On Some Geometric Transformation of $t$-norms*

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Abstract

Given a triangular norm $T$, its $t$-reverse $T^*$, introduced by C. Kimberling (Publ. Math. Debrecen 20, 21-39, 1973) under the name invert, is studied. The question under which conditions we have $T^{**} = T$ is completely solved. The $t$-reverses of ordinal sums of $t$-norms are investigated and a complete description of continuous, self-reverse $t$-norms is given, leading to a new characterization of the continuous $t$-norms $T$ such that the function $G(x, y) = x + y - T(x, y)$ is a $t$-conorm, a problem originally studied by M.J. Frank (Acta Mathematica 19, 194-226, 1979). Finally, some open problems are formulated.

1 Introduction

Triangular norms ($t$-norms) and the corresponding $t$-conorms play a fundamental role in several branches of mathematics, e.g., in probabilistic metric spaces [6], in the theory of generalized measures and games [1] and in fuzzy logic [5]. In [3], the $t$-reverse $T^*$ of a $t$-norm $T$ was introduced (under the name invert). We somewhat extend and complete the study of $t$-reverses done there.

A triangular norm ($t$-norm for short) is a function $T : [0, 1]^2 \rightarrow [0, 1]$ which is commutative, associative, non-decreasing in both components, and satisfies the boundary condition $T(x, 1) = x$ for each $x \in [0, 1]$. Given a $t$-norm $T$, its dual $t$-conorm $S_T$ is defined by

$$S_T(x, y) = 1 - T(1 - x, 1 - y).$$

The most important $t$-norms, together with their dual $t$-conorms are

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\[ T_M(x, y) = \text{min}(x, y), \quad S_M(x, y) = \text{max}(x, y); \]

\[ T_P(x, y) = xy, \quad S_P(x, y) = x + y - xy; \]

\[ T_L(x, y) = \text{max}(0, x + y - 1), \quad S_L(x, y) = \text{min}(1, x + y); \]

\[ T_W(x, y) = \begin{cases} 
\text{min}(x, y) & \text{if } \text{max}(x, y) = 1, \\
0 & \text{otherwise}, 
\end{cases} \quad S_W(x, y) = \begin{cases} 
\text{max}(x, y) & \text{if } \text{min}(x, y) = 0, \\
1 & \text{otherwise}. 
\end{cases} \]

It is obvious that these \( t \)-norms satisfy the inequality \( T_W \leq T_L \leq T_P \leq T_M \). Moreover, for each \( t \)-norm \( T \) we have \( T_W \leq T \leq T_M \). A continuous \( t \)-norm is called Archimedean if for each \( x \in \mathbb{R} \) we have \( T(x, x) < x \).

An interesting family of \( t \)-norms \( \{ T^F_s \}_{s \in [0, +\infty]} \) was studied in [2]:

\[ T^F_s(x, y) = \begin{cases} 
T_M(x, y) & \text{if } s = 0, \\
T_P(x, y) & \text{if } s = 1, \\
T_L(x, y) & \text{if } s = \infty, \\
\log_s \left[ 1 + \frac{(s-1)(x^s-1)}{s^s-x^s} \right] & \text{otherwise}. 
\end{cases} \]

These \( t \)-norms will be referred to as the Frank \( t \)-norms, the family of the dual Frank \( t \)-conorms will be denoted \( \{ S^F_s \}_{s \in [0, +\infty]} \). The family \( \{ T^F_s \}_{s \in [0, +\infty]} \) of Frank \( t \)-norms is decreasing (see [1] and [4]) and continuous in the sense that we have

\[ (s_n)_{n \in \mathbb{N}} \uparrow t \Rightarrow (T^F_{s_n})_{n \in \mathbb{N}} \downarrow T^F_t. \]

2 Definition of the \( t \)-reverse

Let \( T \) be a \( t \)-norm. Then the function \( T^* : [0, 1]^2 \rightarrow [0, 1] \) defined by

\[ T^*(x, y) = \text{max}(0, x + y - 1 + T(1 - x, 1 - y)) \]  \hspace{1cm} (1)

is called the \( t \)-reverse of \( T \). This definition goes back to [3] where the name invert was used for \( T^* \).

