Didactical Note: Probabilistic Conditionality in a Boolean Algebra

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Abstract

This note deals with two logical topics and concerns Boolean Algebras from an elementary point of view. First we consider the class of operations on a Boolean Algebra that can be used for modelling "If-then" propositions. These operations, or Conditionals, are characterized under the hypothesis that they only obey to the **Modus Ponens**-Inequality, and it is shown that only six of them are boolean two-place functions.

Is the Conditional Probability the Probability of a Conditional? This problem will be only considered, with the Material Conditional Operation, on a Boolean Algebra endowed with a finite probability and in three different cases: with the Internal-Conditional Probability, with the External-Conditional Probability and with both the External-Conditional and the "a priori" probabilities. It is shown when and how the problem can have some solution.

Keywords. Boolean Algebras. Conditional Operations. Finite probabilities. Conditional probabilities. Probability of a Conditional.

1 Conditional operations on a Boolean algebra

Let $E = \{a, b, c, ...\}$ be a Boolean Algebra (see [3]) with least element 0, greatest element 1, complement $'$, union $+$, intersection $\cdot$, and the partial order $\leq$ defined, as usual, by: $a \leq b$ **iff** $a \cdot b = a$ **iff** $a + b = b$.

It is well-known that the operation $a \to b := a' + b$ is usually taken to represent in E the Material Conditional of Logic (see [6], [7], [8]) since affirming $a \to b$ is equivalent to stating that a and b are in the relation $\leq$.

Proposition 1.1. $a \leq b$ **iff** $a \to b = 1$.

Proof. From $1 = a' + b$ it follows $a \cdot 1 = a = a \cdot (a' + b) = a \cdot a' + a \cdot b = a \cdot b$, or $a \leq b$. Reciprocally from $a \leq b$ it follows $1 = a' + a \leq a' + b$, or $1 = a' + b$.

It is usual to see Conditionals as those operations $R : E \times E \to E$ such that the **Modus Ponens**-Inequality [see 8]:

$$f(a) \cdot R(a, b) \leq f(b), \tag{1}$$

holds for some function $f : E \to E$ (called a Logical-State) and any a, b in E. We have that:

Proposition 1.2. *Equation (1) holds if and only if $R(a, b) \leq f(a) \to f(b)$, for all the a, b in E.*

Proof. From (1) it follows $f(b)' \cdot f(a) \cdot R(a, b) = 0$, this means that $R(a, b) \leq (f(b)' \cdot f(a))' = f(a)' + f(b)$. Viceversa, from this last inequality it follows $f(a) \cdot R(a, b) \leq f(a) \cdot f(b) \leq f(b)$.

The only functions $f : E \to E$ which are boolean are the following ones: $f_1(x) = id(x) = x$, $f_2(x) = id'(x) = x'$, $f_3(x) = 0$ and $f_4(x) = 1$. Consequently, constant functions being excluded as proper Logical-States (see [8]), the only candidates to be boolean proper Logical-States for an operation R are id and id'.

Corollary 1.3. *The only Conditionals with boolean proper Logical-States in a Boolean Algebra are those $R : E \times E \to E$ such that $R(a, b) \leq a \to b$, or $R(a, b) \leq b \to a$, for all a, b in E.*

Hence, operation $\to$ gives the largest Conditional on E.

Proposition 1.4. *The only non-trivial boolean functions R that are Conditionals in a Boolean Algebra are: $R_6(a, b) = a \to b$, $R_5(a, b) = a \cdot b$, $R_4(a, b) = a' \cdot b$, $R_3(a, b) = a' \cdot b'$, $R_2(a, b) = b$ and $R_1(a, b) = a \cdot b + a' \cdot b'$.*

Proof. All functions R_i verify $a \cdot R(a, b) \leq b$. Reciprocally, since any boolean two-place function R can be written as $R(a, b) = \alpha \cdot a \cdot b + \beta \cdot a' \cdot b + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b'$, with α, β, γ and δ in $\{0, 1\} \subset E$, from the **Modus Ponens**-Inequality succesively follows $\alpha \cdot a \cdot b + \gamma \cdot a \cdot b' \leq b$, $\gamma \cdot a \cdot b' = 0$ and $\gamma = 0$. Thus, variying α, β and δ in $\{0, 1\}$ the only different R obtained are the R_i, $1 \leq i \leq 6$, plus those that are identical, respectively, to 0 and 1, to be excluded as Conditional Operations.

