the objects and events observed. But instruments have their limits. Parikh’s theorem -for more details see [6]- shows that for α -connected spaces, instruments of observation or measurement cannot clarify the extension of a vague predicate: all observations, however fine, have their twilight zone.

Another way of sharpening a vague predicate such as “tall” is to treat it as expressing a relativistic concept, comparing, for example, the height of one individual with that of another. In this case the semantics of the vague predicate is not represented by a degree of truth but by an ordered set of values. But since the predicate, being vague, admits more than one instantiation, certain bearers of vague predicates with different heights cannot be mutually discriminated or distinguished by means of the predicate “tall”. In other words, they are not mutually comparable.

In view of these difficulties we may feel tempted to wonder what a tall individual really is. It would be helpful to try to answer this question in terms of a precise measure; for example, by taking a rule and using it to measure an appropriate individual. Suppose that the result is a height of 184.0 cm. If the individual we have measured appears to be truly tall, we might say that a tall individual is someone who measures 184.0 cm. But this does not answer the original question;

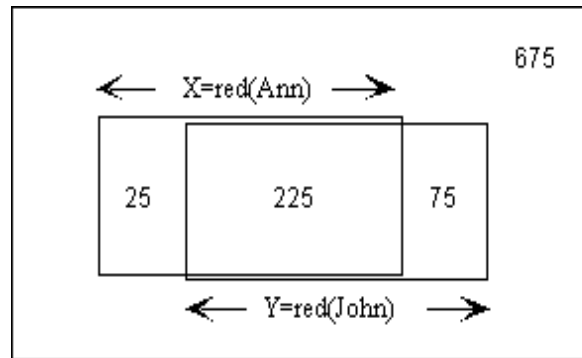
at most, it answers the question “What is it to measure 184.0 cm?”, which is quite distinct, at least in theory, from the question “What is it to measure 184.1 cm?” It would thus seem decidedly difficult to determine precisely what tallness is, in spite of our all knowing how to employ the word “tall” in our social dealings.

So a scientific definition of “tall” does not coincide with a public definition, and *vice versa*. To say of someone that they are tall is a cultural assertion, but science is a limited or special form of culture, involving a specific apprenticeship undertaken in restricted academic quarters and particular communities. Natural language, on the other hand, is the fruit of evolution and part of our organic and social development. Anyone, through merely living in community, learns the meaning of “tall”, although each individual interprets it in his or her own terms. As Quine [11] pointed out, the usual means by which the meaning of observational predicates is acquired is ostension, so nobody can be sure that what a word means for them coincides exactly with what it means for someone else, even though they *are* confident of gross coincidence (as is shown by the failure of communication to break down whenever we use these predicates).

Coarse meaning is thus a characteristic of certain words in ordinary language. Although 184.0 cm is perfectly distinguishable from 184.1 cm, we would all agree that if someone said to measure 184.0 cm were classified as tall, this classification would not be altered if they turned out to measure 184.1 cm. Even if variation of linguistic conventions from one community to another are ignored, an individual’s being tall is, as it were, not a representative of something unique, but of a collection of instances fitting the description given by the predicate. The conventions governing predication of tallness are useful, and capture the world, but do not need to be immutable; they can change if their users so wish. This is not the case of Pythagoras’ Theorem, a mathematical proposition considered to be universal and timeless, at least in the appropriate theoretical framework.

For vague predicates, the consensus of the community is extremely important in attributing meaning [9]. In order to be able to know that the car in front is braking, there must be agreement that red is red, even though the measured wavelengths of the braking lights of any two cars selected at random would doubtless almost invariably be different. What is needed is a pragmatic theory of the use of the word “red”, not an ideal theory of it (which is already available).

Parikh [8] has devised a very apt illustration of the utility of a pragmatic theory of the use of observational terms. It goes like this. Ann and John teach in the same school and share a flat. One day, Ann phones John from school to say that she has left at home a book on logic she needs for her next class, and can he please find it and take it to her. Since John does not know what the book looks like, he asks her what colour it is. Ann replies with a vague observational predicate: “red”. The problem is that Ann has 1000 books, of which 250 are red for Ann ($\text{set}(X)$), 300 red for John ($\text{set}(Y)$) and only 225 red for both; 25 are red for Ann but not for John, 75 red for John but not for Ann, and 675 red for neither of them. This situation is represented in the following figure.