Using the dual \( t \)-conorm \( S_T \) of \( T \), this definition can be rewritten as

\[ T^*(x, y) = \text{max}(0, x + y - S_T(x, y)). \]  \hspace{1cm} (2)

The construction of \( T^* \) can be conceived geometrically as follows (it is visualized in Figure 1):

(i) The graph of \( T \) is rotated \( 180^\circ \) around the vertical symmetry axis of the unit cube

(ii) The plane \( z = x + y - 1 \) is added to the rotated graph (this implies that the boundary conditions \( T^*(x, 1) = x \) and \( T^*(x, 0) = 0 \) are satisfied).

(iii) Any negative values are replaced by zero.
Figure 1: Visualization of the reversion: a t-norm (top left), rotating it around the vertical symmetry axis (top right), adding the plane $x + y - 1$ (bottom left), cutting off negative values (bottom right).
It is clear that $T^*$ satisfies the symmetry and boundary conditions required for $t$-norms. The monotonicity and associativity, however, may not hold for $T^*$:

**Example 2.1.** (i) $T^*_W = T_L$.

(ii) $T^*_L = T_L$.

(iii) If $T$ is the $t$-norm given by

$$T(x, y) = \begin{cases} \frac{xy}{x+y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

then $T^*$ is not associative, since, e.g., $T^*(T^*(0.2, 0.9), 0.9) \approx 0.1952$ and $T^*(0.2, T^*(0.9, 0.9)) \approx 0.1948$.

(iv) Let $T$ be the ordinal sum \{\{0, 0.5, T_W\}, \{0.5, 1, T_L\}\} (for the general definition of ordinal sums see Section 4). Then $T^*$ is not non-decreasing, since, e.g., $T^*(0.4, 0.6) = 0.4 > 0.2 = T^*(0.6, 0.6)$.

Examples 2.1 (iii) and (iv) both show that the $t$-reverse $T^*$ of a $t$-norm $T$ not necessarily is a $t$-norm. We shall say that a $t$-norm $T$ is $t$-reversible if its $t$-reverse $T^*$ is also a $t$-norm, and we shall denote the family of all $t$-reversible $t$-norms by $R$.

### 3 General properties

In [3] it was conjectured that a $t$-norm $T$ is $t$-reversible only if $T$ equals one of the basis $t$-norms $T_M, T_P, T_L, T_W$ or a specific ordinal sum (for the general definition of ordinal sums see again Section 4) thereof. However, this conjecture turns out to be incorrect, as a consequence of the following result.

**Theorem 3.1.** For all $t$-norms $T$ with $T \leq T_L$ we have $T^* = T_L$.

**Proof.** If $x + y \leq 1$ then $x + y = S_L(x, y) \leq S_T(x, y)$, where $S_T$ is the dual $t$-conorm of $T$, in which case we have $x + y - S_T(x, y) \leq 0$ and, therefore, $T^*(x, y) = 0$. If $x + y > 1$ then $1 = S_L(x, y) \leq S_T(x, y)$, implying $S_T(x, y) = 1$ and, consequently, $T^*(x, y) = x + y - 1$.

Theorem 13 in [3] claims that for a $t$-norm $T$ we always have $T^{**} = T$. This is not true since $T$ may not be $t$-reversible, in which case $T^{**} = (T^*)^*$ is not properly defined. Even if $T$ is $t$-reversible, this claim is wrong: from Example 2.1 (i) and (ii) we have $T^*_W = T_L$ and $T^*_L = T_L$, showing that $T^*_W \neq T_W$. However, we get the following result:

**Theorem 3.2.** Let $T$ be a $t$-reversible $t$-norm. Then $T^{**} = T$ if and only if $T \geq T_L$. 
Proof. By definition we have

\[ T^*(x, y) = \max[0, x + y - S_T(x, y)], \]

where \( S_T^* \) is the dual of the \( t \)-norm \( T^* \), for which we get

\[
S_T^*(x, y) = 1 - T^*(1 - x, 1 - y) = 1 - \max[0, 1 - x + 1 - y - S_T(1 - x, 1 - y)] = 1 - \max[0, T(x, y) + 1 - x - y] = \min[1, x + y - T(x, y)].
\]

This implies

\[
T^{**}(x, y) = \max[0, x + y - \min(1, x + y - T(x, y))] = \max[0, \max(x + y - 1, T(x, y))] = \max(T_L(x, y), T(x, y)).
\]

Now it is clear that \( T^{**} = T \) if and only if \( T \geq T_L \).