Remarks. 1) Any of the functions R_i formally allow to represent, in the Propositional Calculus, conditional propositions like "If a, then b". Nevertheless, the only well-known function usually considered in such Calculus is R_6. In fact, let's denote by $\hat{R}_i \subset E \times E$ the binary relations:

$$(a,b) \in \hat{R}_i \quad \text{or} \quad a\hat{R}_i b \quad \textbf{iff} \quad R_i(a,b) = 1,$$

Proposition 1.2 shows $\hat{R}_6 = \leq$ and it is quite obvious that $\hat{R}_5 = \{(1,1)\}$, $\hat{R}_4 = \{(0,1)\}$, $\hat{R}_3 = \{(0,0)\}$, $\hat{R}_2 = \{(x,1); x \in E\}$ and $\hat{R}_1 = \{(x,x); x \in E\}$. Then it seems that R_6 is the most convenient Conditional, but the world of Logic has not Propositional Calculus as a fence.

It is easy to see that each R_i can be written as a function of the others. For example, it is $R_6(a,b) = R_5(a,b')' = R_4(b,a)' = R_3(a',b) = R_2(1,a'+b) = R_1(1,a'+b)$.

2) The operations R_6, R_5, R_2 and R_1 verify the **Modus Ponens**-Inequality because $a \cdot R_i(a,b) = a \cdot b$, and R_4 and R_3 through $a \cdot R_i(a,b) = 0$.

3) The only R_i that are commutative operations are R_5, R_3 and R_1. It is $R_6(a,b) = R_6(b',a')$, $R_4(a,b) = R_4(b,a')$ and $R_2(a,b) = R_2(b,a)$ **iff** $a = b$.

4) Defining R as reflexive if $R(a,a) = 1$ for any a in E, and as transitive if $R(a,b) \cdot R(b,c) \leq R(a,c)$ for any a,b,c in E (see [8]), R_6 and R_1 are reflexive and transitive but the other ones are only transitive. Hence, the only "preorders" are R_6 and R_1, and then $\hat{R}_6$ and $\hat{R}_1$ are classical preorders ([8]).

5) Defining R to be **Monotonic** if $R(a,b) \leq R(a \cdot c, b)$ for any a,b,c in E ([8]), it is easy to see that R_6, R_4, R_3 and R_2 are Monotonic, but R_5 and R_1 are Non-Monotonic.

2 Conditioning in Boolean algebras

2.1. Given a in E, let $aE = \{x \in E; x = a \cdot y$, for some $y \in E\}$, i.e.,

$$aE = \{z \in E;\ z \leq a\},$$

Subsets aE of E are characterized by a, that is: $aE = bE$ **iff** $a = b$. In fact, if $a = b$ it is obvious that $aE = bE$, and reciprocally as $a = a \cdot 1 \in aE$ it follows that $a \in bE$ and $b \in aE$, or $a \leq b$ and $b \leq a$.

Proposition 2.1.1. *aE is a Boolean Algebra with least element 0 and greatest element a, endowed with the restrictions of $+$ and $\cdot$, and with the complement defined by $x^{\neq} = a \cdot x'$.*

It should be pointed out that in general aE is not a subalgebra of E and for a fixed a in E, it is $R_5(a,x) = a \cdot x \in aE$ when $x \in E$. Since this reason we may say that the Boolean Algebra aE is obtained by "conditioning" the initial algebra E with R_5. Because of this aE is called the Conditioned Algebra of E by a.

2.2. In Probability Theory one usually writes E/a instead of aE and the elements $x \in aE$ are also written as y/a, provided that $y \cdot a = x$. In this quotient-style notation we have that $y/a = z/a$ **iff** $y \cdot a = z \cdot a$, and as:

- $y \cdot a + z \cdot a = (y + z) \cdot a$, it is $y/a + z/a = (y + z)/a$
- $(y \cdot a) \cdot (z \cdot a) = (y \cdot z) \cdot a$, it is $y/a \cdot z/a = (y \cdot z)/a$
- $(y \cdot a)^{\text{æ}} = a \cdot (y \cdot a)' = a \cdot (y' + a') = a \cdot y'$, it is $(y/a)^{\text{æ}} = y'/a$.