There are 100 books to which Ann and John assign different colours, but it is assumed that neither of them is aware of this when Ann makes her request. John tries to find the book Ann wants with the stated colour as his only external clue (he has to look at the title page to make sure he has the right red book). It is instructive to calculate the average number of books he has to open in different circumstances.

- a) In the absence of any information at all about the external appearance of the book, if he is lucky it will be the first one he opens, otherwise it will be the last. On average, then, he will have to open 500.5 books.
- b) If John's concept of red coincides with X, then by analogous reasoning he will have to open, on average, 125.5 books, since his search will be limited to 250 books.
- c) If Ann's and John's concepts of red overlap, as is the case, let us suppose that Ann believes the book to be in X with probability 0.9 and John believes it to be in Y with the same probability. John will first search Y and then, if the search is unsuccessful, the complement of Y. He will therefore first look at an average 150.5 books with probability 0.9, and then, if he looks at all 300 in X without finding it, at an average 350 more with probability 0.1, making an overall average of $0.9 \times 150 + 0.1 \times (300 + 350) = 200$ books.

To sum up, John is helped to no small extent by Ann's telling him the colour or probable colour of the book, even though their concepts of red do not exactly coincide. In other words, if the vague predicate is used, it endows Ann's message with greater probability of success than if it is not used: the search time - a pragmatic measure relevant to the utility of the message - is significantly reduced thanks to the information received.

The above example shows that the success of Ann's message does not depend on her and John both knowing precisely how they use the word "red", nor on their usage being the same in different situations. For finding the requested object it is sufficient for them to share part of the referential meaning of the word.

Thus, in spite of vague predicates having different values, their use does not lead to the disaster foretold by the classical contradictions (although it does lead to difficulties concerning the propagation of degrees of truth through reasoning processes; I shall not dwell on this important point). Their massive use in language is an example of evolutionary, adaptive linguistic behaviour reinforced by the benefits afforded by the achievement of understanding among different people when there is linguistic cooperation. Cooperation involves consensus, but consensus prevents the drawing of a sharp, unalterable line between cases to which the vague predicate is applicable and cases to which it is not applicable.

Science nevertheless aspires to provide objective knowledge, and so too does a formal theory of propositions with vague meanings, such as fuzzy logic. How can naked truth, which offers objectivity, be reconciled with utility, which stresses the importance of vagueness in communication? I shall now attempt to comment this question.

4 Final reflexions

Going back to Sorites, what would justify a sentence being assigned one degree of truth and not another, almost indistinguishable from the first? As noted above, it could be argued that the choice of the former degree of truth has proved useful for the required purposes, and no reduction in utility would result from replacing it by the latter.

An approach based on utility allows the value of a sentence to be represented in multiple ways, at the agent's convenience, without there inevitably arising a contradiction that aborts communication, as would occur if the value were treated as the negation of a minimally different one. An interpretation of the semantics of fuzzy logic in terms of utility thus affords the theorist some freedom in modelling problems, and this adaptability to specific contexts is one of the key factors in its success [5]. But at the same time it may be an excuse for avoiding possible refutations. Since affairs are dealt with in partial, subjective scenarios, in many cases it is difficult to tell whether a proposal is the best solution or even a good one. In some cases the problem may not have been modelled as well as it might, though the contextuality makes it difficult to prove the contrary.

Flexibility is undoubtedly useful, but appears to be incompatible with the rigour that formal languages are supposed to possess. It would therefore be desirable to seek a certain objectivity in the undeniable utility offered by fuzzy logic, understood as a logic of assertions. This objectivity is usually provided by a logical criterion.