Corollary 3.3. Suppose that both \( T \) and \( T^* \) are \( t \)-reversible \( t \)-norms. Then we have \( T^{***} = T^* \).

Proof. This is obvious since we always have

\[ T^*(x, y) = \max(0, x + y - S_T(x, y)) \geq \max(0, x + y - 1) = T_L(x, y). \]

\[ \blacksquare \]

Theorem 3.4. Let \( T \) be a continuous Archimedean, \( t \)-reversible \( t \)-norm. Then \( T^* \) is a continuous Archimedean \( t \)-norm.

Proof. Continuity follows from the definition. That \( T^* \) is Archimedean is a consequence of the fact that for all \( x \in [0, 1] \)

\[ T^*(x, x) = \max(0, x + x - S_T(x, x)) < x, \]

since the dual \( t \)-conorm \( S_T \) of \( T \) satisfies \( S_T(x, x) > x \) for all \( x \in [0, 1] \).

\[ \blacksquare \]

4 \( t \)-reverses of ordinals sums

An important way to construct new \( t \)-norms from given ones is that of an ordinal sum: let \( \{[\alpha_k, \beta_k]\}_{k \in K} \) be a non-empty countable family of pairwise disjoint open subintervals of \([0,1]\) and let \( \{T_k\}_{k \in K} \) be a family of corresponding \( t \)-norms. Then the ordinal sum \( \{[\alpha_k, \beta_k, T_k]\}_{k \in K} \) is the function \( T : [0, 1]^2 \to [0, 1] \) defined by

\[
T(x, y) = \begin{cases} 
\alpha_k + (\beta_k - \alpha_k) \cdot T_k \left( \frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k} \right) & \text{if } x, y \in [\alpha_k, \beta_k], \\
\min(x, y) & \text{otherwise,}
\end{cases}
\]
which is always a $t$-norm. In order to keep the notation short, we also consider here the trivial ordinal sum $T = \{0, 1, T_1\}$, i.e., where $K = \{1\}$ is a one point set and $\alpha_1 = 0$ and $\beta_1 = 1$, in which case we have $T = T_1$.

Ordinal sums of $t$-conorms are defined in the same way as ordinal sums of $t$-norms, only replacing min by max. Observe, however, that the dual $t$-conorm of an ordinal sum $\{(\alpha_k, \beta_k, T_k)\}_{k \in K}$ of $t$-norms is the ordinal sum $\{(1 - \beta_k, 1 - \alpha_k, S_{T_k})\}_{k \in K}$ of $t$-conorms which, in general, is different from the ordinal sum $\{(\alpha_k, \beta_k, S_{T_k})\}_{k \in K}$.

Each continuous $t$-norm can be written as an ordinal sum $\{(\alpha_k, \beta_k, T_k)\}_{k \in K}$ such that all $T_k$ are continuous Archimedean $t$-norms.

Denote by $\mathcal{F}$ the family of $t$-norms $T$ such that the function $G : [0, 1]^2 \rightarrow [0, 1]$ given by

$$
G(x, y) = x + y - T(x, y)
$$

is associative, i.e., a $t$-conorm.

Each element of $\mathcal{F}$ can be written as an ordinal sum $\{(\alpha_k, \beta_k, T_k)\}_{k \in K}$ such that all $T_k$ are Frank $t$-norms (see [2]). For more details about ordinal sums, see, e.g., [6].

In [3] the class of all $t$-norms satisfying the condition

$$
x \leq u \text{ and } y \leq v \Rightarrow u + v - T(u, v) \geq x + y - T(x, y)
$$

was denoted by $\mathcal{M}$ (in the language of [3], these $t$-norms are said to be of moderate growth). In [3, Theorem 12] it is shown that, given $T \in \mathcal{M}$, then $T^*$ is necessarily non-decreasing in each component, so only the associativity of the $t$-reverse can be a problem. Finally, Theorem 16 in [3] proves that if $T \in \mathcal{M}$ is an ordinal sum of $t$-reversible $t$-norms, i.e., $T = \{(\alpha_k, \beta_k, T_k)\}_{k \in K}$, with $T_k$ reversible, then $T$ itself is $t$-reversible, and $T^*$ equals the ordinal sum $\{(1 - \beta_k, 1 - \alpha_k, T_k')\}_{k \in K}$.