Evidently it is $y/a = a \cdot y/a = (a' + y)/a$.

Proposition 2.2.1. *For a fixed $a \in E$ the only functions $F : E \times E \to E$ such that*

$$y/a = F(a, y)/a,$$

are those functions satisfying,

$$a \cdot y \leq F(a, y) \leq a' + y.$$

Proof. From $y \cdot a = F(a, y) \cdot a$ it follows $F(a, y) \cdot a \cdot y' = 0$, or $F(a, y) \leq (a \cdot y')' = a' + y$. Analogously it also follows $y \cdot a \cdot F(a, y)' = 0$, or $F(a, y)' \leq (y \cdot a)'$, then $y \cdot a \leq F(a, y)$. Viceversa, from the inequalities we obtain $a \cdot y \leq a \cdot F(a, y) \leq a \cdot y$, or $a \cdot y = a \cdot F(a, y)$.

Proposition 2.2.2. *For a fixed $a \in E$ the only boolean functions $F : E \times E \to E$ such that $y/a = F(a, y)/a$, are R_6, R_5, R_2 and R_1.*

Proof. Like before it should be $F(a, y) = \alpha \cdot a \cdot y + \beta \cdot a' \cdot y + \gamma \cdot a \cdot y' + \delta \cdot a' \cdot y'$. From Proposition 2.2.1 it follows $a \cdot y \leq \alpha \cdot a \cdot y + \gamma \cdot a \cdot y' \leq a \cdot y$, or $\alpha \cdot a \cdot y + \gamma \cdot a \cdot y' = a \cdot y$, for any y in E. Hence $\alpha = 1$, $\gamma = 0$ and the functions are $F(a, y) = a \cdot y + \beta \cdot a' \cdot y + \delta \cdot a' \cdot y'$. Varying β and δ in $\{0, 1\}$ the proof is complete.

Given a, y in E there are only four boolean ways of writting y/a:

$$y/a = a \cdot y/a = a' + y/a = a \cdot y + a' \cdot y'/a,$$

and the upper term of each "quotient" is always the a-Section of some boolean Conditional.

2.3. As it was already stated, in a Boolean Algebra of propositions the operation R_6 is used to model the conditional propositions of the type "If x, then y". In the Conditioned Algebra aE this operation becomes $x \to_a y = x^{\text{æ}} + y = a \cdot x' + y =$

$(a+y)\cdot(x'+y) = a\cdot(x \to y)$. Consequently, when $x \to y$ in E we have also $x \to_a y$ in aE. Nevertheless, when affirming $x \to_a y$ in some aE we can only deduce in general that $a \leq x \to y$ in E, because $a = a \cdot (x \to y)$.

3 The case with finite probabilities

In what follows a probabilized Boolean Algebra (E, p) will be considered, i.e., a Boolean Algebra E endowed with a finite probability $p : E \to [0, 1]$ such that:

1) $p(1) = 1$ (axiom of normalization)

2) $p(x + y) = p(x) + p(y)$, whenever $x \cdot y = 0$ (axiom of finite additivity).

From these axioms it is easy to deduce the following well-known properties of any probability: $p(x') = 1 - p(x)$; $p(0) = 0$; $p(x + y) = p(x) + p(x' \cdot y)$; $p(x) = p(x \cdot y) + p(x' \cdot y')$; $p(x) + p(y) = p(x + y) + p(x \cdot y)$, and $p(x) \leq p(y)$ when $x \leq y$.

3.1. Let aE be a Conditioned Boolean Algebra in E. The restriction p_{aE} of p to aE is not a probability for aE unless $a = 1$, and the necessary and sufficient condition for p_{aE} to be identically zero is $p(a) = 0$. To avoid this, from now on we will consider $a \in E^+ = \{x \in E; p(x) > 0\}$.

A **finite** probability on aE is a function $\hat{p} : aE \to [0, 1]$ verifying the above axioms of normalization and finite addivitity:

1) $\hat{p}(a) = 1$, and 2) $\hat{p}(x + y) = \hat{p}(x) + \hat{p}(y)$, whenever $x \cdot y = 0$, for any x, y in aE.

