Mathware & Soft Computing 5 (1998) 233-242

Generation of Multi-dimensional Aggregation Functions

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Abstract

In this paper we study two ways of generating multi-dimensional aggregation functions. First of all we obtain multi-dimensional OWA operators in two different ways, one of them through quantifiers and the other through sequences. In the first case, we see that all the operators we obtain are multi-dimensional aggregation functions. We then characterize the multi-dimensional aggregation functions that are generated by quantifiers. In the second case, we characterize the sequences that provide multi-dimensional aggregation functions and give examples and properties.

Keywords. OWA operators, multi-dimensional aggregation functions, quantifiers, weighting triangles.

1 Introduction

The problem of aggregating information has been studied during the last years from different points of view. Particularly the $n-$dimensional OWA operators introduced by Yager in [5] have proved to be interesting in this process. These operators have been extensively studied by many authors ([1], [2], [4], [5], [6], [7], [8]). In this way, the WOWA operators are studied in [4] as a generalization of the OWA operators and the weighted means.

However, the common fact of these operators is their n-dimensionality. On the other hand, [1] introduces the multi-dimensional (extended) OWA operators (EOWA). The main contribution of these operators is their multi-dimensional nature, that is, they can be applied to lists of n elements for any value of $n \geq 1$. A step forward is the introduction in [3] of the multi-dimensional (extended) aggregation functions (EAF) which, besides of being multi-dimensional operators, are monotonic with respect to certain orders that allow to compare lists of elements of different length. The relationship between these two types of operators is given in [3] where the EOWA which are EAF are characterized through a property on the weights (see proposition 2).

In this paper, we present two ways to obtain EOWA operators that are EAF. The first way is through the so-called quantifiers (fuzzy quantifiers, linguistic quantifiers) represented by fuzzy subsets of the real line, that is, by monotonic mappings $Q : [0,1] \to [0,1]$ (see [6], [7]). These references show how these quantifiers are used to interpret concepts like "most", "many", etc. In this part of the paper we verify that the way of defining OWA operators through fuzzy quantifiers given in [7] can be generalized to define EOWA operators and that these operators are always multi-dimensional aggregation functions. We characterize the EOWA operators generated by fuzzy quantifiers that are EAF and study their properties.

The second way of obtaining EAF consists of generating EOWA operators through sequences of real numbers. In this case not all the operators we obtain are EAF; in any case we characterize the sequences that generate EAF and give some examples, like the decreasing sequences, the arithmetic and geometric progressions, the Fibonacci sequence, etc.

2 Preliminaries

In this section we present the definitions and basic results we will use throughout this paper.

Given a lattice $(L, \leq)$ with minimum 0 and maximum 1, let us denote by E the set $E = \bigcup_{n \geq 1} L^n$ of all the ordered lists formed by elements of L.

Definition 1 The following relations $\leq_\pi, \leq_\alpha$ and $\leq_\beta$ define orders on the set E. Given $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ elements of E:

- $x \leq_\pi y$ if and anly if $n = m$ and $x_1 \leq y_1, \ldots, x_n \leq y_n$ (*product order*)
- $x \leq_\alpha y$ if and only if $n \leq m$, $x_1 \leq y_1, \ldots, x_n \leq y_n$, and if $n < m$ then $sup(x_1, \ldots, x_n) \leq inf(y_{n+1}, \ldots, y_m)$
- $x \leq_\beta y$ if and only if $n \geq m$, $x_1 \leq y_1, \ldots, x_n \leq y_n$, and if $n > m$ then $sup(x_{m+1}, \ldots, x_n) \leq inf(y_1, \ldots, y_m)$

The following result gives a quite useful characterization of the monotonicity with respect to the orders α and β.

