Mathware & Soft Computing 2 (1995) 61-70

On Inner Product Sapces - II

Pl. Kannappan
Dpt. of Pure Mathematics
University of Waterloo
Waterloo, ON N2L 3G1
Canada

Abstract

Among normal linear spaces, the inner product spaces (i.p.s.) are particularly interesting. Many characterizations of i.p.s. among linear spaces are known [2, 3, 4, 5, 6, 7, 8] using various functional equations. Three functional equations characterizations of i.p.s. are based on the Fréchet condition, the Jordan and von Neumann identity and the Ptolemaic inequality respectively. The object of this paper is to solve generalizations of these functional equations.

1991 Mathematics subjection classification 15A63, 39B52, 39B22.

1 Introduction

After the introduction of normal linear spaces (n.l.s.) by Banach, several basic characterizations of i.p.s. were given. It was Fréchet who in 1935 proved in [6] that a n.l.s. is an i.p.s if, and only if, every 3 dimensional subspace is an Euclidean 3 dimensional space. The functional equation resulting from *the Fréchet condition* is [see 9]

$$p(xyz) = p(xy) - p(z) + p(yz) - p(x) + p(zx) - py + \alpha. \qquad (F)$$

We study a generalization of (F) in the next section 2. After Fréchet's result, Jordan and von Neumann in [7] proved that a n.l.s. is an i.p.s. if, and only if, every 2 dimensional subspace is an Euclidean plane, that is, the norm satisfies the parallelogram identity, that is the *quadratic functional equation*,

$$q(x+y)+q(x-y)=2q(x)+2q(y). \qquad (QE)$$

A functional equation connected to (QE) is solved in section 3. Finally, in section 4 we solve a functional equation related to the Ptolematic inequality [see 3].

Let $\mathbb{R}$ denote the set of reals, $\mathbb{R}^+$ the set of non-negative reals, $\mathbb{R}^-$, the set of non-positive reals G denote a group and H a commutative group. A map $A: G \to H$ is called *additive*, provided $A(xy) = A(x)+A(y)$ holds for all $x, y \in G$. A map $B: G\times G \to H$ is *biadditive* if B is additive in each variable. We make use of the following two results in section 2.

Result 1: [8,9]. Every function $p: G \to H$ satisfying the functional equation (F) satisfies also

$$8p(x) = 4A(x) + B(x,x) + 8\alpha, \tag{1}$$

where $A: G \to H$ is additive and $B: G\times G \to H$ is biadditive.

Result 2: Suppose $f, g, h: G \to H$ satisfy the *Pexider functional equation*

$$f(xy) = g(x) + h(y), \quad \text{for } x, y \in G. \tag{2}$$

Then there exists and additive function $A: G \to H$ such that

$$f(x) = A(x) + b + c,\ g(x) = A(x) + b,\ h(x) = A(x) + c, \tag{3}$$

where a, b, c are arbitrary constants.

2 Solution of a generalization of the equation (F).

This section is devoted to the study of the following functional equation which is a generalization of the Fréchet condition (F):

$$p(xyz) = p_1(xy) - p_2(z) + p_3(yz) - p_4(x) + p_5(zx) - p_6(y). \qquad (4)$$

We prove the following theorem regarding (4).

Theorem 1 *Suppose that* $p, p_i : G \to H$, $(i = 1$ *to 6) satisfy the functional equation (4). Let* G *be a group (need not be Abelian), and* H *an Abelian group. Then there exist additive functions* $A, A_i : G \to H$ $(i = 1, 2, 3)$ *and a biaditive map* $B : G \times G \to H$ *such that* p, p_i*'s satisfy*