Next definition gives a $\hat{p}$ depending on both p and a (see [2]).

Definition 3.1.1. *The function $p^i_a : aE \to [0, 1]$, $p^i_a(x) = p(x)/p(a)$, with $a \in E^+$ is a finite probability on aE and it will be called the Internal-Conditional Probability.*

Definition 3.1.2. *The function $p^e_a : E \to [0, 1]$, $p^e_a(y) = p(a \cdot y)/p(a)$ is a finite probability on E and it will be called the External-Conditional Probability.*

As $p^e_a(1) = p(a)/p(a) = 1$, and when $x\cdot y = 0$ it is also $a\cdot(x\cdot y) = (a\cdot x)\cdot(a\cdot y) = 0$ and the $p^e_a(x+y) = p(a\cdot(x+y))/p(a) = p(a\cdot x+a\cdot y)/p(a) = (p(a\cdot x)+p(a\cdot y))/p(a) = p^e_a(x) + p^e_a(y)$, so p^e_a is a finite probability on E.

These probabilities are formally different since they are defined in different Boolean Algebras and, of course, the restriction of p^e_a to aE is p^i_a. In the Theory of Probability[1], $p^e_a(y)$ has been traditionally called the Conditional Probability of y given a, and it is usually written as $p(y/a)$. From now on this notation will be assumed and instead of finite probabilities we will say just probabilities. Nevertheless, it should be noticed that $p(y/a)$ is actually a function of only $y \in E$

as a is fixed in E^+ and, if considered as a two-place function $p(\ /\ ) : E \times E^+ \to [0,1]$ it is not a probability, at least because of $E \times E^+$ does not inherite the Boolean Algebra structure of E as there will be not, in general, a least element for the product order $(x, y) \leq (x_1, y_1)$ **iff** $x \leq x_1$ and $y \leq y_1$. This function $p(\ /\ )$ is just a Fuzzy Relation between E^+ (also not a Boolean Algebra) and E (see [4], [5]).

Remark. Since $p(y/a) = p(a \cdot y)/p(a) = p(R_5(a, y))/p(a) := p_5(y/a)$, it is natural to think about the possibility of using in the same way the other conditionals R_i. But a probability is rarely obtained since, for example: $p_6(1/a) = p_2(1/a) = 1/p(a)$, $p_4(1/a) = p_1(0/a) = p(a')/p(a)$ and $p_3(1/a) = 0$.

3.2. Let us now study the possible solutions, of the equations:

$$p^i_a(x) = p^i_a(z \to_a x) \quad \text{and} \quad p(y/a) = p(z \to y/a),$$

in aE and E respectively.

Proposition 3.2.1. *Given $x \in aE$ a necessary and sufficient condition for the existence of $z \in aE$ such that $p^i_a(x) = p^i_a(z \to_a x)$ is given by $p(a \cdot x' \cdot z') = 0$. Two sufficient conditions are $p(a \cdot z') = 0$ and $z = a$.*

Proof. The equation holds if and only if $p(x) = p(z^{\not a} + x)$, or $p(z^{\not a}) - p(z^{\not a} \cdot x) = 0$ equivalent to $p(a \cdot z' \cdot x') = 0$. Since $a \cdot x' \cdot z' \leq a \cdot z'$ the first sufficient condition follows and, for this it is sufficient, e.g., to have $a \cdot z' = 0$, or $a \leq z$ that actually means $z = a$.

Corollary 3.2.2. *For the equation $p^i_a(x) = p^i_a(z \to_a x)$ to be verified for any $x \in aE$, a necessary and sufficient condition is $p(a \cdot z') = 0$ and a sufficient condition is $z = a$.*

Analogously we have:

Proposition 3.2.3. *Given $y \in E$ a necessary and sufficient condition for the existence of $z \in E$ such that $p(y/a) = p(z \to y/a)$ is that $p(a \cdot y' \cdot z') = 0$. Two sufficient conditions are $p(a \cdot z') = 0$ and $a \leq z$.*

Corollary 3.2.3. *For the equation $p(y/a) = p(z \to y/a)$ to be verified for any $y \in E$ a necessary and sufficient condition is $p(a \cdot z') = 0$ and a sufficient condition is $a \leq z$.*