An interesting question is now the relation between the three families $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{F}$, i.e., of the families of $t$-norms which are $t$-reversible, of moderate growth, and which are solutions of the problem of Frank [2], respectively. Here are some simple observations concerning this problem.

**Example 4.1.** (i) The monotonicity of $t$-conorms implies that all elements of $\mathcal{F}$ belong to $\mathcal{M}$, i.e., $\mathcal{F}$ is a subfamily of $\mathcal{M}$.

(ii) Conversely, an element of $\mathcal{M}$ need not be an element of $\mathcal{F}$: the $t$-norm $T$ mentioned in Example 2.1 (iii) is an example for this, showing that $\mathcal{F}$ is a proper subfamily of $\mathcal{M}$.

(iii) Not each $t$-reversible $t$-norm belongs to $\mathcal{M}$: $T_W$ is an example for this. Hence, $\mathcal{R}$ is not a subfamily of $\mathcal{M}$.

The exact relationship between the three families $\mathcal{R}$, $\mathcal{M}$ and $\mathcal{F}$ is given as follows.

**Theorem 4.2.** A $t$-norm $T$ is both $t$-reversible and an element of $\mathcal{M}$ if and only if $T$ is an element of $\mathcal{F}$ (this means that we have $\mathcal{F} = \mathcal{R} \cap \mathcal{M}$).
Proof. Assume first that $T = \{ (\alpha_k, \beta_k, T_{S_k}) \}_{k \in K}$ is an element of $\mathcal{F}$ and, consequently, of $\mathcal{M}$. Let $S_T$ be the dual $t$-conorm of $T$, i.e., $S_T$ is the ordinal sum \( \{ (1 - \beta_k, 1 - \alpha_k, \textsc{S}_{\text{st}_k}) \}_{k \in K} \). Then from [2] we know that the expression
\[
x + y - S_T(x, y)
\]
is always nonnegative and defines a $t$-norm. Taking into account
\[
T^*(x, y) = \max(0, x + y - S_T(x, y)) = x + y - S_T(x, y),
\]
it is clear that $T$ is $t$-reversible.

If, conversely, $T \in \mathcal{R} \cap \mathcal{M}$, observe first that (4) implies the inequality
\[
1 = 1 + 1 - T(1, 1) \geq 1 - x + 1 - y - T(1 - x, 1 - y),
\]
from which we get
\[
0 \leq x + y - 1 + T(1 - x, 1 - y) = x + y - S_T(x, y).
\]
Now, using $T \in \mathcal{R}$ and (2), we get
\[
T^*(x, y) = x + y - S_T(x, y)
\]
or, equivalently,
\[
S_T(x, y) = x + y - T^*(x, y),
\]
which, as a consequence of the results in [2], means that $S_T$ can be written as an ordinal sum \( \{ (\alpha_k, \beta_k, \textsc{S}_{\text{st}_k}) \}_{k \in K} \), implying that we have $T = \{ (1 - \beta_k, 1 - \alpha_k, T_{\text{st}_k}) \}_{k \in K}$, i.e., $T \in \mathcal{F}$.

Remark 4.3. (i) Note that from the proof of Theorem 4.2 we can conclude that for $T \in \mathcal{F}$ we have
\[
T^*(x, y) = 1 - S(1 - x, 1 - y),
\]
where $S$ is the $t$-conorm defined by $S(x, y) = x + y - T(x, y)$.

(ii) Let $T$ be an ordinal sum of Frank $t$-norms, i.e., $T = \{ (\alpha_k, \beta_k, T_{\text{st}_k}) \}_{k \in K}$. Using the fact that for each pair $(T_{\text{st}}^{\text{F}}, S_{\text{st}}^{\text{F}})$ we have
\[
T_{\text{st}}^{\text{F}}(x, y) + S_{\text{st}}^{\text{F}}(x, y) = x + y
\]
(see again [2]), we see that $T^*$ equals the ordinal sum \( \{ (1 - \beta_k, 1 - \alpha_k, T_{\text{st}_k}^{\text{F}}) \}_{k \in K} \), the dual $t$-conorm $S_T^*$ of which is just given by $S_T^*(x, y) = x + y - T(x, y)$.

