Didactical Note: Probabilistic Conditionality in a Boolean Algebra

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Abstract

This note deals with two logical topics and concerns Boolean Algebras from an elementary point of view. First we consider the class of operations on a Boolean Algebra that can be used for modelling “if-then” propositions. These operations, or Conditionals, are characterized under the hypothesis that they only obey to the Modus Ponens-Inequality, and it is shown that only six of them are boolean two-place functions.

Is the Conditional Probability the Probability of a Conditional? This problem will be only considered, with the Material Conditional Operation, on a Boolean Algebra endowed with a finite probability and in three different cases: with the Internal-Conditional Probability, with the External-Conditional Probability and with both the External-Conditional and the “a priori” probabilities. It is shown when and how the problem can have some solution.


1 Conditional operations on a Boolean algebra

Let $E = \{a, b, c, ...\}$ be a Boolean Algebra (see [3]) with least element 0, greatest element 1, complement $'$, union $\cup$, intersection $\cap$, and the partial order $\leq$ defined, as usual, by: $a \leq b$ iff $a \cdot b = a$ iff $a + b = b$.

It is well-known that the operation $a \rightarrow b := a' + b$ is usually taken to represent in $E$ the Material Conditional of Logic (see [6], [7], [8]) since affirming $a \rightarrow b$ is equivalent to stating that $a$ and $b$ are in the relation $\leq$.

Proposition 1.1. $a \leq b$ iff $a \rightarrow b = 1$. 

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Proof. From 1 = \alpha' + b it follows \alpha \cdot 1 = \alpha = \alpha \cdot (\alpha' + b) = \alpha \cdot \alpha' + \alpha \cdot b = \alpha \cdot b, or \alpha \leq b. Reciprocally from \alpha \leq b it follows 1 = \alpha' + \alpha \leq \alpha' + b, or 1 = \alpha' + b.

It is usual to see Conditionals as those operations \( R : E \times E \to E \) such that the \textbf{Modus Ponens}-Inequality [see 8]:

\[
f(a) \cdot R(a, b) \leq f(b),
\]
holds for some function \( f : E \to E \) (called a Logical-State) and any \( a, b \in E \). We have that:

\textbf{Proposition 1.2.} Equation (1) holds if and only if \( R(a, b) \leq f(a) \to f(b) \), for all the \( a, b \) in \( E \).

Proof. From (1) it follows \( f(b)' \cdot f(a) \cdot R(a, b) = 0 \), this means that \( R(a, b) \leq (f(b)' \cdot f(a))' \), \( f(a)' = f(a)' + f(b) \). Viceversa, from this last inequality it follows \( f(a) \cdot R(a, b) \leq f(a) \cdot f(b) \leq f(b) \).

The only functions \( f : E \to E \) which are boolean are the following ones: \( f_1(x) = id(x) = x \), \( f_2(x) = id'(x) = x' \), \( f_3(x) = 0 \) and \( f_4(x) = 1 \). Consequently, constant functions being excluded as proper Logical-States (see [8]), the only candidates to be boolean proper Logical-States for an operation \( R \) are \( id \) and \( id' \).

\textbf{Corollary 1.3.} The only Conditionals with boolean proper Logical-States in a Boolean Algebra are those \( R : E \times E \to E \) such that \( R(a, b) \leq a \to b \), or \( R(a, b) \leq b \to a \), for all \( a, b \) in \( E \).

Hence, operation \( \to \) gives the largest Conditional on \( E \).

\textbf{Proposition 1.4.} The only non-trivial boolean functions \( R \) that are Conditionals in a Boolean Algebra are: \( R_0(a, b) = a \to b \), \( R_5(a, b) = a \cdot b \), \( R_4(a, b) = a' \cdot b \), \( R_3(a, b) = a' \cdot b' \), \( R_2(a, b) = b \) and \( R_1(a, b) = a \cdot b + a' \cdot b' \).

