On a new class of distances between fuzzy numbers

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Abstract

In the course of the studies on fuzzy regression analysis, we encountered the problem of introducing a distance between fuzzy numbers, which replaces the classical \((x - y)^2\) on the real line. Our proposal is to compute such a function as a suitable weighted mean of the distances between the \(\alpha\)-cuts of the fuzzy numbers. The main difficulty is concerned with the definition of the distance between intervals, since the current definitions present some disadvantages which are undesirable in our context. In this paper we describe an approach which removes such drawbacks.

Keywords: Fuzzy numbers; Fuzzy distance; Topological equivalence

1 Basic definitions and notations

We recall here some classical definitions and results of the fuzzy set and fuzzy number theories. We will identify the fuzzy quantities with
their membership functions and represent both of them by means of an overlining tilde.

**Definition 1.1 (6.3)** A fuzzy number is a convex and normal fuzzy subset of the real line \( \mathbb{R} \), that is, a map (membership function)

\[
\tilde{A} : \mathbb{R} \rightarrow [0, 1]
\]

with the following properties

- \( \tilde{A}[(\lambda x + (1 - \lambda)y) \geq \inf[\tilde{A}(x), \tilde{A}(y)] \),
- \( \exists \pi \in \mathbb{R} \) such that \( \tilde{A}(\pi) = 1 \),
- \( \text{Supp}(\tilde{A}) \) is a compact subset of \( \mathbb{R} \).

It is easy to recognize that all the \( \alpha \)-cuts \( \{ A_\alpha | \alpha \in (0, 1] \} \) of a fuzzy number are intervals, and so is the crisp subset \( A_0 = \text{Supp}(\tilde{A}) \); let us denote by \( a^{(1)}(\alpha) \) and \( a^{(2)}(\alpha) \) respectively, the first and second endpoints of these intervals, no matter if they belong or not to the \( \alpha \)-cut. Functions \( a^{(i)} \), \( i = 1, 2 \) satisfy the following properties

\[
\text{Dom}[a^{(i)}] = [0, 1] \quad (1.2.a)
\]

- \( a^{(1)} \) is not decreasing
- \( a^{(2)} \) is not increasing
- \( a^{(1)}(1) \leq a^{(2)}(1) \)
- \( a^{(i)} \in L^2([0, 1]) \)

Properties (1.2.a)(1.2.d) are evident; (1.2.b)(1.2.c) follow from the fact that \( \alpha < \beta \) implies \( A_\beta \subseteq A_\alpha \). If \( A_0 \) is compact, then (1.2.e) follows from the monotonicity; otherwise condition (1.2.e) will be imposed in this context. We can also note that functions \( a^{(1)}, a^{(2)} \) are continuous almost everywhere.

A fuzzy number \( \tilde{A} \) completely determines the two functions \( a^{(1)}, a^{(2)} \), but on the contrary the knowledge of the two functions \( a^{(i)}, i = 1, 2 \) with the properties (1.2), does not allow us to uniquely determine the corresponding fuzzy number; if \( a^{(i)} \) takes the same
value \( \bar{x} \) on an interval \([\alpha, \beta]\), then \( \bar{A}(\bar{x}) \) may be any value between \( \alpha \) and \( \beta \). Nevertheless, since \( a^{(i)} \) are monotonic, the number \( \bar{A} \) is determined except for, at most, a countable subset of \( \mathbb{R} \). In order to introduce the notion of distance, this property allows us to represent a fuzzy number by means of the endpoints of its \( \alpha \)-cuts,

\[
\bar{A} = \{ [a^{(1)}(\alpha), a^{(2)}(\alpha)] \mid \alpha \in [0, 1] \} \tag{1.3}
\]

because in our definition, as well as in the ones usually considered, two numbers \( \bar{A}, \bar{B} \) which differ at most in a countable number of points will have distance zero.

