On some constructions of new triangular norms

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Abstract

We discuss the properties of two types of construction of a new t-norm from a given t-norm proposed recently by B. Demant, namely the dilatation and the contraction. In general, the dilatation of a t-norm is an ordinal sum t-norm and the continuity of the outgoing t-norm is preserved. On the other hand, the contraction may violate the continuity as well as the non-continuity of the outgoing t-norm. Several examples are given.

Keywords: contraction, dilatation, ordinal sum, triangular norm.

1 Introduction

Among several constructions of the new t-norms form given ones [2,3,4], we recall two basic constructions arisen from the semigroup interpretation of a triangular norm, see Schweizer and Sklar [3].

Ordinal sum: Let \([a_k, b_k]; \ k \in \mathcal{K}\] be a disjoint system of open subintervals of the unit interval \([0,1]\] and let \([T_k; \ k \in \mathcal{K}]\] be a system
of given t-norms. For $x, y \in [0, 1]$, put

$$
T(x, y) = \begin{cases} 
  a_k + (b_k - a_k)T_k ((x - a_k)/(b_k - a_k)), & \text{if } x, y \in [a_k, b_k] \\
  (y - a_k)/(b_k - a_k) & \text{for some } k \in K \\
  \min(x, y) & \text{otherwise}
\end{cases}
$$

Then $T$ is a t-norm and it is called an ordinal sum with summands $<a_k, b_k, T_k>$, $k \in K$, briefly $T \sim [<a_k, b_k, T_k>; k \in K]$.

**Semigroup deformation:** let $\phi : [0, 1] \to [a, 1]$, $a \in [0, 1]$, be an increasing bijection. Let $T$ be a given t-norm. For $x, y \in [0, 1]$ put

$$
T_{\phi}(x, y) = \phi^{-1}(T(\phi(x), \phi(y)),
$$

where $\phi^{-1} : [0, 1] \to [0, 1]$ is the pseudo-inverse of $\phi$, $\phi^{-1}(x) = \phi^{-1}(\max(a, x))$. Note that if $a = 0$ then $T_{\phi}$ is called a $\phi$-transformation of $T$ and $T$ and $T_{\phi}$ are isomorphic (and hence the properties such as continuity, strictness, etc., are preserved). If $a > 0$ then the deformation $T_{\phi}$ preserves the continuity and the Archimedean property (and the nilpotency) but the strictness may be violated. Take, e.g., the product t-norm $T_P$ and let $\phi(x) = 2^{x-1}$, i.e., $a = 1/2$ and $\phi^{-1}(x) = 1 + \log_2 x$. Then

$$
(T_P)_{\phi}(x, y) = 1 + \log_2 \max(1/2, 2^{x-1} \cdot 2^{y-1}) = \max(0, x + y, -1).
$$

Hence the $\phi$-deformation of the strict product t-norm $T_P$ is the nilpotent Lukasiewicz t-norm $T_L$, $(T_P)_{\phi} = T_L$.

Recently, Demant [1] has suggested two new types of t-norm constructions. Let $\phi : [0, 1] \to [0, a]$, $a \in [0, 1]$, be a given increasing bijection. For $x \in [0, 1]$ we define the pseudo-inverse of $\phi$ by $\phi^{-1}(x) = \phi^{-1}(\min(a, x))$. Let $T$ be a given t-norm. For $x, y \in [0, 1]$, we define:

**Contraction:**

$$
T^{(\phi)}(x, y) = \begin{cases} 
  \phi^{-1}(T(\phi(x), \phi(y)) & \text{if } \max(x, y) < 1 \\
  T(x, y) & \text{otherwise}
\end{cases}
$$
Dilatation:

\[ T_{(\phi)}(x, y) = \begin{cases} 
\phi \left( T(\phi^{-1}(x), \phi^{-1}(y)) \right) & \text{if } T(x, y) < a \\
T(x, y) & \text{otherwise}
\end{cases} \]

Both the contraction and the dilatation of a t-norm \( T \) are again t-norms, see [1]. Note that in the case \( a = 1 \) both the dilatation and the contraction are the usual semigroup transformation of Schweizer and Sklar [3], \( T^{(\phi)} = T_\phi \) and \( T_{(\phi)} = T_{\phi^{-1}} \). Further note that the only t-norms preserved by arbitrary deformation, transformation, and contraction are the limit t-norms \( T_M \) and \( T_W \). However, the only t-norm preserved by an arbitrary dilatation is \( T_M \), while \( (T_W)_{(\phi)} \neq T_W \) whenever \( \phi(1) \neq 1 \).