Proof. All functions \( R_i \) verify \( a \cdot R(a, b) \leq b \). Reciprocally, since any boolean two-place function \( R \) can be written as \( R(a, b) = \alpha \cdot a \cdot b + \beta \cdot a \cdot b' + \gamma \cdot a \cdot b' + \delta \cdot a' \cdot b' \), with \( \alpha, \beta, \gamma \) and \( \delta \) in \( \{0, 1\} \subseteq E \), from the \textbf{Modus Ponens}-Inequality successively follows \( \alpha \cdot a \cdot b + \gamma \cdot a \cdot b' \leq b \), \( \gamma \cdot a \cdot b' = 0 \) and \( \gamma = 0 \). Thus, varying \( \alpha, \beta \) and \( \delta \) in \( \{0, 1\} \) the only different \( R \) obtained are the \( R_i \), \( 1 \leq i \leq 6 \), plus those that are identical, respectively, to 0 and 1, to be excluded as Conditional Operations.

\textbf{Remarks.} 1) Any of the functions \( R_i \) formally allow to represent, in the Propositional Calculus, conditional propositions like “If \( a \), then \( b \)”. Nevertheless, the only
well-known function usually considered in such Calculus is \( R_6 \). In fact, let’s denote by \( R_i \subseteq E \times E \) the binary relations:

\[
(a, b) \in R_i \quad \text{or} \quad aR_ib \quad \text{iff} \quad R_i(a, b) = 1,
\]

Proposition 1.2 shows \( R_6 = \leq \) and it is quite obvious that \( R_5 = \{(1,1)\} \), \( R_4 = \{(0,1)\} \), \( R_3 = \{(0,0)\} \), \( R_2 = \{(x,1); x \in E\} \) and \( R_1 = \{(x,x); x \in E\} \). Then it seems that \( R_6 \) is the most convenient Conditional, but the world of Logic has not Propositional Calculus as a fence.

It is easy to see that each \( R_i \) can be written as a function of the others. For example, it is \( R_6(a, b) = R_5(a, b')^\dagger = R_4(b, a')^\dagger = R_3(a', b) = R_2(1, a' + b) = R_1(1, a' + b) \).

2) The operations \( R_6, R_5, R_2 \) and \( R_1 \) verify the Modus Ponens-Inequality because \( a \cdot R_i(a, b) = a \cdot b \), and \( R_4 \) and \( R_3 \) through \( a \cdot R_i(a, b) = 0 \).

3) The only \( R_i \) that are commutative operations are \( R_6, R_3 \) and \( R_1 \). It is \( R_6(a, b) = R_6(b', a')^\dagger, \ R_4(a, b) = R_4(b, a')^\dagger \) and \( R_2(a, b) = R_2(b, a) \) if \( a = b \).

4) Defining \( R \) as reflexive if \( R(a,a) = 1 \) for any \( a \) in \( E \), and as transitive if \( R(a,b) \cdot R(b,c) \leq R(a,c) \) for any \( a, b, c \) in \( E \) (see [8]), \( R_6 \) and \( R_1 \) are reflexive and transitive but the other ones are only transitive. Hence, the only “preorders” are \( R_6 \) and \( R_1 \), and then \( R_6 \) and \( R_1 \) are classical preorders ([8]).

5) Defining \( R \) to be Monotonic if \( R(a,b) \leq R(a \cdot c, b) \) for any \( a, b, c \) in \( E \) ([8]), it is easy to see that \( R_6, R_4, R_3 \) and \( R_2 \) are Monotonic, but \( R_5 \) and \( R_1 \) are Non-Monotonic.

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2 Conditioning in Boolean algebras

2.1 Given \( a \) in \( E \), let \( aE = \{x \in E; x = a \cdot y, \text{ for some } y \in E\} \), i.e.,

\[
aE = \{z \in E; z \leq a\}.
\]

Subsets \( aE \) of \( E \) are characterized by \( a \), that is: \( aE = bE \) if and only if \( a = b \). In fact, if \( a = b \) it is obvious that \( aE = bE \), and reciprocally as \( a = a \cdot 1 \in aE \) it follows that \( a \in bE \) and \( b \in aE \), or \( a \leq b \) and \( b \leq a \).

Proposition 2.1.1. \( aE \) is a Boolean Algebra with least element \( \emptyset \) and greatest element \( a \), endowed with the restrictions of \( + \) and \( \cdot \), and with the complement defined by \( x^\dagger = a \cdot x^\dagger \).