I have already noted that if the user wishes to have a logical basis for the use of degrees of truth, their assignment must be unique and concern unique referents. In this case (and leaving to one side the difficulty of constructing a list of sentences), it would be proper to claim the validity of the T-schema for a vague natural language. But if that were so, the equivalence that the T-schema requires between metalanguage and object language could only be substantiated by clarifying the state of things referred to by a sentence containing vague predicates, i.e. by elucidating the notion of vague object. If that were achieved, we would have a precise

graded logic, since each sentence in the metalanguage would have a single referent in the object language.

But what a vague object is is studied, in the framework of fuzzy logic, at the representational level, not the ontic level [4]. There is therefore no ontology of vague objects allowing the Sorites paradox to be avoided by marking the transition from a predicate to its negative. And this paradox constitutes an unsurmountable obstacle to a logical interpretation, since it leads to contradictions that are not just partial but absolute, even though they are anything but reasonable.

From a representational point of view, vagueness is there in language and it would seem to be both desirable and useful to have a logic that treats ordinary language as it is, i.e. as vague. In view of the T-schema, this can only come about if multiple-valuedness is an attribute of the predicate “true” itself; that is, if there is a multiple-valued logic available. If this logic is not to break down, it must be based on a criterion of utility, and will consequently have a mutable, insecure logical foundation, since assertions signify contextually. But as a formal language, a logic of vagueness is a precise language; its symbols are well defined and its logical meaning is characterized in a definite way. For such a language, an ideal definition of truth allowing the development of a precise metalogic would be valid.

Metalogic can give a certain rigour and objectivity to the multiplicity that stems from the interpretation of degree of truth in terms of utility. Delimitation of the properties of certain formalisms that are necessary to characterize vagueness is a task that reinforces the credibility of fuzzy logic. Because of the computational slant that Zadeh gave to fuzzy logic [2], metalogic has not been the field to which most effort has been devoted in the fuzzy community, but a change in this trend may benefit both its foundations and the achievement of a certain objectivity.

Acknowledgements

The author is indebted to Settimo Termini, Senén Barro and Alberto Bugarín for their comments.

Work supported by PGIDT99PXI10502B project, Xunta de Galicia, Galicia, Spain.

References

- [1] Austin, J. L., ‘Truth’, in J.O. Urmson y G. J. Warnock, (comp.) (1961), *Philosophical Papers of J. L. Austin*, Oxford Univ. Press.
- [2] Bellman, R. E. and Zadeh, L. A., ‘Local and Fuzzy Logics’, in J. Dunn and G. Epstein, *Moderns Uses of Multiple-Valued Logic*. Reidel, 1977, 105-165.
- [3] Dubois, D. and Prade, H., ‘The three semantics of fuzzy logic’, *Fuzzy Sets and Systems*, 90, 1997, 141-150.
- [4] Evans, G., Can there be vague objects?, *Analysis*, 38, 1978, 208.

- [5] Giles, R., 'A Utility-Valued Logic for Decision Making', *International Journal of Approximate Reasoning*, 1988, 2, 113-141.
- [6] Goguen, J., 'Fuzzy Sets and the Social Nature of Truth', in M. M. Gupta et al. (eds), *Advances in Fuzzy Set Theory and Applications*, North-Holland Pub. Co., 1979, 49-67.
- [7] Parikh, R., 'The problem of vague predicates', R. S. Cohen and M. W. Wartofsky (eds.), *Language, Logic and Method*, Reidel. Dordrecht, 1983, 241- 261.
- [8] Parikh, R., 'Vagueness and utility: The semantics of common nouns', *Linguistic and Philosophy*, 17, 1994, 521-535.
- [9] Parikh, R., 'Vague predicates and language games', *Theoria* (España), vol. 11, 1996, 27, 97-107.
- [10] Putnam, H., 'Vagueness and alternative logics', *Erkenntnis*, 19, 1983, 297-314.
- [11] Quine, W. V. O, *Word and Object*, Cambridge, Mass., 1960
- [12] Tarski, A. 'The concept of truth in formalized languages', en *Logic, Semantics, Metamathematics*, 1936. Trans. by J. H. Woodger, Oxford Univ. Press, 152-278.