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The Embedding of the Formal Concept Analysis into the L-Fuzzy Concept Theory

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Abstract

In this work, we study the relation between the concept lattice of Wille ([5], [6]) and the $L-Fuzzy$ concept lattice ([2]) developed by us. To do it, we have defined an application g that associates to each concept of Wille an $L-Fuzzy$ concept. The main point of this work is to prove that if we are working with a crisp relation between an object set and an attribute set, the concept lattice of Wille is a sublattice of the $L-Fuzzy$ concept lattice.

At the end, we show a typical example in the formal concept theory where we have constructed the $L-Fuzzy$ concept lattice.
Key words: Concepts, $L-Fuzzy$ concept analysis, hierarchies of concepts, $L-Fuzzy$ sets, conceptual knowledge.

1 Introduction.

The first publication relating to the formal concept analysis was written by R. Wille ([5]) in 1982.

Wille defines a context as a tuple (G, M, I), where G is the object set, M is the attribute set and I is a binary relation between G and M.

For every $A \subseteq G$ and $B \subseteq M$, we can define the sets:

$$A^* = \{m \in M \,/\, gIm \,\forall g \in A\}$$

$$B^* = \{g \in G \,/\, gIm \,\forall m \in B\}$$

that represent the attribute set shared by all the objects of A and the object set shared by all the attributes of B.

Wille calls *([5]) derivation operator.

One of the most important definitions in the formal concept theory is the concept definition.

A concept of a context (G, M, I) is a pair (A, B) where $A \subseteq G$ and $B \subseteq M$ satisfying that $A^* = B$ and $B^* = A$.

A and B are the extension and the comprehension of the concept.

Wille ([5]) proves that the set of all the concepts of a context (G, M, I), wich he denotes $\mathcal{L}(G, M, I)$ or $\underline{\mathcal{L}}$, is a complete lattice with the order definition:

$$(A, B) \leq (C, D) \Longleftrightarrow A \leq C (\text{or equiv.} B \geq D).$$

$\mathcal{L}(G, M, I)$ is said to be concept lattice of the context (G, M, I), and $0_{\underline{\mathcal{L}}}$, $1_{\underline{\mathcal{L}}}$ denote the minimum and maximum elements of the lattice $\underline{\mathcal{L}}$.

Taking as departure point the formal concept analysis of Wille ([5]) and the $L - Fuzzy$ set theory, we have proved in [2] a new model from $L - Fuzzy$ relations between the objects and the attributes. We are going to look at the main aspects:

Let $(L, \leq, ', \top)$ be the algebraic system determined by a complete lattice $(L, \leq)$, a complementation $'$ in L and a t-conorm $\top$ in L([1]).

We defined an $\underline{L - Fuzzy \text{ context}}$ as a tuple $(L, X, Y, \underset{\smile}{R})$ where L is a complete lattice, X and Y are the object and attribute sets respectively, and $\underset{\smile}{R} \in L^{X \times Y}$ is an $L - Fuzzy$ relation.

In ([2]) we extended the derivation operator definitions gived by Wille([5]) to $L - Fuzzy$ contexts, and then we defined the derivation operators weighted by a complementation:

Given $\underset{\smile}{A} \in L^X$, we associate to it the $L - Fuzzy$ set $\underset{\smile}{A}_1$ of L^Y such that

$$\underset{\smile}{A}_1(y) = \inf_{x \in X} (\underset{\smile}{A}'(x) \top \underset{\smile}{R}(x, y))$$

In the same way, given $\underset{\smile}{B} \in L^Y$ we associate to it $\underset{\smile}{B}_2 \in L^X$ such that

$$\underset{\smile}{B}_2(x) = \inf_{y \in Y} (\underset{\smile}{B}'(y) \top \underset{\smile}{R}(x, y))$$

It is very easy to prove that these definitions recover, in the case of $L = \{0, 1\}$, the derivation operator concepts defined by Wille.