$$\left.\begin{array}{l}
8p(x) = 4A(x) + B(x,x) + 8c, \\
8p_1(x) = 4A(x) + B(x,x) - 8A_1(x) - 8(c_1 + c_2 - c - b_2), \\
8p_2(x) = 4A(x) + B(x,x) - 8A_2(x) - 8A_3(x) + 8b_2, \\
8p_3(x) = 4A(x) + B(x,x) - 8A_3(x) - 8(c_5 + c_6 - c - b_4), \\
8p_4(x) = 4A(x) + B(x,x) - 8A_1(x) - 8A_2(x) + 8b_4, \\
8p_5(x) = 4A(x) + B(x,x) - 8A_2(x) - 8(c_3 + c_4 - c - b_6), \\
8p_6(x) = 4A(x) + B(x,x) - 8A_1(x) - 8A_3(x) + 8b_6,
\end{array}\right\} \qquad (5)$$

where $c, b_2, b_4, b_6, c_1, c_2, c_3, c_4, c_5, c_6$ *are constants.*

Proof. Letting $z = 3$ in (4), we obtain

$$p(xy) - p_1(xy) + b_2 = p_3(y) - p_6(y) + p_5(x) - p_4(x)$$

where $p_2(e) = b_2$, which is a Pexider equation (2). So, by result 2, there is an additive A_1 such that

$$\begin{array}{rcl}
(p - p_1)(x) & = & A_1(x) + c_1 + c_2 - b_2 \\
(p_3 - p_6)(x) & = & A_1(x) + c_1, \; (p_5 - p_4)(x) = A_1(x) + c_2,
\end{array} \qquad (6)$$

where c_1, c_2 are arbitrary constants.

Similarly, setting $y = e$ and $x = e$ separately in (4), results in the Pexider equations

$$p(xz) - p_5(zx) + b_6 = (p_1 - p_4)(x) + (p_3 - p_2)(z),$$

and

$$p(yz) - p_3(yz) + b_4 = (p_1 - p_6)(y) + (p_5 - p_2)(z),$$

respectively with $p_4(e) = b_4$, $p_6(e) = b_6$ and the solutions

$$\left.\begin{array}{l} (p - p_5)(y) = A_2(y) + c_3 + c_4 - b_6, \\ (p_1 - p_4)(y) = A_2(y) + c_4,\ (p_3 - p_2)(y) = A_2(y) + c_3, \end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l} (p - p_3)(z) = A_3(z) + c_5 + c_6 - b_4, \\ (p_1 - p_6)(z) = A_3(z) + c_5,\ (p_5 - p_2)(z) = A_3(z) + c_6, \end{array}\right\} \tag{8}$$

where A_2, A_3 are additive, c_3, c_4, c_5, c_6 are constants.

Now use (6), (7) and (8) to express p_1 to p_6 in terms of p as.

$$\left.\begin{array}{ll} p_1(x) & = p(x) - A_1(x) - c_1 - c_2 + b_2, \\ p_5(y) & = p(y) - A_2(y) - c_3 - c_4 + b_6, \\ p_3(z) & = p(z) - A_3(z) - c_5 - c_6 + b_4, \\ p_2(y) & = p_3(y) - A_2(y) - c_3 = \\ & = p(y) - A_3(y) - A_2(y) - c_3 - c_5 - c_6 + b_4, \\ p_4(y) & = p_1(y) - A_2(y) - c_4 = \\ & = p(y) - A_1(y) - A_2(y) - c_4 - c_1 - c_2 + b_2, \\ p_6(x) & = p_3(x) - A_1(x) - c_1 = \\ & = p(x) - A_3(x) - A_1(x) - c_1 - c_5 - c_6 + b_4. \end{array}\right\} \tag{9}$$

Substitution of these p_1 to p_2 given by (9) into (4), using A_1, A_2 and A_3 additive result in

$$p(xyz) = p(xy) - p(z) + p(yz) - p(x) + p(zx) - p(y) + c \tag{10}$$

with $p(e) = c$, $b_2 = c - c_3 - c_5 - c_6 + b_4$, $b_6 = c - c_1 - c_5 - c_6 + b_4$ and $c_1 + c_2 + c_4 = c_3 + c_5 + c_6$. Equation (10) is precisely (F). Thus by result 1, there exists an additive A and a biadditive B such that

$$8p(x) = 4A(x) + B(x, x) + 8c. \tag{11}$$

Now the sought for (5) is obtained from (11) and (9). This completes the proof of this theorem.

