

On Boolean Modus Ponens

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Abstract

An abstract form of modus ponens in a Boolean algebra was suggested in [1]. In this paper we use the general theory of Boolean equations (see e.g. [2]) to obtain a further generalization. For a similar research on Boolean deduction theorems see [3].

The starting point of the paper [1] by E.Trillas and S.Cubillo is the inequality $x \bullet (x \rightarrow y) \leq y$ (where $x \rightarrow y = x' \vee y$), valid in an arbitrary Boolean algebra $(B, \vee, \bullet, ', 0, 1)$ and viewed as a Boolean analogue of modus ponens. The authors determine other Boolean variants of modus ponens by replacing conjunction (\bullet) and implication (\rightarrow) by other truth functions f and g , respectively. More exactly, they determine those functions $f, g : B^2 \rightarrow B$ that can be written in the form

$$(1) f(x, y) = axy \vee bxy' \vee cx'y \vee dx'y',$$

$$(2) g(x, y) = pxy \vee qxy' \vee rx'y \vee sx'y',$$

where $a, \dots, d, p, \dots, s \in \{0, 1\} \subseteq B$ and which satisfy the identity

$$(3) f(x, g(x, y)) \leq y$$

and $f(1, 1) = 1$. A similar problem is solved for the identity

$$(4) f(x, g(x, y)) = xy.$$

The aim of this Note is to take a further step by allowing the coefficients a, \dots, s to be any constants from B (not only 0 and 1). It will be seen that by applying the general theory of Boolean equations (see e.g. [2]), the functional equations (3) and (4) can be solved easily in a compact way.

We insist on the distinction introduced in [2] between *Boolean functions* and *simple Boolean functions*. In the case of two variables this amounts to the fact that functions of the form (1) where $a, b, c, d \in B$ (or (2) where $p, q, r, s \in B$) are Boolean functions, while if $a, b, c, d \in \{0, 1\}$ (or $p, q, r, s \in \{0, 1\}$) they are termed simple Boolean functions. In particular any *truth function* (also called a *switching function*), i.e. any function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, is a simple Boolean function.

In a Boolean algebra $B \neq \{0, 1\}$ there are Boolean functions that are not simple Boolean functions and there are functions that are not Boolean.

So we are going to solve each of the functional equations (3) and (4) with respect to the unknown Boolean functions (in the above general sense) f and g . Then solutions satisfying $f(1, 1) = 1$ will be obtained by introducing this condition into the solutions previously obtained. Finally the results of [1] will be recaptured by letting the parameters which occur in the solutions run over $\{0, 1\}$.

For a similar research on Boolean deduction theorems see [3].

Lemma. *The identities (1) and (2) imply*

$$f(x, g(x, y)) = (ap \vee bp')xy \vee (aq \vee bq')xy' \vee (cr \vee dr')x'y \vee (cs \vee ds')x'y'.$$

Proof. A well known property (see e.g. Theorem 1.5(1.53) in [2], where ' should read)') yields the intermediate step

$$\begin{aligned} f(x, g(x, y)) &= (ax \vee cx')(pxy \vee qxy' \vee rx'y \vee sx'y') \vee \\ &\quad \vee (bx \vee dx')(p'xy \vee q'xy' \vee r'x'y \vee s'x'y'). \end{aligned}$$

Proposition 1. *The functions*

$$(1) f(x, y) = axy \vee bxy' \vee cx'y \vee dx'y',$$

$$(2) g(x, y) = pxy \vee qxy' \vee rx'y \vee sx'y',$$

satisfy the identity

$$(3) f(x, g(x, y)) \leq y$$

if and only if the following relations hold:

$$(5.1) ab = cd = 0,$$

$$(5.2) b \leq q \leq a',$$

$$(5.3) d \leq s \leq c'.$$

Proof. In view of the Lemma, identity (3) is equivalent to the system of equalities

$$(6.1) aq \vee bq' = 0,$$

$$(6.2) cs \vee ds' = 0.$$

As is well known (see e.g. Theorem 2.2 and Lemma 2.1 in [2]), equation (6.1) is consistent with respect to q if and only if $ab = 0$, in which case its solutions are given by (5.2). Similarly, equation (6.2) is equivalent to $cd = 0$ and (5.3).

Remark 1. Conditions (5) can be expressed in the form

$$b = a'a_1, \quad d = c'd_1, \quad q = a'(a_1 \vee q_1), \quad s = c'(d_1 \vee s_1),$$

where a, c, a_1, d_1, q_1, s_1 are arbitrary parameters in B , because (i) $xy = 0$ iff $x \leq y'$ and (ii) if $x \leq y'$ then $x \leq z \leq y$ iff $z = x \vee t$ for some $t \leq y$.

Proposition 2. The Boolean functions f and g satisfy the identity

$$(3) \quad f(x, g(x, y)) \leq y$$

and $f(1, 1) = 1$ if and only if they are of the form

$$(8) \quad f(x, y) = xy \vee cx'y \vee dx'y',$$

$$(9) \quad g(x, y) = pxy \vee rx'y \vee sx'y',$$

where

$$(10) \quad cd = 0 \text{ and } d \leq s \leq c'.$$

Proof. It follows from (1) that $f(1, 1) = a$. Taking $a = 1$ in (5) yields $b = q = 0$, so that (1), (2) and (5) reduce to (8), (9) and (10), respectively.