Proposition 1 Let $A : E \to L$ be a monotonic with respect to the product order mapping. Then:
a) A is monotonic with respect to $\leq_\alpha$ if and only if $A(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n, sup(x_1, \ldots, x_n))$,
b) A is monotonic with respect to $\leq_\beta$ if and only if $A(x_1, \ldots, x_n, inf(x_1, \ldots, x_n)) \leq A(x_1, \ldots, x_n)$
for all $(x_1, \ldots, x_n) \in E$.

Definition 2 A mapping $A : E \to L$ is a *multi-dimensional (Extended) Aggregation Function* (A is an EAF) if it satisfies the following conditions:

1) A is monotonic with respect to the orders $\leq_\alpha$ and $\leq_\beta$

2) A is idempotent, that is, $A(\overbrace{x, \ldots, x}^{n}) = x$ for all $x \in L$ y $n \geq 1$.

Definition 3 (Yager) A mapping $A : [0,1]^n \to [0,1]$ is an $n-$dimensional OWA (*Ordered Weighted Averaging*) operator if there exists $W = (w_1, \ldots, w_n) \in [0,1]^n$ (called the weighting list) with $\sum_{i=1}^{n} w_i = 1$ and such that

$$A(x_1, \ldots, x_n) = W \cdot R(x_1, \ldots, x_n) \quad \text{for all} \quad (x_1, \ldots, x_n) \in [0,1]^n$$

where $R(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ is the initial list $(x_1, \ldots, x_n)$ realigned decreasingly and $W \cdot R(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i y_i$.

Definition 4 It is said that an application $A : \bigcup_{n \geq 1} [0,1]^n \to [0,1]$ is an EOWA operator (a multi-dimensional (Extended) OWA operator) if the restriction of A to $[0,1]^n$ is an $n-$dimensional OWA operator. That is, if for all $n \geq 1$ there exists an $n-$dimensional weighting list, $W_n = (w_1^n, \ldots, w_n^n)$, such that $A(x_1, \ldots, x_n) = W_n \cdot R(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in [0,1]^n$.

Remark 1 Notice that an EOWA operator is always idempotent, symmetrical and monotonic with respect to the product order. Notice as well that an EOWA operator can be represented through a *weighting (or probabilistic) triangle* in such a way that every row n of this triangle comprises the corresponding $n-$dimensional weighting list W_n. That is,

$$\begin{array}{c} 1 \\ w_1^2 \quad w_2^2 \\ w_1^3 \quad w_2^3 \quad w_3^3 \\ w_1^4 \quad w_2^4 \quad w_3^4 \quad w_4^4 \\ \ldots \end{array}$$

Reciprocally, every weighting triangle (it means that the elements of any row add up to 1) defines obviously an EOWA operator. From now on, we will represent an EOWA operator or a weighting triangle by $\wedge w_j^n$.

The following examples show how some classical operators are EOWA operators as well as Multi-dimensional Aggregation Functions.

Example 1 $A(x_1, \ldots, x_n) = min(x_1, \ldots, x_n)$ is an EOWA operator with $W_n = (\overbrace{0, \ldots, 0}^{n-1}, 1)$ for every n. Similarly, $A(x_1, \ldots, x_n) = max(x_1, \ldots, x_n)$ is also an EOWA operator with $W_n = (1, \overbrace{0, \ldots, 0}^{n-1})$ for each n. It is easy to see that both operators are EAF.

Example 2 The arithmetic mean, $A(x_1, \ldots, x_n) = \frac{x_1+\ldots+x_n}{n}$ is an EOWA operator with $W_n = (\overbrace{1/n, \ldots, 1/n}^{n})$ for each n. It is easy to see that A is an EAF.

Next result (see [3]) gives a characterization of the EOWA operators that are EAF.

Proposition 2 Let A be an EOWA operator with $W_n = (w_1^n, \ldots, w_n^n)$ for each $n \geq 1$. Then A is an EAF if and only if the following inequalities hold for all $n \geq 1$ and $p = 1, \ldots, n$

$$\sum_{i=1}^{p} w_i^{n+1} \leq \sum_{i=1}^{p} w_i^n \leq \sum_{i=1}^{p+1} w_i^{n+1} \ . \tag{1}$$

Definition 5 It is said that a weighting triangle $\wedge w_j^n$ is *regular* if it satisfies the condition (1).