(iii) This means that all Frank $t$-norms are self-reverse, i.e., we have $(T_{\text{st}}^{\text{F}})^* = T_{\text{st}}^{\text{F}}$ for all $s \in [0, +\infty]$ (for a more detailed discussion see Section 5).

Example 2.1 (iv) and Theorem 3.1 show that ordinal sums of $t$-reversible $t$-norms, in general, need not be $t$-reversible (this fact is visualized in Figure 2). The following proves that a $t$-reversible ordinal sum can have at most one summand which is smaller than $T_L$. 
Figure 2: Ordinal sum \( \{0.3, 0.9, T\} \) with \( T(x, y) = 1 - \min[1 - (\sqrt{1 - x} + \sqrt{1 - y})^2] \), i.e., \( T < T_L \) (top left) whose \( t \)-reverse (top right) is not monotone and, therefore, not a \( t \)-norm. The \( t \)-reverse (bottom right) of the ordinal sum \( \{0.4, 1, T\} \) (bottom left), however, is a \( t \)-norm, namely, the ordinal sum \( \{0, 0.6, T_L\} \).

**Theorem 4.4.** Let \( T \) be the ordinal sum \( \{\alpha_k, \beta_k, T_k\} \) \( \forall k \in K \) such that \( T \) is \( t \)-reversible and \( T_{k_0} < T_L \) for some \( k_0 \in K \). Then we have \( \beta_{k_0} = 1 \) (as a consequence, there is at most one summand \( T_k \) with \( T_k < T_L \)).
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Proof. Let \( (x, y) \in [0, 1]^2 \) be a point such that \( T_{h_0}(x, y) < T_L(x, y) \), i.e.,
\[
x + y - 1 - T_{h_0}(x, y) > 0.
\]
(5)
Assume that \( \beta_{h_0} < 1 \) is true. Then, on the one hand, we have
\[
T^* (1 - \beta_{h_0}, 1 - \beta_{h_0}) = 1 - \beta_{h_0}.
\]
(6)
On the other hand, observe that
\[
1 - \alpha_{h_0} + (\alpha_{h_0} - \beta_{h_0}) \cdot x > 1 - \beta_{h_0},
\]
(7)
\[
1 - \alpha_{h_0} + (\alpha_{h_0} - \beta_{h_0}) \cdot y > 1 - \beta_{h_0},
\]
(8)
implying that
\[
T^* (1 - \alpha_{h_0} + (\alpha_{h_0} - \beta_{h_0}) \cdot x, 1 - \alpha_{h_0} + (\alpha_{h_0} - \beta_{h_0}) \cdot y)
= \max \left( 0, 1 - \alpha_{h_0} + (\alpha_{h_0} - \beta_{h_0}) \cdot x + 1 - \alpha_{h_0} + (\alpha_{h_0} - \beta_{h_0}) \cdot y - 1 + \alpha_{h_0} \cdot (\alpha_{h_0} - \beta_{h_0}) \cdot T_{h_0}(x, y) \right)
= \max \left( 0, 1 - \alpha_{h_0} + (\beta_{h_0} - \alpha_{h_0}) \cdot (T_{h_0}(x, y) - x - y) \right)
= \max \left( 0, 1 - \beta_{h_0} + (\beta_{h_0} - \alpha_{h_0}) \cdot (x + y - 1 - T_{h_0}(x, y)) \right)
< 1 - \beta_{h_0},
\]
where the inequality follows from (5). This, together with (6), (7) and (8), violates the monotonicity of the t-norm \( T^* \), and therefore \( \beta_{h_0} < 1 \) cannot be true. ■

Conversely, it is not difficult to see that the each ordinal sum of some special form is \( t \)-reversible allowing us to formulate the following result:

Corollary 4.5. Let the t-norm \( T \) be the ordinal sum \( \{ (\alpha_k, \beta_k, T_k) \}_{k \in K} \) of Frank t-norms up to possibly one summand, say \( T_{h_0} \), with \( T_{h_0} < T_L \) and \( \beta_{h_0} = 1 \). Then \( T \) is \( t \)-reversible and its t-reverse \( T^* \) equals the t-reverse of \( T \), where \( T \) is the ordinal sum \( \{ (\alpha_k, \beta_k, T_k) \}_{k \in K} \) with \( \tilde{T}_k = T_k \) for all \( k \neq h_0 \) and \( \tilde{T}_{h_0} = T_L \).