3 An equation to (QE).

The functional equation related

$$g(g(x)y + g(y)x) = g(x)g(y)g(x + y), \tag{12}$$

has been studied by many authors in connection with a characterization of i.p.s. under some additional conditions satisfied by g, see [2,5,10]. Here we obtain the continuous solution of (12) under different conditions first by a simple and direct method and then the general solution by reducing it to the quadratic equation (QE). As a matter of fact we prove the following theorem.

Theorem 2 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that g is 1-1 both on the non-negative reals and on the non-positive reals, that is, $\{g/\mathbb{R}^+ : \mathbb{R}^+ \to \mathbb{R}$ and $g/\mathbb{R}^- : \mathbb{R}^- \to \mathbb{R}$ are injective $\}$. Then g is a solution of the functional equation (12) holding for all $x, y \in \mathbb{R}$, if, and only if,*

$$g(x) = cx^2, \text{ for } x \in \mathbb{R} \tag{13}$$

where c is a constant.

Proof. Setting $x = 0 = y$ in (12) gives $g(0) = g(0)^3$. So, either $g(0) = 1$ or $g(0) = -1$ or $g(0) = 0$.

Let $g(0) = 1$. Then $x = 0$ in (12) yields $g(y) = g(y)^2$ for all $y \in \mathbb{R}$. Hence $g(y) = 0$ or 1, contradicting g is 1-1 on $\mathbb{R}^+$.

Now consider the case $g(0) = -1$. With $x = 0$ in (12), we get $g(-y) = -g(y)^2$, for all $y \in \mathbb{R}$, that is, $g(y) = -g(-y)^2 = -g(y)^4$, again contradicting the one-oneness of g on $\mathbb{R}^+$.

Thus $g(0) = 0$. Taking $y = -x$ in (12), we have $g(-g(x)x + g(-x)x) = 0$, that is, $-xg(x) + xg(-x) = 0$ for all x, since $g(0) = 0$ and g is 1-1 on $\mathbb{R}^+$ and $\mathbb{R}^-$. Therefore, g is even, that is,

$$g(-x) = g(x), \text{ for } x \in \mathbb{R} \tag{14}$$

First set $y = x$ and then separately $y = -2x$ in (12) and use (14) to obtain

$$g(2xg(x)) = g(x)^2 g(2x) = g(-2xg(x) + g(2x)x).$$

Since g is even and 1-1 on non-negative reals, we can conclude either $-2xg(x) = -2xg(x) + g(2x)x$, that is, $g(2x) = 0$, for $x \neq 0$ which is not possible, or $2xg(x) = -2xg(x) + g(2x)x$, that is,

$$g(2x) = 2^2 g(x), \text{ for all } x \in \mathbb{R}. \tag{15}$$

Now put $y = 2x$ and $y = -3x$ separately in (12) and use (14) and (15) to have

$$\begin{aligned} g(2g(x)x + 4g(x)x) &= 4g(x)^2 g(3x) = \\ &= g(-3xg(x) + xg(3x)), \text{ for all } x \in \mathbb{R}. \end{aligned} \tag{16}$$

As before, either $6xg(x) = 3xg(x) - xg(3x)$, in which case $g(3x) = -3g(x)$ for all $x \in \mathbb{R}$. With $y = 3x$ in (12), we get $g(3xg(x)+g(3x)x) = 16g(x)^2 g(3x)$, that is, $-48g(x)^3 = 0$ for all x, which cannot be. So, $6xg(x) = -3xg(x) + xg(3x)$, or

$$g(3x) = 3^2 g(x), \text{ for all } x \in \mathbb{R}.$$

By induction, it follows that

$$g(nx) = n^2 g(x), \tag{17}$$

for all integers n and $x \in \mathbb{R}$. Now (17) yields $g\left(\frac{x}{n}\right) = \frac{1}{n^2} g(x)$, for $n \neq 0$ and

$$g\left(\frac{m}{n}x\right) = \left(\frac{m}{n}\right)^2 g(x), \text{ for rational } \frac{m}{n} \text{ and } x \in \mathbb{R}. \tag{18}$$

We invoke the continuity of g to obtain (13) where $c = g(1)$ is a constant. This completes the proof of this theorem.