It should be pointed out that in general \( aE \) is not a subalgebra of \( E \) and for a fixed \( a \) in \( E \), it is \( R_6(a, x) = a \cdot x \in aE \) when \( x \in E \). Since this reason we may say that the Boolean Algebra \( aE \) is obtained by “conditioning” the initial algebra \( E \) with \( R_6 \). Because of this \( aE \) is called the Conditioned Algebra of \( E \) by \( a \).
2.2. In Probability Theory one usually writes $E/a$ instead of $aE$ and the elements $x \in aE$ are also written as $y/a$, provided that $y \cdot a = x$. In this quotient-style notation we have that $y/a = z/a \iff y \cdot a = z \cdot a$, and as:

- $y \cdot a + z \cdot a = (y + z) \cdot a$, it is $y/a + z/a = (y + z)/a$
- $(y \cdot a) \cdot (z \cdot a) = (y \cdot z) \cdot a$, it is $y/a \cdot z/a = (y \cdot z)/a$
- $(y \cdot a)^* = a \cdot (y' + a') = a \cdot y'$, it is $(y/a)^* = y'/a$.

Evidently it is $y/a = a \cdot y/a = (a' + y)/a$.

**Proposition 2.2.1.** For a fixed $a \in E$ the only functions $F : E \times E \to E$ such that

$y/a = F(a, y)/a$,

are those functions satisfying,

$a \cdot y \leq F(a, y) \leq a' + y$.

**Proof.** From $y \cdot a = F(a, y) \cdot a$ it follows $F(a, y) \cdot a \cdot y' = 0$, or $F(a, y) \leq (a \cdot y')' = a' + y$. Analogously it also follows $y \cdot a \cdot F(a, y)' = 0$, or $F(a, y)' \leq (y \cdot a)'$, then $y \cdot a \leq F(a, y)$. Viceversa, from the inequalities we obtain $a \cdot y \leq a \cdot F(a, y) \leq a \cdot y$, or $a \cdot y = a \cdot F(a, y)$.

**Proposition 2.2.2.** For a fixed $a \in E$ the only boolean functions $F : E \times E \to E$ such that $y/a = F(a, y)/a$, are $R_0, R_5, R_2$ and $R_1$.

**Proof.** Like before it should be $F(a, y) = \alpha \cdot a \cdot y + \beta \cdot a' \cdot y + \gamma \cdot a \cdot y' + \delta \cdot a' \cdot y'$. From Proposition 2.2.1 it follows $a \cdot y \leq \alpha \cdot a \cdot y + \gamma \cdot a \cdot y' \leq a \cdot y$, or $\alpha \cdot a \cdot y + \gamma \cdot a \cdot y' = a \cdot y$, for any $y$ in $E$. Hence $\alpha = 1, \gamma = 0$ and the functions are $F(a, y) = a \cdot y + \beta \cdot a' \cdot y + \delta \cdot a' \cdot y'$. Varying $\beta$ and $\delta$ in $(0, 1)$ the proof is complete.

Given $a, y$ in $E$ there are only four boolean ways of writing $y/a$:

$y/a = a \cdot y/a = a' + y/a = a \cdot y + a' \cdot y'/a$,

and the upper term of each “quotient” is always the a-Section of some boolean Conditional.

2.3. As it was already stated, in a Boolean Algebra of propositions the operation $R_6$ is used to model the conditional propositions of the type “If $x$, then $y$”. In the Conditioned Algebra $aE$ this operation becomes $x \rightarrow_a y = x^* y = a \cdot x' + y = x \cdot a' + y = x'$. 

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(a + y) \cdot (x' + y) = a \cdot (x \to y)$. Consequently, when $x \to y$ in $E$ we have also $x \to_a y$ in $aE$. Nevertheless, when affirming $x \to_a y$ in some $aE$ we can only deduce in general that $a \leq x \to y$ in $E$, because $a = a \cdot (x \to y)$.

3 The case with finite probabilities

In what follows a probabilized Boolean Algebra $(E, p)$ will be considered, i.e., a Boolean Algebra $E$ endowed with a finite probability $p : E \to [0, 1]$ such that:

1) $p(1) = 1$ (axiom of normalization)

2) $p(x + y) = p(x) + p(y)$, whenever $x \cdot y = 0$ (axiom of finite additivity).

From these axioms it is easy to deduce the following well-known properties of any probability: $p(x') = 1 - p(x)$; $p(0) = 0$; $p(x + y) = p(x) + p(x' \cdot y)$; $p(x) = p(x \cdot y) + p(x' \cdot y')$; $p(x) + p(y) = p(x + y) + p(x \cdot y)$, and $p(x) \leq p(y)$ when $x \leq y$.