The study of these equations shows that we have other solutions, besides the trivial one $z = a$, depending, in the internal case, on the probability p: if there are elements $z \in aE$ such that $p(a \cdot z') = 0$ these elements are solutions of the equation $p^i_a(x) = p^i_a(z \to_a x)$ and the Internal-Conditional Probability is the Internal-

Conditional Probability of a Conditional. In the External case, in addition to those $z \in E$ such that $p(a \cdot z') = 0$, any $z \geq a$ is a solution. Consequently, the general question (see [6], [7]) "**Is a Conditional Probability the Probability of a Conditional?**" has a positive answer, at least when the "probabilities" appearing on both sides of the equation are the same kind of Conditional Probability.

Remark. Proposition 3.2.1 is not formally a particular case of Proposition 3.2.3 as the restriction of $\rightarrow$ to aE is not $\rightarrow_a$. In fact, when $x = a \cdot x_1 \in aE$ and also $y = a \cdot y_1 \in aE$ it is:

- $x \rightarrow_a y = a \cdot (x \rightarrow y) = a \cdot x_1' + a \cdot y_1 = a \cdot (x_1 \rightarrow y_1)$

- $x \rightarrow y = (a \cdot x_1)' + a \cdot y_1 = a' + x_1' + a \cdot y_1$,

both expressions being different unless $a = 1$.

3.3. Let's consider a different but well-known problem (see [6], [7]). Given $a \in E^+$ to solve the equation $p(y/a) = p(z \rightarrow y)$ for $y \in E$, where the unknown is $z \in E$, considering the Conditional Probability $p(\ /a)$ and the "a priori" probability p. The possibility of solutions, for a given y or for all them, depend not only on the existence of z but also on the effective range of p in [0,1].

Proposition 3.3.1. *A necessary but not sufficient condition for the existence of some solution of* $p(y/a) = p(z \rightarrow y)$ *is* $p(y) \leq p(y/a)$*, for each* $y \in E$*.*

Proof. From the equation follows $p(y) \leq p(z' + y) = p(y/a)$. The reciprocal is evidently not true.

Without sastisfying this condition the equation has no solutions for the given y. In fact, from $p(y/a) < p(y)$ and the equation, it follows $p(z + y) > 1$, because of $p(y/a) = p(z' + y) = p(y) + p(z' \cdot y') = p(y) + 1 - p(z + y)$.

In any case, the equation is equivalent to $1 - p(y/a) = 1 - p(z' + y) = p(z \cdot y')$, or to $p(a \cdot y')/p(a) = p(z \cdot y')$. Thus, if the necessary condition is satisfied, the solutions are to be searched among the $z \in E$ verifying this equation for the given y, and depend on the available values of p. For example, to be $z = a$ a solution it should be $p(a \cdot y') = p(a) \cdot p(a \cdot y')$, and this when $p(a) < 1$ implies $p(a \cdot y') = 0$ only for particular values of y: with $y = a'$ it is $p(a) = 0$.

Let's consider the peculiar case $p(a) = 1$. The necessary condition is $p(y) \leq p(a \cdot y)$; since $p(a') = 0$ we have that $p(a' \cdot y) = 0$ and $p(y) = p(a \cdot y)$, and it is then satisfied. The equation is $p(a \cdot y) = p(z' + y) = p(z') + p(y) - p(z' \cdot y)$, which is equivalent to $p(z') - p(z' \cdot y) = p(a \cdot y) - p(y)$ and to $p(z' \cdot y') = -p(a' \cdot y) = 0$. The solutions then are to be searched among those z such that $p(z + y) = 1$, and for obtaining this it is sufficient to have $z + y = 1$, or $y' \leq z$.

Examples. Let it be $E = 2^3$ with atoms a_1, a_2, a_3 and the uniform probability $p(a_i) = 1/3$, $1 \le i \le 3$.

- If $a = a_1$, $y = a_2$ it is $p(a_2/a_1) = 0 < 1/3 = p(a_3)$. There is no solution for the equation $p(a_2/a_1) = p(z \to a_2)$.