2 Contractions of t-norms

For a given bijection \( \phi : [0, 1] \to [0, a] \) with \( a < 1 \) and a given t-norm \( T \), the values of the contraction \( T^{(\phi)} \) on the half-open square \([0, 1]^2\) depend on the values of \( T \) on the half-open square \([0, a]^2\) only (the remainder of the domain is contained in the borders of the unit square where all t-norms coincide). Hence the non-continuity (and the absence of the Archimedean property) of \( T \) need not be true for its contraction \( T^{(\phi)} \). On the other hand, the continuity of \( T \) may be violated by \( T^{(\phi)} \), too, while the Archimedean property remains preserved.

Example 1

i) Let \( \phi(x) = a \cdot x, \ a \in [0, 1], \) and let \( T = T_P \). Then the contraction \( T^{(\phi)} \) is defined by

\[ T^{(\phi)}(x, y) = \begin{cases} 
 a \cdot x \cdot y & \text{if } \max(x, y) < 1 \\
x \cdot y & \text{otherwise}
\end{cases} \]

Note that \( T^{(\phi)} \) is not continuous although \( T \) is continuous. Further, the strictness \( T^{(\phi)}(x, y) < T^{(\phi)}(x, z) \) for each \( x > 0, \ y < z, \) holds true.
ii) Let \( \phi(x) = a \cdot x, a \in [0,1] \), and let \( T \sim \left[ < 0, a, T_P >, < a, 1, T_W > \right] \) be an ordinal sum t-norm. Then \( T \) is non-continuous (and non-Archimedean) but \( T^{(\phi)} = T_P \) is continuous and Archimedean.

For a composition law of two contractions we have the following result.

**Proposition 1** Let \( \phi : [0,1] \rightarrow [0,a] \) and \( \psi : [0,1] \rightarrow [0,b] \) be two bijections with \( a \leq 1 \) and \( b \leq 1 \). Let \( T \) be a given t-norm. Then
\[
\left[ T^{(\phi)} \right]^{(\psi)} = T^{(\phi \circ \psi)},
\]
i.e., the \( \psi \)-contraction of a \( \phi \)-contraction of \( T \) is the \( \phi \circ \psi \)-contraction of \( T \).

The problem of t-norms invariant under given \( \phi \)-contraction will be partially solved in the next section.

## 3 Dilatations of t-norms

Non-trivial dilatations (i.e., when \( a < 1 \)) are always ordinal sums with two summands.

**Proposition 2** Let \( \phi : [0,1] \rightarrow [0,a] \) be a given increasing bijection where \( a \in [0,1] \) and let \( T \) be a given t-norm. Then the \( \phi \)-dilatation of \( T \) is an ordinal sum with two summands,
\[
T_{(\phi)} \sim \left[ < 0, a, T_{\phi/\alpha} >, < a, 1, T_a > \right],
\]
where \( T_{\phi/\alpha} \) is the transformation \( T \) of with respect to the mapping \( \phi/\alpha : [0,1] \rightarrow [0,1] \), while \( T_a \) is the deformation of \( T \) with respect to the linear transformation \( \lambda_a : [0,1] \rightarrow [a,1], \lambda_a(x) = a + (1 - a) \cdot x \), depending only on \( T \) and \( a \) (independent of \( \phi \) up to the value \( a = \phi(1) \)), \( T_a = T_{\lambda_a} \),
\[
T_a(x, y) = \left( \max \left[ 0, T(a + (1 - a) \cdot x, a + (1 - a) \cdot y) - a \right] \right) / (1 - a).
\]
Remark 1 For Archimedean continuous t-norms we have the following result: let \( f \) be an additive generator of a given t-norm \( T \) \([5]\) and let the left derivative of \( f \) in the point 1 be non-trivial, \( f'_L(1) \in ]-\infty, 0[ \). Then \( \lim_{a \to 1^-} T_a = T_L \), where \( T_L \) is the Lukasiewicz t-norm. The proof follows from the fact that if \( T \) has an additive generator \( f \) then \( T_a \) has an additive generator \( f(a + (1-a) \cdot x) \). 