Remark 1. Traditionally the solution (13) is associated to the quadratic equation (QE): so, we will derive (QE) from (12) and the hypothesis of theorem 2. Indeed we prove theorem 2 by linking (12) to the quadratic equation (QE).

Note that g is even (14) and that (15) $g(2x) = 4g(x)$.

Changing y into $y - x$ in (12) we have

$$\begin{aligned} g(g(x)(y-x) + g(y-x)x) &= g(x)g(y-x)g(y) \\ \text{also} &= g(-g(x)y + g(y)x). \end{aligned}$$

Now since g is even and 1-1 on $\mathbb{R}^+$ we can conclude that either $g(x)(y-x)+xg(y-x)=g(x)y-xg(y)$, that is, $xg(y-x)=x(g(x)-g(y))$ for all $x,y\in\mathbb{R}$ which for $y=2x$ gives $xg(x)=x(g(x)-4g(x))$, that is, $g(x)=0$ for $x\neq 0$ which cannot be. So,

$$xg(x-y)=x(g(x)+g(y))-2yg(x),\ \text{for } x,y\in\mathbb{R}. \tag{19}$$

Replacing y by $-y$ in (19) and adding the resultant equation to (19) and using g even, we get

$$x(g(x+y)+g(x-y))=2x(g(x)+g(y)),\ \text{for all } x,y\in\mathbb{R}.$$

Hence g satisfies the equation (QE), the continuous solution of which is given by (13) [see 1].

Remark 2: General solution of (12). The general solution $g:\mathbb{R}\to\mathbb{R}$ satisfying (12) with g one-one on $\mathbb{R}^+$ and $\mathbb{R}^+$ is given by

$$g(x)=B(x,x)$$

where $B:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a biadditive map.

Proof. From the proof in Remark 1, we can conclude that g satisfies the quadratic equation (QE). Then from [2, p. 166] we get that $g(x)=B(x,x)$ where B is biadditive.

Note that here we have obtained the solution of (12) without assuming any regularity condition on g.

4 Solution of an equation related to Ptolemaic inequality.

In [3], the authors motivated by the Ptolemaic inequality studied the functional equation

$$f(x-y)=f(x)f(y)f\left(\frac{x}{f(x)^2}-\frac{y}{f(y)^2}\right),\ \ \text{for } x,y(\neq 0)\in\mathbb{R}, \tag{20}$$

in connection with i.p.s. under some additional conditions satisfied by f. In this section we obtain the solution of (20) under a slightly different condition and then under the same conditions as in [3] by presenting a simple and direct proof.

Theorem 3 *Let $f : \mathbb{R} \to \mathbb{R}^+$ be strictly increasing on $\mathbb{R}^+$ and satisfy the functional equation (20) for all non zero reals x, y,*

$$f(x)f\left(\frac{2x}{f(x)^2}\right) = 2, \quad \textit{for } x \neq 0, \tag{21}$$

and

$$f(-x) = f(x), \quad \textit{for all } x \in \mathbb{R} \textit{ (that is, } f \textit{ is even).} \tag{22}$$

Then f has the form

$$f(x) = c|x|, \ \textit{for positive } c \textit{ and } x \in \mathbb{R}. \tag{23}$$