**Definition 1.2** The sum of two fuzzy numbers is defined, using the extension principle ([4]), by means of their \( \alpha \)-cuts as

\[
(\bar{A} + \bar{B})_\alpha = [a^{(1)}(\alpha) + b^{(1)}(\alpha), a^{(2)}(\alpha) + b^{(2)}(\alpha)] \tag{1.4}
\]

It is easy to recognize that the subset of the fuzzy numbers compounded by the functions

\[
\tilde{\delta}_\alpha(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{if } x \neq a
\end{cases} \tag{1.5}
\]

is isomorph to the algebra of the real numbers, so we can identify the real line with this particular subset.

## 2 The distance

In order to define the distance \( D(\bar{A}, \bar{B}) \) between two fuzzy numbers \( \bar{A} \) and \( \bar{B} \), we introduce a distance \( d(A_\alpha, B_\alpha) \) between their corresponding \( \alpha \)-cuts and then define \( D(\bar{A}, \bar{B}) \) as a suitable weighted mean of the values \( d(A_\alpha, B_\alpha) \). The crucial point of this process is the definition of \( d(A_\alpha, B_\alpha) \); so our first task consists on defining a measure of the distance between two intervals. In order to justify our proposal we present the reasons which prevent us for utilizing the two
best known kinds of distance at least as long as we are constraint to a fuzzy statistical regression context.

(a) The first kind of distance is the Hausdorff one \(d_H(A, B)\), defined by
\[
d_H(A, B) = \max[\delta(A, B), \delta(B, A)]
\]
where \(\delta(X, Y) = \sup_{x \in X} \inf_{y \in Y} |x - y|\). It is easy to realize that it assigns the same distance to, for instance, the two pairs of intervals \(A = [0, 5]\), \(B = [6, 7]\) and \(\hat{A} = [0, 5]\), \(\hat{B} = [6, 10]\) (fig. 1a below), whereas it seems natural for the distance between the second pair to be greater than the distance between the first one.

(b) The second kind of distance between intervals \(d^\ast(A, B)\), is a suitable “combination” of the two differences \(|a^{(1)} - b^{(1)}|\) and \(|a^{(2)} - b^{(2)}|\). The best knowns are
\[
d_1(A, B) = \frac{1}{2} |a^{(1)} - b^{(1)}| + |a^{(2)} - b^{(2)}|
\]
\[
d_2(A, B) = \sqrt{\frac{1}{2}[(a^{(1)} - b^{(1)})^2 + (a^{(2)} - b^{(2)})^2]}
\]
Both expressions, as well as any other “combination” of \(|a^{(1)} - b^{(1)}|\) and \(|a^{(2)} - b^{(2)}|\), attribute the same distance to the two pairs \(A = [-2, 2]\), \(B = [-1, 1]\) and \(\hat{A} = [-2, 1]\), \(\hat{B} = [-1, 2]\) (fig. 1b above), but once again we think that the distance between the second pair has to be greater.

In fact, the intervals \(A\) and \(B\) are both centered at the same value (zero), whereas the intervals \(\hat{A}\) and \(\hat{B}\) are left and right shifted, respectively. We think that some difference between these two situations has to be underlined at least when \(d\) is used to construct a distance between fuzzy numbers.

Let us consider, for example, the two pairs of fuzzy numbers \((\hat{A}, \hat{B})\) and \((\hat{A}', \hat{B}')\) showed in fig. 2.
In the first pair, both numbers are centered at zero (they represent, with more or less precision, the same value “0”). In the second pair the fuzzy number $\tilde{A}'$ is tendentially negative, whereas $\tilde{B}'$ is tendentially positive (they represent respectively “$0^-$” and “$0^+$”). It seems quite natural to pretend that some difference between the distance of the two pairs of numbers exists, but the function $d^*$ assigns the same distance to the pair $\tilde{A}$ and $\tilde{B}$ as to $\tilde{A}'$ and $\tilde{B}'$, because $d^*(A_\alpha, B_\alpha) = d^*(A'_\alpha, B'_\alpha)$ for all $\alpha \in [0, 1]$.