It is easy to see that for arbitrary dilatation the t-norm \( T_M \) remains stable. Further, for each \( a \in ]0, 1[ \) it is \( [T_W]_a = T_W \) and hence for arbitrary \( \phi \) it is \( [T_W]_{\phi} \sim \left[ <0, a, T_W >, < a, 1, T_W > \right] \). Applying the \( \phi \)-dilatation to \( T_W \) infinitely many times we get a new t-norm \( T_a^* \) depending only on \( a \) and invariant under \( \phi \)-dilatation,

\[
T_a^* \sim \left[ < a^n, a^{n-1}, T_W >; \; n \in \mathbb{N} \right].
\]

A natural question arises: for a given transformation \( \phi \), are there some other \( \phi \)-dilatation invariant t-norms up to \( T_M \) (a continuous t-norm) and \( T_a^* \) (a discontinuous t-norm)? It is obvious that each \( \phi \)-dilatation invariant t-norm \( T^* \) different from \( T_M \) should be an ordinal sum of type

\[
T^* \sim \left[ < a^n, a^{n-1}, T >; \; n \in \mathbb{N} \right],
\]

where \( T \) is a t-norm such that \( T = T_a = T_{\phi/a} \). Requiring the continuity of \( T \), we have the following result.

Proposition 3 Let \( T \) be a continuous t-norm and let \( a \in ]0, 1[ \). Then \( T_a \) equals \( T \) if and only if \( T \) is the member of the extended Yager’s family \( \left\{ T_p, p \in ]0, \infty[ \right\} \) \([6]\), i.e., \( T_\infty = T_M \) and for \( p \in ]0, \infty[ \), the t-norm \( T_p \) is generated by an additive generator \( f_p \), \( f_p(x) = (1 - x)^p \), \( x \in [0, 1] \).

Note that the proof is based on a modified Cauchy functional equation. Further, let \( f \) be an additive generator of a given t-norm \( T \). Then \( T_{\phi/a} \) has an additive generator \( f \circ (\phi/a) \), see \([5]\), and thus \( T \) equals...
$T_{\phi/a}$ if and only if $f$ differs from $f \circ (\phi/a)$ only by a multiplicative constant. For nilpotent t-norm $T$ with the normed generator $f$ (this is the case of the Yager’s t-norms) this means that $\phi/a$ is the identity, i.e., $\phi(x) = a \cdot x$. We have just shown the next result.

**Proposition 4** Let $\phi : [0, 1] \rightarrow [0, a]$, where $a \in [0, 1]$, be an increasing bijection. If $\phi$ is not linear then the only continuous t-norm invariant under $\phi$-dilatation is the strongest t-norm $T_M$. If $\phi$ is linear, then the only continuous t-norms invariant under $\phi$-dilatation are the members of the family $[T_{a,p}^*; p \in [0, \infty]]$, where $T_{a,p}^* \sim \left< a^n, a^{n-1}, T_p^* \right>; n \in \mathbb{N}$.

Note that $T_{a,1}^* = T_M$. Further, it is usual to put $T_{0}^* = T_W$ (the left limit member of the Yager family). Then each member of the family $[T_{a,p}^*; p \in [0, \infty]]$, where $T_{a,0}^* = T_{a}^*$, is invariant under the $\phi$-dilatation for $\phi(x) = a \cdot x$.

**Remark 2** Note that the $\phi$-contraction acts as an inverse of the $\phi$-dilatation, the opposite being not true, i.e., for arbitrary t-norm $T$ it is $\left[T_{(\phi)}^* \right]^{(\phi)} = T$. Now, it is obvious that if a given t-norm $T$ is invariant with respect to a given $\phi$-dilatation it has to be invariant also with respect to the corresponding $\phi$-contraction.

**References**