If we write $\underset{\smile}{A}_{12}$ and $\underset{\smile}{B}_{21}$ to represent the $L - Fuzzy$ sets $(\underset{\smile}{A}_1)_2$ and $(\underset{\smile}{B}_2)_1$ respectively, then we can define the operators φ and ψ :

$$\varphi : L^X \to L^X \,/\, \varphi(\underset{\smile}{A}) = \underset{\smile}{A}_{12}$$

$$\psi : L^Y \to L^Y \,/\, \psi(\underset{\smile}{B}) = \underset{\smile}{B}_{21}$$

which we call constructor operators.

These operators preserve the order and we used them to give the following definition:

If $\underset{\smile}{M} \in fix(\varphi)$, then the pair $(\underset{\smile}{M}, \underset{\smile}{M}_1)$ is said to be $\underline{L - fuzzy \text{ concept}}$ of the $L - Fuzzy$ context $(L, X, Y, \underset{\smile}{R})$.

We proved in [2], using the theorem of Tarski ([4]), that the $L-Fuzzy$ concept set $\underset{\sim}{\mathcal{L}}(L,X,Y,\underset{\sim}{R})$ with the order relation $\preceq$ defined as:

$$(\underset{\sim}{A},\underset{\sim}{B}) \preceq (\underset{\sim}{C},\underset{\sim}{D}) \text{ if } \underset{\sim}{A} \leq \underset{\sim}{C} (\text{or equiv.} \underset{\sim}{B} \leq \underset{\sim}{D}),$$
$$\forall (\underset{\sim}{A},\underset{\sim}{B}),(\underset{\sim}{C},\underset{\sim}{D}) \in \underset{\sim}{\mathcal{L}}(L,X,Y,\underset{\sim}{R})$$

is a complete lattice.

The set $\underset{\sim}{\mathcal{L}}(L,X,Y,\underset{\sim}{R})$ with the order definition $\preceq$ is said to be <u>$L-Fuzzy$ concept lattice</u> of the $L-Fuzzy$ context $(L,X,Y,\underset{\sim}{R})$.

We will use $(\underset{\sim}{\mathcal{L}}(L,X,Y,\underset{\sim}{R}),\preceq)$ or $\underset{\sim}{\mathcal{L}}$ to denote the $L-Fuzzy$ concept lattice, and $0_{\underset{\sim}{\mathcal{L}}}$, $1_{\underset{\sim}{\mathcal{L}}}$ to denote respectively the minimum and maximum elements of $\underset{\sim}{\mathcal{L}}$.

If L, X and Y are finite sets with cardinality k, m and n respectively, one of the easiest ways to construct this $L-Fuzzy$ concept lattice (but not the only one) is to take the k^m $L-Fuzzy$ sets of L^X and see if they are fixed points of φ:

We can denote $\mathcal{M}$:

$$\mathcal{M} = \{\underset{\sim}{A} \in L^X \;/\; \underset{\sim}{A} = \varphi(\underset{\sim}{A})\}$$

Now, for every $\underset{\sim}{A} \in \mathcal{M}$ we can construct the concept $(\underset{\sim}{A},\underset{\sim}{A}_1)$ and calculate the whole lattice.

2 Relation between the concept lattice of Wille and the L-Fuzzy concept lattice.

Let $R \subseteq X \times Y$, and let $\mathcal{L}(X,Y,R)$ be the concept lattice obtained by the theory of Wille ([5]).

Let L be a complete lattice with 1 and 0 the maximum and minimum elements respectively. Let $\underset{\sim}{R},\underset{\sim}{A},\underset{\sim}{B}$ be the characteristic functions of the sets $A \subseteq X$, $B \subseteq Y$ and $R \in L^{X \times Y}$.