Remark 2. Conditions (10) can be expressed in the form

$$d = c'd_1, \quad s = c'(d_1 \vee s_1),$$

where c, d_1, s_1 are arbitrary parameters in B .

Proposition 3. There are 16 pairs of simple Boolean functions (f, g) satisfying

$$(3) \quad x, g(x, y)) \leq y$$

and $f(1, 1) = 1$, namely

$$(12.1) \quad f(x, y) = xy \vee cx'y = (x \vee c)y,$$

$$(12.2) \quad g(x, y) = pxy \vee rx'y,$$

and

$$(13.1) \quad f(x, y) = xy \vee dx'y',$$

$$(13.2) \quad g(x, y) = pxy \vee rx'y \vee x'y',$$

where the coefficients are arbitrary in $\{0, 1\}$.

Proof. Taking $s = 0$ in Proposition 2 we obtain $d = 0$; these values satisfy (10) and reduce (8) and (9) to (12.1) and (12.2), respectively. Similarly, for $s = 1$ we obtain $c = 0$ and the solutions (13.1), (13.2).

Remark 3. *The 16 solutions determined in Proposition 3 are the 14 solutions found in [1] plus the solutions $(xy, 0)$ and $(y, 0)$, which don't seem to have a modus ponens interpretation.*

Proposition 4. *The Boolean functions f and g satisfy the identity*

$$(4) f(x, g(x, y)) = xy$$

if and only if they are of the form

$$(14) f(x, y) = axy \vee a'xy' \vee cx'y \vee dx'y',$$

$$(15) g(x, y) = axy \vee a'xy' \vee rx'y \vee sx'y',$$

where

$$(16.1) cd = 0,$$

$$(16.2) d \leq r \leq c',$$

$$(16.3) d \leq s \leq c'.$$

Proof. In view of the Lemma, the identity (4) is equivalent to the system of equalities :

$$(17.1) ap \vee bp' = 1,$$

$$(17.2) aq \vee bq' = 0,$$

$$(17.3) cr \vee dr' = 0,$$

$$(17.4) cs \vee ds' = 0.$$

We apply again Theorem 2.2 and Lemma 2.1 in [2], as well as their duals. So the consistency conditions of the four equations with respect to p, q, r and s , respectively, are $a \vee b = 1$, $ab = 0$ and (16.1). The first two conditions are equivalent to $b = a'$, so that equations (17.1) and (17.2) have the unique solutions $p = a$ and $q = a'$, respectively. This reduces expressions (1) and (2) to (14) and (15), respectively, while the solutions of equations (17.3) and (17.4) are (16.2) and (16.3), respectively.

Proposition 5. *The Boolean functions f and g satisfy the identity*

$$(4) f(x, g(x, y)) = xy$$

and $f(1, 1) = 1$ if and only if they are of the form

$$(18) f(x, y) = xy \vee cx'y \vee dx'y',$$

$$(19) g(x, y) = xy \vee rx'y \vee sx'y',$$

where the coefficients satisfy (16).

Proof. Take $a = 1$ in Proposition 4.

Remark 4. Conditions (16) can be expressed in the form

$$(20) \quad d = c'd_1, \quad r = c'(d_1 \vee r_1), \quad s = c'(d_1 \vee s_1),$$

where c, d_1, s_1 are arbitrary parameters in B .

Proposition 6. There are 6 pairs of simple Boolean functions (f, g) satisfying

$$(4) \quad f(x, g(x, y)) = xy$$

and $f(1, 1) = 1$, namely

$$(21) \quad f(x, y) = y, g(x, y) = xy,$$

$$(22) \quad f(x, y) = xy \vee x'y', g(x, y) = x' \vee y,$$

and

$$(23) \quad f(x, y) = xy, g(x, y) = xy \vee rx'y \vee sx'y',$$

where r, s are arbitrary in $\{0, 1\}$.

Proof. By taking $c = 1$ in Proposition 5, conditions (16) reduce to $d = r = s = 0$, so that (18) and (19) reduce to (21). For $c = 0$ conditions (16) reduce to $d \leq r$ and $d \leq s$, whose 0 – 1 solutions are $d = r = s = 1$ and $d = 0$, for which the functions (18) and (19) reduce to (22) and to (23), respectively.

We have thus recaptured the 6 solutions found in [1].

References

1. E.Trillas, S.Cubillo, Modus ponens on Boolean algebra revisited. *Mathware & Soft Computing* **3** (1996), 105-112.
2. S. Rudeanu, *Boolean Functions and Equations*. North-Holland, Amsterdam 1974.
3. S. Rudeanu, *On deduction theorems and functional Boolean equations*. An. Univ. Timișoara Ser.Ști.Mat. 28(1990), 197-210.