Let us remark that the observations made below are meaningful when dealing with a distance between fuzzy numbers, but they do not refer to other situations where the two classical distances have been successfully employed.

The definition we propose here does not have the above quoted disadvantages; it is a generalization of that introduced in [1,5], which allowed us to guarantee the existence of the best interpolating fuzzy polynomial in a regression context.

**Definition 2.1** Let $\mu$ be a normalized weight measure on $(\mathbb{R}, \mathcal{B}([0,1]))$. The squared distance $d^2$ between two intervals $A = [a^{(1)}, a^{(2)}], \quad B = [b^{(1)}, b^{(2)}]$ is given by

$$d^2(A, B) = \int_0^1 [t \cdot |a^{(1)} - b^{(1)}| + (1 - t)|a^{(2)} - b^{(2)}|^2] \, dg(t) \quad (2.1)$$

**Remark 1.** As $d$ also depends on the choice of measure $g$, a more appropriate notation for these distances would be $d_{\alpha^*}$. Nevertheless, subindex $g$ will be omitted as long as no equivocations appear.
It is easy to recognize that the distance \( d \) between the second pair of intervals of the above examples is strictly greater than that between the first one, unless (in the second case) the probability measure \( g \) is concentrated on 0 and 1, since then \( d \) reduces to \( d^2_2 \).

In this paper we restrict ourselves to weight measures which are the sum of a term which is continuous with respect to the Lebesgue measure and of a finite weight distribution placed at \( S \) points \( t_1 \ldots t_S \), that is

\[
 dg = \gamma(t) \, dt \quad \text{with} \\
\gamma(t) = \tau(t) + \sum_{s=1}^{S} \delta(t - t_s) \tag{2.2}
\]

where \( \tau \) is a Lebesgue measurable function, and \( \delta \) is the Dirac distribution \( \delta(t - t_s) = 0 \forall t \neq t_s \) and \( \int \delta(t - t_s) \, dt = 1 \) in all the intervals containing \( t_s \). In this case (2.1) becomes

\[
d^2(A, B) = \int_0^1 \gamma(t) [t(a^{(1)} - b^{(1)}) + (1-t)(a^{(2)} - b^{(2)})]^2 \, dt + \sum_{s=1}^{S} k_s \left| a_s - b_s \right|^2 \tag{2.4}
\]

where \( a_s = t_s a^{(1)} + (1 - t_s) a^{(2)} \), \( b_s = t_s b^{(1)} + (1 - t_s) b^{(2)} \), and the function \( \gamma \) satisfies the following properties

\[
\begin{align*}
\gamma(t) &\geq 0 \quad \text{(2.5.a)} \\
\int_0^1 \gamma(t) \, dt &= 1 \quad \text{(2.5.b)} \\
\gamma(0) &> 0 \ , \ \gamma(1) > 0 \quad \text{(2.5.c)} \\
t_1 = 0 \ , \ t_S = 1 \quad \text{if } S > 1 \quad \text{(2.5.d)}
\end{align*}
\]

The conditions (2.5.a)(2.5.b) have to be imposed if we want the restriction of \( d \) to the family of improper intervals \([a, a]\) and \([b, b]\) (which represent the real numbers \( a \) and \( b \)) to coincide with the usual distance \(|b - a|\); (2.5.c) assures that \( d \) is a true distance (these are necessary conditions for \( d \) could be used in every construction of fuzzy number distance). The other is a property that underlines the importance of the end points.
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When no reason exists for preferring the left half side of the interval with respect to the right one, we will impose the supplementary condition

$$\gamma(t) = \gamma(1 - t) \quad \text{(2.5.e)}$$

It is easy to prove that the function $d$ defined by (2.1) is really a distance. In fact, using the Minkowski inequalities