We are going to see how we can embed the concept lattice $\mathcal{L}(X,Y,R)$ into the lattice $\underset{\sim}{\mathcal{L}} = \underset{\sim}{\mathcal{L}}(L,X,Y,\underset{\sim}{R})$:

At this point, we define the auxiliary functions g^1 and g^2:

$$g^1 : \mathcal{P}(X) \longrightarrow L^X \qquad g^2 : \mathcal{P}(Y) \longrightarrow L^Y$$
$$A \longrightarrow \underset{\sim}{A} \qquad\qquad B \longrightarrow \underset{\sim}{B}$$

From these functions we take the function:

$$\begin{array}{rcl} g : \underline{\mathcal{L}} & \longrightarrow & L^X \times L^Y \\ (A,B) & \longrightarrow & (g^1(A), g^2(B)) = (\underset{\sim}{A},\underset{\sim}{B}) \end{array}$$

We are going to prove that the set $g(\underline{\mathcal{L}})$ is a subset of $\underset{\sim}{\mathcal{L}}$; that is, if (A, B) is a concept, then the pair $(\underset{\sim}{A}, \underset{\sim}{B})$ is an $L-Fuzzy$ concept:

$$\begin{aligned}
\underset{\sim}{A}_1(y) &= \inf_{x\in X}(\underset{\sim}{A}'(x) \top \underset{\sim}{R}(x,y)) = \\
&= \inf_{x\in A}(0 \top \underset{\sim}{R}(x,y)) \wedge \inf_{x\notin A}(1 \top \underset{\sim}{R}(x,y)) = \\
&= \inf_{x\in A}\underset{\sim}{R}(x,y) \wedge 1 = \inf_{x\in A}\underset{\sim}{R}(x,y) = \\
&= \begin{cases} 1 & \text{if } \underset{\sim}{R}(x,y) = 1, \forall x \in A \\ 0 & \text{otherwise} \end{cases} = \\
&= \begin{cases} 1 & \text{if } y \in A^* \\ 0 & \text{otherwise} \end{cases} = \underset{\sim}{B}(y)
\end{aligned}$$

taking into account that $(A, B) \in \underline{\mathcal{L}}$, thus it is true that $A^* = B$.

In the same way, as $B^* = A$

$$\begin{aligned}
\underset{\sim}{A}_{12}(x) &= \inf_{y\in Y}(\underset{\sim}{A}'_1(y) \top \underset{\sim}{R}(x,y)) = \inf_{y\in B}\underset{\sim}{R}(x,y) = \\
&= \begin{cases} 1 & \text{if } \underset{\sim}{R}(x,y) = 1, \forall y \in B \\ 0 & \text{otherwise} \end{cases} = \\
&= \begin{cases} 1 & \text{if } x \in B^* \\ 0 & \text{otherwise} \end{cases} = \underset{\sim}{A}(x).
\end{aligned}$$

Obviously, the function g is inyective and preserves the order.
We can use it to embed the concept lattice of the theory of Wille into the $L-Fuzzy$ concept lattice.

Theorem 1. *$g(\underline{\mathcal{L}})$ is a complete sublattice of $\underset{\sim}{\mathcal{L}}$.*

Proof: To prove that it is a complete sublattice of $\underset{\sim}{\mathcal{L}}$ we can see that for every family $\{(\underset{\sim}{A}_i, \underset{\sim}{B}_i), i \in I\} \subseteq g(\underline{\mathcal{L}})$ it is true that

$$\bigvee_{\underset{\sim}{\mathcal{L}}}(\underset{\sim}{A}_i, \underset{\sim}{B}_i) \in g(\underline{\mathcal{L}}) \qquad\qquad \bigwedge_{\underset{\sim}{\mathcal{L}}}(\underset{\sim}{A}_i, \underset{\sim}{B}_i) \in g(\underline{\mathcal{L}})$$

In [2] we proved that the join operator in the lattice $\underset{\sim}{\mathcal{L}}$ satisfies:

$$\bigvee_{\underset{\sim}{\mathcal{L}}}(\underset{\sim}{A}_i, \underset{\sim}{B}_i) = (\bigvee_{\Omega}\underset{\sim}{A}_i, \bigwedge_{\Sigma}\underset{\sim}{B}_i) = (luis(\varphi)(\bigvee \underset{\sim}{A}_i), llis(\psi)(\bigwedge \underset{\sim}{B}_i)).$$