Definition 6 Given an EOWA operator A with $W_n = (w_1^n, \ldots, w_n^n)$ for each $n \geq 1$, let us define the *reciprocal of* A (represented by A^r) to be the EOWA operator defined by $U_n = (u_1^n, \ldots, u_n^n)$ for every $n \geq 1$, where $u_j^n = w_{n-j+1}^n$ for all $n \geq 1$ and for all $j = 1, \ldots, n$. On the other hand, it is said that an EOWA operator A is *symmetrical* when $A = A^r$.

3 Generation through quantifiers

Definition 7 A *quantifier* is a mapping $Q : [0,1] \to [0,1]$ such that

i) $Q(0) = 0, \ Q(1) = 1$
ii) $r < t \Longrightarrow Q(r) \leq Q(t)$

This definition can be already found in [6] and [7] and in both cases these quantifiers are used to represent concepts like “most”, “many”. Particularly, it is shown in [7] how, given a quantifier $Q : [0,1] \to [0,1]$, it can be constructed an $n-$dimensional OWA operator, A_Q, with the weights given by

$$w_j = Q\left(\frac{j}{n}\right) - Q\left(\frac{j-1}{n}\right) \quad \text{for all} \ \ j = 1, \ldots, n.$$

Thus, for example, to evaluate a sentence like

“Most students are poor”

we would proceed in the following way: We represent the concept of “most” by a quantifier Q, to be “poor” by a fuzzy set B and then, given a sample of students $x_1, \ldots, x_n$, the value of the previous sentence would be given by $A_Q(b_1, \ldots, b_n)$ where A_Q is the $n-$dimensional OWA operator obtained from Q and each $b_i = B(x_i)$.

However, there is no reason to be limited to $n-$dimensional operators. The same process can be generalized to multi-dimensional operators and thus construct an EOWA operator A_Q from a quantifier Q by defining the weights as

$$\forall n \geq 1, \quad \omega_j^n = Q(\frac{j}{n}) - Q(\frac{j-1}{n}), \quad j = 1, \ldots, n$$

and then $A_Q(x_1, \ldots, x_n) = \sum_{j=1}^{n} w_j^n y_j$, where $(y_1, \ldots, y_n) = R_n(x_1, \ldots, x_n)$, and R_n is the increasing realignment over $[0,1]^n$.

Remark 2 Notice that following this procedure we really obtain a weighting triangle

- $w_j^n \in [0,1], \ \forall j = 1, \ldots, n, \ \forall n \geq 1$
- $\sum_{j=1}^{n} w_j^n = \sum_{j=1}^{n} \left[Q(\frac{j}{n}) - Q(\frac{j-1}{n})\right] = Q(1) - Q(0) = 1$

Observe also that $\sum_{j=1}^{k} w_j^n = \sum_{j=1}^{k} \left[Q(\frac{j}{n}) - Q(\frac{j-1}{n})\right] = Q(\frac{k}{n})$.

Example 3 From the quantifiers

$$Q_1(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0,1] \end{cases} \qquad Q_2(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$$

$$Q_3(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

we obtain the EOWA operators

$$A_{Q_1} = Max, \quad A_{Q_2} = Min \quad \text{and} \quad A_{Q_3} = Median,$$

respectively.