$$\sqrt{\sum_i (a_i + b_i + \ldots + c_i)^2} \leq \sqrt{\sum_i a_i^2} + \sqrt{\sum_i b_i^2} + \ldots + \sqrt{\sum_i c_i^2}$$

$$\sqrt{\int (u + v + \ldots + w)^2 \, dt} \leq \sqrt{\int u^2 \, dt} + \sqrt{\int v^2 \, dt} + \ldots + \sqrt{\int w^2 \, dt}$$

we can recognize that the positive function, defined on the family of the intervals $A = [a^{(1)}, a^{(2)}]$ by means of

$$\nu(A) = \nu([a^{(1)}, a^{(2)}]) = \left(\int_0^1 \gamma(t) [t a^{(1)} + (1 - t) a^{(2)}]^2 \, dt \right)^{1/2} \quad \text{(2.6)}$$

is a norm, provided that $\gamma$ is a function of type (2.3), and therefore $d$ is a distance.

The general form (2.1) is too complicated from the operational point of view. We think that we can obtain a sufficiently good measure, if we choose in (2.4) $\gamma(t) = 0$, $S = 3$ (that is if we choose a probability measure concentrated on $0$, $0.5$, $1$). In this case (2.4) reduces to

$$d^2(A, B) = k [a^{(1)} - b^{(1)}]^2 + h [a^{(G)} - b^{(G)}]^2 + k [a^{(2)} - b^{(2)}]^2 \quad \text{(2.7)}$$

where $a^{(G)} = \frac{a^{(1)} + a^{(2)}}{2}$ and $2k + h = 1$.

Remark 2. It is easy to recognize that (2.7), which corresponds to the simplest and most useful form of the function $\gamma$, eliminates all the drawbacks showed before for the classical distances. So the use of a complicated function $\gamma$ would seem quite artful. Nevertheless, in many cases further informations regarding the intervals may be available. For example, if we know that the left hand side of the intervals is
more interesting than the right one (it has been measured with more accuracy), then we will choose the values of $\gamma$ on the interval $[0, \frac{1}{2}]$ greater than those on $[\frac{1}{2}, 1]$. A possible choice of $\gamma$, compatible with the conditions (2.5.a)–(2.5.d), is the following:

$$\gamma(t) = \overline{\gamma}(t) = 2 - \beta - 2(1 - \beta)t, \quad \text{with } 0 < \beta < 1.$$ 

**Remark 3.** Let us examine in detail the distance $d$. It proceeds as follows:

- each of the intervals $[a^{(1)}, a^{(2)}]$, $[b^{(1)}, b^{(2)}]$ is parametrized by means of the linear map $\theta : [0, 1] \to [x^{(1)}, x^{(2)}]$ defined by

$$\theta(t; X) = tx^{(1)} + (1 - t)x^{(2)} \quad (X = A \text{ or } B).$$

- the squared distance between the points corresponding to the same value of $t$ is computed,

- the weighted mean of these squared distances is performed in order to obtain the squared distance between the two intervals (the weight is the function $\gamma$).

Let us remark that the parametrizer $\theta$ is “uniform”. More precisely $\theta(0; X)$ always corresponds to the second end point of the interval, $\theta(1; X)$ to the first one, $\theta(\frac{1}{2}; X)$ to the middle point, $\theta(\frac{1}{4}; X)$ to the third quarter and so on.

However in some cases it may happen that there exist some characteristic points in both intervals which do not occupy the corresponding positions in the uniform distribution, although they correspond one to the other in some sense. In this case it is better to choose different parametrizations for the two intervals so that the corresponding characteristic points would be associated with the same value of the parameter. Obviously the parametrization has to be a decreasing or increasing one to one mapping from $[0, 1]$ to $[x^{(1)}, x^{(2)}]$ and $\theta(1; X)$, $\theta(0; X)$ have to be associated with the two end points of the interval.