5 Self-reverse t-norms

We are now interested in studying t-norms which are self-reverse, i.e., satisfy the equality \( T^* = T \). From Remark 4.3(iii) we know that all Frank t-norms \( T_s^F \), \( s \in [0, +\infty) \) have this property. We are now able to characterize all continuous self-reverse t-norms.

Theorem 5.1. Let \( T \) be a continuous t-norm. Then \( T^* = T \) if and only if \( T \) is an ordinal sum \( \{ (\alpha_k, \beta_k, T_k^F) \}_{k \in K} \) of Frank t-norms such that for each \( k \in K \) with \( T_{s_k}^F \neq T_M \) there is a \( j \in K \) with \( s_j = s_k \), \( \alpha_j = 1 - \beta_k \) and \( \beta_j = 1 - \alpha_k \).

Proof. Assuming \( T^* = T \) then we have \( T^{**} = T \) and, by Theorem 3.2, \( T \geq T_L \).
Then for the dual t-conorm \( S_T \) of \( T \) we obtain
\[
S_T(x, y) \leq S_L(x, y) \leq x + y.
\]
implying
\[ x + y - S_T(x, y) \geq 0 \]
and, taking into account \( T^* = T \),
\[ T(x, y) = x + y - S_T(x, y). \]

Because of [2], this means that \( T \) must be an ordinal sum \( \{ (\alpha_k, \beta_k, T_{s_k}^F) \}_{k \in K} \) of Frank \( t \)-norms. From Remark 4.3(ii) we know that \( T \) has to be symmetric in the sense that for each \( k \in K \) with \( T_{s_k}^F \neq T_M \) (\( T_M \) acts like a neutral element when constructing ordinal sums and does not influence this symmetry) there exists a \( j \in K \) such that \( s_j = s_k, \alpha_j = 1 - \beta_k \) and \( \beta_j = 1 - \alpha_k \).

Recall that in the trivial case \( K = \{1\} \), \( \alpha_1 = 0 \) and \( \beta_1 = 1 \), i.e., if \( T \) itself is a Frank \( t \)-norm, the symmetry condition is always satisfied. In the light of this theorem we can give the following variation of the results of [2]:

**Corollary 5.2.** For a continuous \( t \)-norm \( T \) the function \( G : [0, 1]^2 \to [0, 1] \) given by \( G(x, y) = x + y - T(x, y) \) is a \( t \)-conorm if and only if \( T \) is an ordinal sum \( \{ (\alpha_k, \beta_k, T_{s_k}^F) \}_{k \in K} \) of Frank \( t \)-norms, in which case the \( t \)-conorm \( G \) is dual to the \( t \)-reverse \( T^* \), i.e.,
\[ G(x, y) = 1 - T^* (1 - x, 1 - y). \]

### 6 Concluding remarks

Some questions concerning \( t \)-reverses of \( t \)-norms remain still open. The most important open problem is the complete characterization of all \( t \)-reversible \( t \)-norms. Other related questions can be formulated as follows:

**Question 1.** Is a continuous \( t \)-norm \( T \) \( t \)-reversible if and only if \( T \) is an ordinal sum whose summands are Frank \( t \)-norms up to possibly one summand in the upper right corner of the unit square which is weaker than \( T_L \)?

**Question 2.** If \( T \) is a \( t \)-reversible \( t \)-norm, is \( T^* \) necessarily \( t \)-reversible?

**Question 3.** If \( T \) is a \( t \)-reversible \( t \)-norm, is \( T^* \) necessarily continuous?

**Question 4.** If \( T \) is a \( t \)-reversible \( t \)-norm, is \( T^* \) necessarily an ordinal sum of Frank \( t \)-norms?

We conjecture that there is an affirmative answer to each of these questions. However, we have not proven this claim so far (nor do we have counterexamples). Obviously, if there is a positive answer to Question 4, this would imply positive answers to both Questions 2 and 3.

References