Proof. First $y = x$ in (20) gives $f(0) = f(0)f(x)^2$. Since f is strictly increasing, f is 1-1 on $\mathbb{R}^+$, $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$. With $y = -x$ (20) using (21) and (22) gives

$$f(2x) = f(x)^2 f\left(\frac{2x}{f(x)^2}\right) = 2f(x), \ \text{for } x \in \mathbb{R}. \tag{24}$$

Set $y = -3x$ in (20) and utilize (21), (22) and (24) to get

$$\begin{aligned} f(x)f(3x)f\left(\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2}\right) &= f(4x) = 4f(x) \\ &= 2f(x)f(3x)f\left(\frac{6x}{f(3x)^2}\right), \end{aligned}$$

that is,

$$f\left(\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2}\right) = f\left(\frac{12x}{f(3x)^2}\right),$$

since $f(x) \neq 0$ for $x \neq 0$. Then since f is 1-1 on $\mathbb{R}^+$, either $\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2} = -\frac{12x}{f(3x)^2}$ which cannot be since $f(x) > 0$ for $x \neq 0$ or $\frac{x}{f(x)^2} + \frac{3x}{f(3x)^2} = \frac{12x}{f(3x)^2}$, that is, since $f(x) > 0$ for $x \neq 0$,

$$f(3x) = 3f(x), \ \text{for } x \in \mathbb{R}. \tag{25}$$

From (24) and (25) we obtain for all integers m, n

$$f(2^n 3^m) = 2^n 3^m f(1). \tag{26}$$

The set $\{2^n 3^m : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}^+$, and since f is strictly increasing (this is the only place we need this) $f(x) = cx$, for $x \in \mathbb{R}^+$ and positive c. From (22) now we obtain (23). This completes the proof of this theorem.

Remark 3: Suppose (21)' $f(x)f\left(\frac{x}{f(x)^2}\right) = 1$, for $x \neq 0$ holds instead of (21) in Theorem 3. Then

$$f(x) = f(2x - x) = f(2x)f(x)f\left(\frac{2x}{f(2x)^2} - \frac{x}{f(x)^2}\right),$$

that is, since $f(x) \neq 0$ for $x \neq 0$,

$$f(2x)f\left(\frac{2x}{f(2x)^2} - \frac{x}{f(x)^2}\right) = 1 = f(2x)f\left(\frac{2x}{f(2x)^2}\right)$$

from which follows that $f(2x) = 2f(x)$. Now (21) holds and we obtain the solution (23) as in [3].

Acknowledgment

I thank the referee for pointing out an error in the original theorem 2. I thank Professor J. Aczél for his useful commets in preparing this paper.

References

[1] Aczél, J., *Lectures on functional equations and their applications*, Academic Press, New York-London, 1966.

[2] Aczél, J. and Dhombres, J., *Functional equations in several variables*, Cambridge University Press, Cambridge, 1989.

[3] Alsina, C., and Garcia-Roig, J.L., On a functional equation related to the Ptolemaic inequality, *Aeq. Math.*, **34** (1987), 298-303.

[4] Amir, D., *Characterizations of inner product spaces*, Birkhauser-Verlag, Basel, 1986.

[5] Dhombres, J., *Lectures on some aspects of functional equations*, Cherlalongkorn University, Bangkok, Thailand, 1979.

[6] Fréchet, M., Sur la definition axiomatique d'une classe d'espaces vertoriels distancies applicables vertoriellement sur l'espace de Hilbert, *Ann. Maths.,* **36** (1935), 705-718.

[7] Jordan, P. and von Neumann, J., On inner products in linear metric spaces, *Ann. Maths.,* **36** (1935), 719-723.

[8] Kannappan, Pl., On inner product spaces - I, to appear in *Math. Jap.*.

[9] Kurepa, S., On bimorphisms and quadratic forms on groups, *Aeq. Math.*, **9** (1973), 30-45.

[10] Volkmann, P., Eine Characterisierung der positiv definiten quadratischen Formen, *Aeq. Math.,* **11** (1974), 174-181.