- If $a = a_1$, $y = a_1 + a_3$, it is $p(a_1 + a_3) = 2/3 < 1 = p(a_1 + a_3/a_1)$: the necessary condition is satisfied. The equation being $1 = p(z' + a_1 + a_3)$, or $p(z, a_2) = 0$, there are the three solutions $z = a_1, a_2$, $a_1 + a_3$.

- If $a = a_1 + a_2$, $y = a_1$ it is $p(a_1) = 1/3 < 1/2 = p(a_1/a_1 + a_2)$: the necessary condition is satisfied. But now the equation $1/2 = p(z' + a_1)$ has no solutions because there is no $z' + a_1 \in E$ with probability 0.5: the "a priori" probability has no enough values.

The first example shows that in the given probabilized Boolean Algebra there is no solution of the equation $p(y/a_1) = p(z \to y)$ for all y in 2^3.

4 Conclussions and final comments

In the setting of Boolean Algebras it was studied the problem of finding two-place functions to be used as Conditional Operations (to represent "If-then" propositions), and the problem of possibilities for a Conditional Probability to become the "a priori" probability of the Material Conditional.

With the above results it is intended to reach some level of clarification for a problem that was, and still is, studied by many logicians, Artificial Intelligence researchers and philosophers (see [6], [7]). Nevertheless, and apart from minor questions, the problem still remains open at least for what concerns σ-additive "a priori" probabilities: when considering a σ-Algebra and not a finitely additive probability the set of probability's values could have a higher cardinality and, presumably, there will be space for new innovative ideas.

Let's end the paper with a final comment. It is easy to see that the mapping $r_5 : E \to aE$, $r_5(y) = a \cdot y$, is an epimorphism between the Boolean Algebra E and its subset $aE = R_5(a, E)$ endowed with the complement $x^{\#} = a \cdot x'$. This gives a method to prove that aE is a Boolean Algebra.

Analogously, defining the mappings $r_6 : E \to R_6(a, E)$, $r_6(y) = a' + y$, and $r_4 : E \to R_r(a, E)$, $r_4(y) = a' \cdot y$, it is easy to see that both are epimorhpisms between E and, respectively, its subsets $a' + E = \{a' + y; y \in E\} = \{z \in E; a' \le z\}$ and $a'E = \{a' \cdot y; y \in E\}$, with the complements $x^{\#} = a' + x'$ and $x^{\#} = a' \cdot x'$. Then also $a' + E$ and $a'E$ are Conditioned Boolean Algebras of E by a.

Nevertheless, the cases $r_3(y) = a' \cdot y'$, $r_2(y) = a$ and $r_1(y) = a \cdot y + a' \cdot y'$, respectively for $R_3(a, E)$, $R_2(a, E)$ and $R_1(a, E)$, don't give Boolean Algebras.

The function $p_a^4(y) = p(a' \cdot y)/p(a')$ is exactly $p(y/a')$, the Conditional Probability of y by a'. But $p_a^6(y) = p(a' + y)$, although verifies $p_a^6(1) = 1$ is not additive as, for example, $p_a^6(0+1) = p_a^6(1) = 1$ and $p_a^6(0) + p_a^6(1) = p(a') + 1$.

Actually, only aE seems to be a boolean right way of Conditioning (E, p).

References

[1] A. Rnyi (1966), Calcul des Probabilits, Dunod (Paris).

[2] D. Hrault (1976), Teora de la probabilidad, Ed. Promocin Cultural (Barcelona).

[3] G.E. Birkhoff (1979 Third Edition), Lattice Theory, AMS Pubs. (Providence).

[4] G.E. von Wright (1979, Second Edition), The Logical Problem of Induction, Greenwood Press (Connecticut).

[5] E. Trillas (1988), Relational Probabilities on Intuitionistic lattices, *Stochastica*, XII-2/3, 85-102.

[6] I.R. Goodman, M. Gupta, H.T. Nguyen and G.S. Rogers (1991), Conditional Logic in Expert systems, North-Holland (New York).

[7] I.R. Goodman, H.T. Nguyen and E.A. Walker (1991), Conditional Inference and Logic for Intelligent Systems, North-Holland (New York).

[8] E. Trillas (1993), On Logic and Fuzzy Logic, *Int. J. of Uncertainty fuzziness and knowledge-based systems*, 1/2, 107-137.