The triangle corresponding to the *Max* operator would be formed by a "1" in the first entry of each row and a "0" anywhere else. The one corresponding to the *Min* operator would have a "1" in the last entry of each row and a "0" anywhere else. Finally, the weighting triangle corresponding to the *Median* operator would be the following:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1/2 & & 1/2 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 1/2 & & 1/2 & & 0 & \\ 0 & & 0 & & 1 & & 0 & & 0 \\ & & & & \cdots & & & & \end{array}$$

Example 4 If we consider the quantifier $Q(x) = x^2$, frequently used (see [6],[7]) to represent the fuzzy concept of "most", we obtain the EOWA operator with weights $w_j^n = \frac{2j-1}{n^2}$ and thus the corresponding triangle would be:

$$\begin{array}{ccccc} & & 1 & & \\ & 1/4 & & 3/4 & \\ & 1/9 & 3/9 & 5/9 & \\ 1/16 & 3/16 & & 5/16 & 7/16 \\ 1/25 & 3/25 & 5/25 & 7/25 & 9/25 \\ & & \cdots & & \end{array}$$

Proposition 3 Let $Q : [0,1] \to [0,1]$ be a quantifier. Then the EOWA operator A_Q is an EAF.

Proof. It is sufficient to prove that the weighting triangle generated by Q is regular, that is,

$$\sum_{j=1}^{p} w_j^{n+1} \leq \sum_{j=1}^{p} w_j^n \leq \sum_{j=1}^{p+1} w_j^{n+1} \quad \forall\, p = 1, \ldots, n.$$

But this is equivalent to prove

$$Q(\frac{p}{n+1}) \leq Q(\frac{p}{n}) \leq Q(\frac{p+1}{n+1}) \quad \forall\, p = 1, \ldots, n$$

and this is true since Q is increasing and $\dfrac{p}{n+1} leq \dfrac{p}{n} \leq \dfrac{p+1}{n+1}$.

Proposition 4 Let $A : \bigcup\limits_{n\geq 1}[0,1]^n \to 0,1]$ be an EOWA operator with weighting triangle $\wedge w_j^n$. Let us suppose that A is an EAF. Then there exists a quantifier Q such that $A = A_Q$ if and only if the following condition holds:

$$k \cdot m = n \cdot r \Longrightarrow \sum_{j=1}^{k} w_j^n = \sum_{j=1}^{r} w_j^m. \tag{2}$$

Proof. Let us suppose firstly that the condition (2) holds. Let us define the mapping Q over the rationals of $[0,1]$ as follows:

$$Q(0) = 0, \quad Q(\frac{k}{n}) = \sum_{j=1}^{k} w_j^n \quad \forall n \geq 1 \quad \text{and} \quad \forall k = 1, \ldots, n$$

Now if $\frac{k}{n} = \frac{r}{m},\ n \neq m,\ 0 \leq k \leq n,\ 0 \leq r \leq m,$ then

$$Q(\frac{k}{n}) = \sum_{j=1}^{k} w_j^n = \sum_{j=1}^{r} w_j^m = Q(\frac{r}{m}).$$

Thus Q is well-defined over $\mathbb{Q} \cap [0,1]$ and, consequently, it can be extended to a mapping Q defined all over the interval $[0,1]$. Observe that $A = A_Q$ since

$$Q(\frac{j}{n}) - Q(\frac{j-1}{n}) = \sum_{i=1}^{j} w_i^n - \sum_{i=1}^{j-1} w_i^n = w_j^n.$$

Reciprocally, let us now suppose that $A = A_Q$ and let us prove the condition (2). According to the previous remark, we have:

$$\sum_{j=1}^{k} w_j^n = Q(\frac{k}{n}) \text{ and } \sum_{j=1}^{r} w_j^m = Q(\frac{r}{m}).$$

If $k \cdot m = n \cdot r$, then $\frac{k}{n} = \frac{r}{m}$ and condition (2) holds because Q is a mapping.

Remark 3 In the first part of the proof, we have seen that Q is defined over $\mathbb{Q} \cap [0,1]$. Let us see that if $A_Q = A_{Q'}$, then $Q = Q'$ over $\mathbb{Q} \cap [0,1]$.