As $(\underset{\sim}{A}_i, \underset{\sim}{B}_i) \in g(\underline{\mathcal{L}})$ it is true that

$$(\bigvee \underset{\sim}{A}_i)(x) = \begin{cases} 1 & \text{if } x \in \bigcup A_i \\ 0 & \text{otherwise} \end{cases} = g^1(\bigcup A_i)(x)$$

$$(\bigwedge \underset{\sim}{B}_i)(y) = \begin{cases} 1 & \text{if } y \in \bigcap B_i \\ 0 & \text{otherwise} \end{cases} = g^2(\bigcap B_i)(y)$$

taking into account the supremum and infimum definitions of $L - Fuzzy$ sets.

We calculate now

$$\begin{aligned} luis(\varphi)(\bigvee \underset{\sim}{A}_i) &= \limsup(\bigvee \underset{\sim}{A}_i, \varphi(\bigvee \underset{\sim}{A}_i), \varphi(\varphi(\bigvee \underset{\sim}{A}_i))) \dots) \\ &= \limsup(g^1(\bigcup A_i), \varphi(g^1(\bigcup A_i)), \varphi(\varphi(g^1(\bigcup A_i)) \dots) \\ &= \limsup(g^1(\bigcup A_i), g^1((\bigcup A_i)^{**}), g^1((\bigcup A_i)^{****}) \dots) \end{aligned}$$

As $\bigcup A_i \subseteq (\bigcup A_i)^{**}, \forall i \in I$ and g preserves the order, $g^1(\bigcup A_i) \leq g^1((\bigcup A_i)^{**})$. Moreover $(\bigcup A_i)^{**} = (\bigcup A_i)^{****} \dots$ hence

$$\begin{aligned} luis(\varphi)(\bigvee \underset{\sim}{A}_i) &= \limsup(g^1(\bigcup A_i), g^1((\bigcup A_i)^{**}), g^1((\bigcup A_i)^{**}) \dots) \\ &= g^1((\bigcup A_i)^{**}). \end{aligned}$$

In a similar way,

$$\begin{aligned} llis(\psi)(\bigwedge \underset{\sim}{B}_i) &= \liminf(\bigwedge \underset{\sim}{B}_i, \psi(\bigwedge \underset{\sim}{B}_i), \psi(\psi(\bigwedge \underset{\sim}{B}_i)) \dots)) = \\ &= \liminf(g^2(\bigcap B_i), \psi(g^2(\bigcap B_i)), \psi(\psi(g^2(\bigcap B_i))) \dots)) = \\ &= \liminf(g^2(\bigcap B_i), g^2((\bigcap B_i)^{**}), g^2((\bigcap B_i)^{****}) \dots). \end{aligned}$$

Also $(\bigcap B_i)^{**} = (\bigcap B_i)^{****} \dots$. Moreover, as $\bigcap B_i \subseteq (\bigcap B_i)^{**}, \forall i \in I$, it is true that $g^2(\bigcap B_i) \leq g^2(\bigcap B_i)^{**}$, and so

$$llis(\psi)(\bigwedge \underset{\sim}{B}_i) = \liminf(g^2(\bigcap B_i), g^2((\bigcap B_i)^{**}) \dots) = g^2(\bigcap B_i).$$

Finally we can say that

$$\begin{aligned} \bigvee_{\underset{\sim}{\mathcal{L}}}(\underset{\sim}{A}_i, \underset{\sim}{B}_i) &= (luis(\varphi)(\bigvee \underset{\sim}{A}_i), llis(\psi)(\bigwedge \underset{\sim}{B}_i)) = \\ &= (g^1((\bigcup A_i)^{**}), g^2(\bigcap B_i)) = g(\bigvee_{\underline{\mathcal{L}}}(A_i, B_i)). \end{aligned}$$