3.1. Let $aE$ be a Conditioned Boolean Algebra in $E$. The restriction $p_{aE}$ of $p$ to $aE$ is not a probability for $aE$ unless $a = 1$, and the necessary and sufficient condition for $p_{aE}$ to be identically zero is $p(a) = 0$. To avoid this, from now on we will consider $a \in E^+ = \{x \in E; p(x) > 0\}$.

A finite probability on $aE$ is a function $\tilde{p} : aE \to [0, 1]$ verifying the above axioms of normalization and finite additivity:

1) $p(a) = 1$, and 2) $p(x + y) = p(x) + p(y)$, whenever $x \cdot y = 0$, for any $x, y$ in $aE$.

Next definition gives a $\tilde{p}$ depending on both $p$ and $a$ (see [2]).

Definition 3.1.1. The function $\tilde{p}_a^r : aE \to [0, 1]$, $\tilde{p}_a^r(x) = p(x)/p(a)$, with $a \in E^+$ is a finite probability on $aE$ and it will be called the Internal-Conditional Probability.

Definition 3.1.2. The function $p_a^r : E \to [0, 1]$, $p_a^r(y) = p(a \cdot y)/p(a)$ is a finite probability on $E$ and it will be called the External-Conditional Probability.

As $p_a^r(1) = p(a)/p(a) = 1$, and when $x \cdot y = 0$ it is also $a \cdot (x \cdot y) = (a \cdot x) \cdot (a \cdot y) = 0$ and the $p_a^r(x + y) = p(a \cdot (x + y))/p(a) = p(a \cdot x + a \cdot y)/p(a) = (p(a \cdot x) + p(a \cdot y))/p(a) = p_a^r(x) + p_a^r(y)$, so $p_a^r$ is a finite probability on $E$.

These probabilities are formally different since they are defined in different Boolean Algebras and, of course, the restriction of $p_a^r$ to $aE$ is $p_{aE}^r$. In the Theory of Probability $^1$, $p_y^r(y)$ has been traditionally called the Conditional Probability of $y$ given $a$, and it is usually written as $p(y/a)$. From now on this notation will be assumed and instead of finite probabilities we will say just probabilities. Nevertheless, it should be noticed that $p(y/a)$ is actually a function of only $y \in E$.
as \( a \) is fixed in \( E^+ \) and, if considered as a two-place function \( p(\cdot / \cdot) : E \times E^+ \to [0,1] \) it is not a probability, at least because of \( E \times E^+ \) does not inherit the Boolean Algebra structure of \( E \) as there will be not, in general, a least element for the product order \( (x,y) \leq (x_1,y_1) \) \( \iff x \leq x_1 \) and \( y \leq y_1 \). This function \( p(\cdot / \cdot) \) is just a Fuzzy Relation between \( E^+ \) (also not a Boolean Algebra) and \( E \) (see [4], [5]).

**Remark.** Since \( p(y/a) = p(a \cdot y)/p(a) = p(R_5(a,y))/p(a) : = p_0(y/a) \), it is natural to think about the possibility of using in the same way the other conditionals \( R_i \). But a probability is rarely obtained since, for example: \( p_0(1/a) = p_2(1/a) = 1/p(a) \), \( p_4(1/a) = p_1(0/a) = p(a^s)/p(a) \) and \( p_3(1/a) = 0 \).

3.2. Let us now study the possible solutions of the equations:

\[
p^R_n(x) = p^R_n(z \rightarrow_a x) \quad \text{and} \quad p(y/a) = p(z \rightarrow y/a),
\]

in \( aE \) and \( E \) respectively.

**Proposition 3.2.1.** Given \( x \in aE \) a necessary and sufficient condition for the existence of \( z \in aE \) such that \( p^R_n(x) = p^R_n(z \rightarrow_a x) \) is given by \( p(a \cdot x',z') = 0 \). Two sufficient conditions are \( p(a \cdot z') = 0 \) and \( z = a \).

**Proof.** The equation holds if and only if \( p(x) = p(x^a + x) \), or \( p(x^a) - p(x^a \cdot x) \neq 0 \) equivalent to \( p(a \cdot z', x) = 0 \). Since \( a \cdot x' \cdot z' \leq a \cdot z' \) the first sufficient condition follows and, for this it is sufficient, e.g., to have \( a \cdot z' = 0 \), or \( a \leq z \) that actually means \( z = a \).