Observe that: $Q(\frac{j}{n}) - Q(\frac{j-1}{n}) = w_j^n = Q'(\frac{j}{n}) - Q'(\frac{j-1}{n})$.
Now if $j = 1$, then $Q(0) = Q'(0) = 1$, and $Q(\frac{1}{n}) = Q'(\frac{1}{n})$, and so on up to $Q(1) = Q'(1) = 1$.

Remark 4 When Q is uniformly continuous over $\mathbb{Q} \cap [0,1]$, there exists a unique extension of Q to $[0,1]$ that is also uniformly continuous.

Proposition 5 Let Q be a quantifier and $\wedge w_j^n$ the weighting triangle generated by Q. Then $\wedge w_j^n$ is symmetrical if and only if Q is symmetrical with respect to $(1/2, 1/2)$ over $\mathbb{Q} \cap [0,1]$ (that is, $Q(1-x) = 1 - Q(x)$ for all $x \in \mathbb{Q} \cap [0,1]$).

Dem. Let us suppose firstly that the given weighting triangle is symmetrical, that is:

$$w_j^n = w_{n-j+1}^n \quad \forall n \geq 2 \quad \text{and} \quad \forall j = 1, \ldots, n.$$

Let $n \geq 2$. We know that $w_j^n = Q(\frac{j}{n}) - Q(\frac{j-1}{n})$. Then

$$Q(\frac{j}{n}) - Q(\frac{j-1}{n}) = Q(\frac{n-j+1}{n}) - Q(\frac{n-j}{n}),$$

and thus

$$Q(\frac{j}{n}) + Q(1 - \frac{j}{n}) = Q(\frac{j-1}{n}) + Q(1 - \frac{j-1}{n}) \quad \forall j = 1, \ldots, n.$$

Now if $j = n$, then:

$$Q(\frac{j}{n}) + Q(1 - \frac{j}{n}) = Q(1) + Q(0) = 1 \quad \forall j = 1, \ldots, n,$$

and thus

$$Q(1 - \frac{j}{n}) = 1 - Q(\frac{j}{n}) \quad \forall j = 1, \ldots, n,$$

that is, Q is symmetrical with respect to $(1/2, 1/2)$ over $\mathbb{Q} \cap [0,1]$.

Reciprocally, let us now suppose that Q is symmetrical with respect to $(1/2, 1/2)$ over $\mathbb{Q} \cap [0,1]$, that is,

$$Q(1-x) = 1 - Q(x) \quad \forall x \in \mathbb{Q} \cap [0,1].$$

Let $n \geq 2$. Then

$$\forall j = 1, \dots, n, \quad Q(1 - \frac{j}{n}) = 1 - Q(\frac{j}{n}).$$

In particular

$$1 = Q(\frac{j}{n}) + Q(1 - \frac{j}{n}) = Q(\frac{j-1}{n}) + Q(1 - \frac{j-1}{n})$$

and thus

$$Q(\frac{j}{n}) - Q(\frac{j-1}{n}) = Q(\frac{n-j+1}{n}) - Q(\frac{n-j}{n}).$$

Then $w_j^n = w_{n-j+1}^n \quad \forall j = 1, \dots, n$, and the triangle is symmetrical.

4 Generation through sequences

Let us consider a sequence of real numbers $\lambda_1, \lambda_2, \ldots \geq 0$ such that $\lambda_1 > 0$. We can generate a weighting triangle from this sequence as follows:

$$\forall n \geq 1, \quad w_j^n = \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_n}$$

Observe that $\sum_{j=1}^{n} w_j^n = \frac{\sum_{j=1}^{n} \lambda_j}{\lambda_1 + \cdots + \lambda_n} = 1$, and thus we really obtain a weighting triangle.

Proposition 6 A triangle generated by a sequence $\{\lambda_n\}$ is regular if and only if the sequence $\left\{\frac{S_{n+1}}{S_n}\right\}$ is decreasing, where $S_n = \lambda_1 + \cdots + \lambda_n$.