At this point, we are going to calculate the infimum:

$$\bigwedge_{\utilde{\mathcal{L}}}(\utilde{A}_i, \utilde{B}_i) = (\bigwedge_{\Omega} \utilde{A}_i, \bigvee_{\Sigma} \utilde{B}_i) = (llis(\varphi)(\bigwedge \utilde{A}_i), luis(\psi)(\bigvee \utilde{B}_i)).$$

In the same way that we have developed $luis(\varphi)(\bigvee \utilde{A}_i)$ and $llis(\psi)(\bigwedge \utilde{B}_i)$, in this case

$$llis(\varphi)(\bigwedge \utilde{A}_i) = g^1(\bigcap A_i) \text{ and } luis(\psi)(\bigvee \utilde{B}_i) = g^2((\bigcup B_i)^{**}))$$

therefore

$$\begin{aligned}\bigwedge_{\utilde{\mathcal{L}}}(\utilde{A}_i, \utilde{B}_i) &= (llis(\varphi)(\bigwedge \utilde{A}_i), luis(\psi)(\bigvee \utilde{B}_i)) = \\ &= (g^1(\bigcap A_i), g^2((\bigcup B_i)^{**})) = g(\bigwedge_{\underline{\mathcal{L}}}(A_i, B_i)). \quad \blacksquare\end{aligned}$$

Moreover, the following proposition proves that $g(\underline{\mathcal{L}})$ is an $\{\utilde{0}_{\utilde{\mathcal{L}}}, \utilde{1}_{\utilde{\mathcal{L}}}\}$-sublattice of $\utilde{\mathcal{L}}$.

Proposition 1. *The function g verifies:*
$g(0_{\underline{\mathcal{L}}}) = \utilde{0}_{\utilde{\mathcal{L}}}$ *and* $g(1_{\underline{\mathcal{L}}}) = \utilde{1}_{\utilde{\mathcal{L}}}$.

Proof: iv) The minimum and maximum elements of the lattices $\underline{\mathcal{L}}$ and $\utilde{\mathcal{L}}$ are calculated, from results [2] and [5] in the following way:
$0_{\underline{\mathcal{L}}} = (\emptyset^{**}, \emptyset^*)$ and $1_{\underline{\mathcal{L}}} = (X, X^*)$, $\utilde{0}_{\utilde{\mathcal{L}}} = (\utilde{0}_{\Omega}, \utilde{1}_{\Sigma})$, $\utilde{1}_{\utilde{\mathcal{L}}} = (\utilde{1}_{\Omega}, \utilde{0}_{\Sigma})$.

Given a function f that preserves the order, P. and R. Cousot ([2]) define $luis(f)(\utilde{A})$ as the limit of a stationary upper iteration sequence for f starting with $\utilde{A}$, and $llis(f)(\utilde{B})$ as the limit of a stationary lower iteration sequence for f starting with $\utilde{B}$. Using these operators we can develop the last expressions:

$$\utilde{0}_{\Omega} = luis(\varphi)(\utilde{0}) = \lim\sup(\utilde{0}, \varphi(\utilde{0}), \varphi(\varphi(\utilde{0}))\dots)$$

where $\utilde{0}(x) = 0, \forall x \in X$, and so $\utilde{0} = g^1(\emptyset)$.