**Corollary 3.2.2.** For the equation \( p^R_n(x) = p^R_n(z \rightarrow_a x) \) to be verified for any \( x \in aE \), a necessary and sufficient condition is \( p(a \cdot z') = 0 \) and a sufficient condition is \( z = a \).

Analogously we have:

**Proposition 3.2.3.** Given \( y \in E \) a necessary and sufficient condition for the existence of \( z \in E \) such that \( p(y/a) = p(z \rightarrow y/a) \) is that \( p(a \cdot y',z') = 0 \). Two sufficient conditions are \( p(a \cdot z') = 0 \) and \( a \leq z \).

**Corollary 3.2.3.** For the equation \( p(y/a) = p(z \rightarrow y/a) \) to be verified for any \( y \in E \) a necessary and sufficient condition is \( p(a \cdot z') = 0 \) and a sufficient condition is \( a \leq z \).

The study of these equations shows that we have other solutions, besides the trivial one \( z = a \), depending, in the internal case, on the probability \( p \) if there are elements \( z \in aE \) such that \( p(a \cdot z') = 0 \) these elements are solutions of the equation \( p^R_n(x) = p^R_n(z \rightarrow_a x) \) and the Internal-Conditional Probability is the Internal-
Conditional Probability of a Conditional. In the External case, in addition to those 
\( z \in E \) such that \( p(a \cdot z') = 0 \), any \( z \geq a \) is a solution. Consequently, the general 
question (see [6], [7]) “Is a Conditional Probability the Probability of a 
Conditional?” has a positive answer, at least when the “probabilities” appearing 
on both sides of the equation are the same kind of Conditional Probability.

**Remark.** Proposition 3.2.1 is not formally a particular case of Proposition 3.2.3 
as the restriction of \( \to \) to \( aE \) is not \( \to_a \). In fact, when \( x = a \cdot x_1 \in aE \) and also 
\[ y = a \cdot y_1 \in aE \] it is:

\[-x \to_a y = a \cdot (x \to y) = a \cdot x_1 + a \cdot y_1 = a \cdot (x_1 \to y_1)\]

\[-x \to y = (a \cdot x_1)' + a \cdot y_1 = a' + x_1 + a \cdot y_1,\]

both expressions being different unless \( a = 1 \).

3.3. Let’s consider a different but well-known problem (see [6], [7]). Given \( a \in E^+ \) to 
solve the equation \( p(y/a) = p(z \to y) \) for \( y \in E \), where the unknown is \( z \in E \), 
considering the Conditional Probability \( p(y/a) \) and the “a priori” probability \( p \). 
The possibility of solutions, for a given \( y \) or for all \( y \), depend not only on the 
effectiveness of \( z \) but also on the effective range of \( p \) in \([0,1]\).

**Proposition 3.3.1.** A necessary but not sufficient condition for the existence of 
some solution of \( p(y/a) = p(z \to y) \) is \( p(y) \leq p(y/a) \), for each \( y \in E \).

**Proof.** From the equation follows \( p(y) \leq p(z' + y) = p(y/a) \). The reciprocal is 
evidently not true.

Without satisfying this condition the equation has no solutions for the given 
\( y \). In fact, from \( p(y/a) < p(y) \) and the equation, it follows \( p(z + y) > 1 \), because of 
\[ p(y/a) = p(z' + y) = p(y) + p(z' \cdot y') = p(y) + 1 - p(z + y) \].

In any case, the equation is equivalent to \( 1 - p(y/a) = 1 - p(z' + y) = p(z \cdot y') \), 
or to \( p(a \cdot y') / p(a) = p(z \cdot y') \). Thus, if the necessary condition is satisfied, the 
solutions are to be searched among the \( z \in E \) verifying this equation for the given 
\( y \), and depend on the available values of \( p \). For example, to be \( z = a \) a solution it 
should be \( p(a \cdot y') = p(a) \cdot p(a \cdot y') \), and this when \( p(a) < 1 \) implies \( p(a \cdot y') = 0 \) 
only for particular values of \( y \); with \( y = a' \) it is \( p(a) = 0 \).