Proof. Let us suppose firstly that the triangle is regular. Then

$$\sum_{j=1}^{p} w_j^{n+1} \leq \sum_{j=1}^{p} w_j^n \leq \sum_{j=1}^{p+1} w_j^{n+1} \quad \forall p = 1, \dots, n.$$

In our case, $w_j^n = \frac{\lambda_j}{S_n}$, and the previous condition will be

$$\sum_{j=1}^{p} \frac{\lambda_j}{S_{n+1}} \leq \sum_{j=1}^{p} \frac{\lambda_j}{S_n} \leq \sum_{j=1}^{p+1} \frac{\lambda_j}{S_{n+1}} \quad \forall p = 1, \dots, n.$$

If we take $p = n - 1$ in the second inequality, then

$$\frac{S_{n-1}}{S_n} \leq \frac{S_n}{S_{n+1}},$$

which shows that the sequence $\left\{\frac{S_{n+1}}{S_n}\right\}$ is decreasing.
Reciprocally, let us now suppose that $\left\{\frac{S_{n+1}}{S_n}\right\}$ is decreasing and we want to prove that

$$\sum_{j=1}^{p} \frac{\lambda_j}{S_{n+1}} \leq \sum_{j=1}^{p} \frac{\lambda_j}{S_n} \leq \sum_{j=1}^{p+1} \frac{\lambda_j}{S_{n+1}} \quad \forall p = 1, \ldots, n,$$

or equivalently

$$\frac{S_p}{S_{n+1}} \leq \frac{S_p}{S_n} \leq \frac{S_{p+1}}{S_{n+1}}.$$

The first inequality holds because S_n is increasing, and the second one because $\left\{\frac{S_{n+1}}{S_n}\right\}$ is decreasing.

Remark 5 Observe that if a weighting triangle is regular, then the corresponding EOWA operator is an EAF.

Example 5 The triangles generated by the following sequences are regular.

a) decreasing sequences,

b) arithmetic progressions,

c) geometric progressions.

Example 6 The Fibonacci sequence: $1, 1, 2, 3, 5, 8, \ldots$ generates the triangle

$$\begin{array}{c} 1 \\ 1/2 \quad 1/2 \\ 1/4 \quad 1/4 \quad 2/4 \\ 1/7 \quad 1/7 \quad 2/7 \quad 3/7 \\ 1/12 \quad 1/12 \quad 2/12 \quad 3/12 \quad 5/12 \\ \ldots \end{array}$$

It is easily proved by induction that this sequence satisfies:

a) $S_n = \lambda_{n+2} - 1$,
b) $(\lambda_{n+3} - 1)(\lambda_{n+1} - 1) \leq (\lambda_{n+2} - 1)^2$

and this last inequality implies that $\left\{\frac{S_{n+1}}{S_n}\right\}$ is decreasing. According to the proposition (6), this fact proves that the weighting triangle generated by the Fibonacci sequence is regular.

We have seen in this paper that we can obtain EAF from quantifiers and also from sequences. It follows from the proposition (4) that there exist EAF generated by quantifiers as well as by sequences. Let us see examples that illustrate these facts.

Example 7 Arithmetic mean.
Let us consider $Q(x) = x$. An easy calculation shows that $w_j^n = 1/n \ \forall n \geq 1, \ \forall j = 1, \ldots, n$.
Observe that if we take the sequence $\lambda_n = 1 \ \ \forall n \geq 1$, we obtain the same EAF.

Example 8 Median.
According to the example (3), this EAF is obtained from a quantifier. On the other hand, it can not be generated by a sequence since $w_1^3 = 0$.

Example 9 Let us consider the sequence $\lambda_n = 2^{n-1} \ \ \forall n \geq 1$. It is easily proved that the weighting triangle generated by this sequence can not be generated by a quantifier.

Acknowledgements
The first three authors have been partially financed by CICYT, project: TIC96-1393-C06-06. The fourth one has been partially supported by the DGES grant PB96-0191-C02-01.

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