By the previous paragraph $\varphi(g^1(\emptyset)) = g^1(\emptyset^{**})$ and applying φ, $\varphi(\varphi(g^1(\emptyset))) = g^1(\emptyset^{****})$, so:

$$\utilde{0}_{\Omega} = \lim\sup(g^1(\emptyset), g^1(\emptyset^{**}), g^1(\emptyset^{****})\dots)$$

By a property of the derivation operator ([4]) $\emptyset \leq \emptyset^{**}$, then $g^1(\emptyset) \leq g^1(\emptyset^{**})$ since g^1 preserves the order. Moreover $\emptyset^{**} = \emptyset^{****}\dots$ so

$$\utilde{0}_{\Omega} = \lim\sup(g^1(\emptyset), g^1(\emptyset^{**}), g^1(\emptyset^{**})\dots) = g^1(\emptyset^{**})$$

In a similar way, from the P. and R. Cousot([2]) theory

$$\underset{\smile}{1}_{\Sigma} = llis(\psi)(\underset{\smile}{1}) = \liminf(\underset{\smile}{1}, \psi(\underset{\smile}{1}), \psi(\psi(\underset{\smile}{1}))\dots)$$

where $\underset{\smile}{1}(x) = 1, \forall y \in Y$, and therefore $\underset{\smile}{1} = g^2(Y)$.

By the concept formal analysis ([4]) $Y = Y^{**} = Y^{****} = \dots$, we can say that:

$$\underset{\smile}{1}_{\Sigma} = \liminf(g^2(Y), g^2(Y)\dots) = g^2(Y)$$

Therefore, we can conclude that:

$$(\underset{\smile}{0}_{\Omega}, \underset{\smile}{1}_{\Omega}) = (g^1(\emptyset^{**}), g^2(Y)).$$

As $\emptyset^* = Y$ ([4]) it is shown that $\underset{\smile}{0}_{\underset{\smile}{\mathcal{L}}} = g(0_{\underline{\mathcal{L}}})$.

In a similar way we prove that $g(1_{\underline{\mathcal{L}}}) = \underset{\smile}{1}_{\underset{\smile}{\mathcal{L}}}$. ■

To conclude, we are going to illustrate the previous theorem through the following:

Example 1.

We take the clasical example of the formal concept analysis relating to the planets of our solar system ([5]), and we will construct the $L - Fuzzy$ concept lattice derived from it.

The object set is formed by the planets of the solar system X={Mercury (Me), Venus (V), Earth (E), Mars (Ma), Jupiter (J), Saturn (S), Uranus (U), Neptune (N) y Pluto (P)} and the attribute set is Y={size-small (ss), size-medium (sm), size-large (sl), distance-near(dn), distance-far (df), moon-yes (my), moon-no (mn)} respectively in the defined order.)

The relation $\underset{\smile}{R}$ will be the same used by Wille and will represent the relationship between the planets of the solar system and the attributes:

	ss	sm	sl	dn	df	my	mn
Me	1	0	0	1	0	0	1
V	1	0	0	1	0	0	1
E	1	0	0	1	0	1	0
Ma	1	0	0	1	0	1	0
J	0	0	1	0	1	1	0
S	0	0	1	0	1	1	0
U	0	1	0	0	1	1	0
N	0	1	0	0	1	1	0
P	1	0	0	0	1	1	0

In this relation, the value 1 indicates that the object has that attribute, and the value 0 the opposite. The concept lattice $\underline{\mathcal{L}}$ determined by the context of Wille([5]), had 12 concepts.

Now, we will work with the lattice $L = \{0, 0.5, 1\}$, the t-conorm $\top = \vee$ and the complementation of Zadeh.

The $L - Fuzzy$ concept lattice obtained has 51 concepts $(\underset{\backsim}{A}_i, \underset{\backsim}{B}_i)$:

1 $\begin{cases} \underset{\backsim}{A}1 = \{Me/0, V/0, E/0, Ma/0, J/0, S/0, U/0, N/0, P/0\} \\ \underset{\backsim}{B}1 = \{ss/1, sm/1, sl/1, dn/1, df/1, my/1, mn/1\} \end{cases}$

2 $\begin{cases} \underset{\backsim}{A}2 = \{Me/0, V/0, E/0, Ma/0, J/0, S/0, U/0, N/0, P/0.5\} \\ \underset{\backsim}{B}2 = \{ss/1, sm/0.5, sl/0.5, dn/0.5, df/1, my/1, mn/0.5\} \end{cases}$