Let’s consider the peculiar case \( p(a) = 1 \). The necessary condition is \( p(y) \leq 
p(a \cdot y) \); since \( p(a') = 0 \) we have that \( p(a' \cdot y) = 0 \) and \( p(y) = p(a \cdot y) \), and it is 
then satisfied. The equation is \( p(a \cdot y) = p(z' + y) = p(z') + p(y) - p(z' \cdot y) \), which 
is equivalent to \( p(z') - p(z' \cdot y) = p(a \cdot y) - p(y) \) and to \( p(z' \cdot y') = -p(a' \cdot y') = 0 \). 
The solutions then are to be searched among those \( z \) such that \( p(z + y) = 1 \), and 
for obtaining this it is sufficient to have \( z + y = 1 \), or \( y' \leq z \).
Examples. Let it be $E = 2^3$ with atoms $a_1, a_2, a_3$ and the uniform probability $p(a_i) = 1/3, 1 \leq i \leq 3$.

- If $a = a_1, y = a_2$ it is $p(a_2/a_1) = 0 < 1/3 = p(a_3)$. There is no solution for the equation $p(a_2/a_1) = p(z \to a_2)$.

- If $a = a_1, y = a_1 + a_3$, it is $p(a_1 + a_3) = 2/3 < 1 = p(a_1 + a_3/a_1)$: the necessary condition is satisfied. The equation being $1 = p(x' + a_1 + a_3)$, or $p(z, a_2) = 0$, there are the three solutions $z = a_1, a_2, a_1 + a_3$.

- If $a = a_1 + a_2, y = a_1$ it is $p(a_1) = 1/3 < 1/2 = p(a_1/a_1 + a_2)$: the necessary condition is satisfied. But now the equation $1/2 = p(x' + a_1)$ has no solutions because there is no $x' + a_1 \in E$ with probability $0.5$: the “a priori” probability has no enough values.

The first example shows that in the given probabilized Boolean Algebra there is no solution of the equation $p(y/a_1) = p(z \to y)$ for all $y$ in $2^3$.

4 Conclusions and final comments

In the setting of Boolean Algebras it was studied the problem of finding two-place functions to be used as Conditional Operations (to represent “If-then” propositions), and the problem of possibilities for a Conditional Probability to become the “a priori” probability of the Material Conditional.

With the above results it is intended to reach some level of clarification for a problem that was, and still is, studied by many logicians, Artificial Intelligence researchers and philosophers (see [6], [7]). Nevertheless, and apart from minor questions, the problem still remains open at least for what concerns $\sigma$-additive “a priori” probabilities: when considering a $\sigma$-Algebra and not a finitely additive probability the set of probability’s values could have a higher cardinality and, presumably, there will be space for new innovative ideas.

Let’s end the paper with a final comment. It is easy to see that the mapping $r_5 : E \to aE$, $r_5(y) = a \cdot y$, is an epimorphism between the Boolean Algebra $E$ and its subset $aE = R_5(a, E)$ endowed with the complement $x^\# = a \cdot x'$. This gives a method to prove that $aE$ is a Boolean Algebra.

Analogously, defining the mappings $r_6 : E \to R_6(a, E), r_6(y) = a' + y$, and $r_4 : E \to R_4(a, E), r_4(y) = a' \cdot y$, it is easy to see that both are epimorphisms between $E$ and, respectively, its subsets $a' + E = \{a' + y; y \in E\} = \{z \in E; a' \leq z\}$ and $a' E = \{a' \cdot y; y \in E\}$, with the complements $x^\# = a' + x'$ and $x^\# = a' \cdot x'$. Then also $a' + E$ and $a' E$ are Conditioned Boolean Algebras of $E$ by $a$.

Nevertheless, the cases $r_5(y) = a' \cdot y'$, $r_2(y) = a$ and $r_1(y) = a \cdot y + a' \cdot y'$, respectively for $R_5(a, E), R_2(a, E)$ and $R_1(a, E)$, don’t give Boolean Algebras.
The function \( p_a^b(y) = p(a' \cdot y)/p(a') \) is exactly \( p(y/a') \), the Conditional Probability of \( y \) by \( a' \). But \( p_a^b(y) = p(a' + y) \), although verifies \( p_a^b(1) = 1 \) is not additive as, for example, \( p_a^b(0 + 1) = p_a^b(1) = 1 \) and \( p_a^b(0) + p_a^b(1) = p(a') + 1 \).

Actually, only \( aE \) seems to be a boolean right way of Conditioning \((E, p)\).

References