3 $\begin{cases} \underset{\backsim}{A}3 = \{Me/0, V/0, E/0, Ma/0, J/0, S/0, U/0, N/0, P/1\} \\ \underset{\backsim}{B}3 = \{ss/1, sm/0, sl/0, dn/0, df/1, my/1, mn/0\} \end{cases}$

4 $\begin{cases} \underset{\backsim}{A}4 = \{Me/0, V/0, E/0, Ma/0, J/0, S/0, U/0.5, N/0.5, P/0\} \\ \underset{\backsim}{B}4 = \{ss/0.5, sm/1, sl/0.5, dn/0.5, df/1, my/1, mn/0.5\} \end{cases}$

5 $\begin{cases} \underset{\backsim}{A}5 = \{Me/0, V/0, E/0, Ma/0, J/0, S/0, U/1, N/1, P/0\} \\ \underset{\backsim}{B}5 = \{ss/0, sm/1, sl/0, dn/0, df/1, my/1, mn/0\} \end{cases}$

6 $\begin{cases} \underset{\backsim}{A}6 = \{Me/0, V/0, E/0, Ma/0, J/0.5, S/0.5, U/0, N/0, P/0\} \\ \underset{\backsim}{B}6 = \{ss/0.5, sm/0.5, sl/1, dn/0.5, df/1, my/1, mn/0.5\} \end{cases}$

7 $\begin{cases} \underset{\backsim}{A}7 = \{Me/0, V/0, E/0, Ma/0, J/0.5, S/0.5, U/0.5, N/0.5, P/0.5\} \\ \underset{\backsim}{B}7 = \{ss/0.5, sm/0.5, sl/0.5, dn/0.5, df/1, my/1, mn/0.5\} \end{cases}$

8 $\begin{cases} \underset{\backsim}{A}8 = \{Me/0, V/0, E/0, Ma/0, J/0.5, S/0.5, U/0.5, N/0.5, P/1\} \\ \underset{\backsim}{B}8 = \{ss/0.5, sm/0, sl/0, dn/0, df/1, my/1, mn/0\} \end{cases}$

9 $\begin{cases} \underset{\backsim}{A}9 = \{Me/0, V/0, E/0, Ma/0, J/0.5, S/0.5, U/1, N/1, P/0.5\} \\ \underset{\backsim}{B}9 = \{ss/0, sm/0.5, sl/0, dn/0, df/1, my/1, mn/0\} \end{cases}$

10 $\begin{cases} \underset{\backsim}{A}10 = \{Me/0, V/0, E/0, Ma/0, J/1, S/1, U/0, N/0, P/0\} \\ \underset{\backsim}{B}10 = \{ss/0, sm/0, sl/1, dn/0, df/1, my/1, mn/0\} \end{cases}$

$\vdots$

51 $\begin{cases} \underset{\backsim}{A}51 = \{Me/1, V/1, E/1, Ma/1, J/1, S/1, U/1, N/1, P/1\} \\ \underset{\backsim}{B}51 = \{ss/0, sm/0, sl/0, dn/0, df/0, my/0, mn/0\} \end{cases}$

This lattice can be represented as we can see in the following picture. To look at the embedding we only have to take into account that the concepts * are those corresponding to the concept lattice of Wille ([5]).

The $L-Fuzzy$ concept lattice allows us to obtain, through an algorithm process like the one described in the introduction, further information with respect to the theory of Wille. Now, we have new concepts that did not appear using the techniques of Wille, which give us new information.

For example, if we compare concepts 5 and 9 (only the first one included in the lattice of Wille), we can see that if the degree of pertenence of sm decreases from 1 to 0.5, then the degree of pertenence of J,S and P increases from 0.5 to 1. In this sense, if the ambiguity of the attribute sm increases, then the ambiguity of the pertenence of the planets J and S to the concept also grows up; but only for these planets since the movement of sm does not have influence in U and N.

We will analyse this new information in following papers.

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