In the present case the intervals we deal with are the $\alpha$-cuts of a fuzzy number and therefore they contain the mode of the number, that is the middle point $\overline{\tau}$ of the values $\xi$ for which $\overline{X}(\xi) = 1$ holds. In
this case we can choose a parametrization which associates the mode with the value \( t = \frac{1}{2} \); for example

\[
\theta(t; X) = \begin{cases} 
2(\pi - x(t))t + x(t) & \text{if } t \in [0, \frac{1}{2}] \\
2(x(t) - \pi)t + 2\pi - x(t) & \text{if } t \in [\frac{1}{2}, 1]
\end{cases}
\]

(we have chosen an increasing form of the parametrization). Then the distance between \( A \) and \( B \) takes the form

\[
d^2(A, B) = \int_0^1 \gamma(t)[\theta_A(t; A) - \theta_B(t; B)]^2 \, dt.
\]

We can choose the function \( \gamma \) with three relative maxima at the points 0, \( \frac{1}{2}, 1 \). In this way we give a special importance to the end points (as natural) and to the mode of the fuzzy number.

Starting from \( d(A_\alpha, B_\alpha) \) we construct \( D(\bar{A}, \bar{B}) \) as follows. Let \( f \) be another normalized weight measure on \((0, 1], \mathcal{B}([0, 1])\) generated by a density function \( \phi(\alpha) \) of the form

\[
\phi(\alpha) = \varphi(\alpha) + \sum_{i=1}^L h_i \delta(\alpha - \alpha_i)
\]

where, as function \( \varphi, \varphi \) is a Lebesgue measurable function.

**Definition 2.2** For every pair of fuzzy numbers \( \bar{A}, \bar{B} \) for which the function \( d^2(\alpha) = d^2(A_\alpha, B_\alpha) \) is \( \varphi \)-integrable, the distance \( D \) between \( \bar{A} \) and \( \bar{B} \) is defined by

\[
D(\bar{A}, \bar{B}) = \sqrt{\int_0^1 [d(A_\alpha, B_\alpha)]^2 \, d\phi(\alpha)} \quad (2.8)
\]

provided the integral exists.

In particular this happens when the functions \( a^{(1)}(\alpha), a^{(2)}(\alpha), b^{(1)}(\alpha), b^{(2)}(\alpha) \) (which define \( \bar{A}, \bar{B} \)) belong to \( L^2([0, 1]) \), as, for instance, those corresponding to fuzzy numbers with bounded support. Function \( \varphi \) is a suitable normalized weight function that looks like a “probability density”; consequently it has to satisfy conditions similar
to (2.5.a,b) of function $\gamma$. Moreover we will impose another supplementary condition (the non-decreasing monotonicity) due to the fact that, on computing the distance, intervals with greater membership degree count more than those with lower ones. Those conditions are:

$$\varphi(\alpha) \geq 0$$
$$\alpha_1 < \alpha_2 \implies \varphi(\alpha_1) \leq \varphi(\alpha_2)$$
$$\int_0^1 \varphi(\alpha) \, d\alpha = 1$$

The latter is to guaranteeing that $D(\bar{A}, \bar{B})$ reduces to the usual distance $|b - a|$ on the real line, when $\bar{A} = a$ and $\bar{B} = b$ are fuzzy numbers corresponding to real ones.

Note that we can obtain an infinite distance between “numbers” or “intervals”, but these cases are not taken into account for practical purposes, even though they are theoretically possible. In particular we always have finite distance when the numbers have bounded support; this is the case of the numerical observations as those for a regression analysis. From now on, we will reduce to those fuzzy numbers for which we are able to compute their distance, and in particular to those with bounded support.

**Remark 4.** The definition 2.2 provides the distance between two fuzzy numbers, that is the distance between two pairs of functions $[a^{(1)}(\alpha), a^{(2)}(\alpha)]$ and $[b^{(1)}(\alpha), b^{(2)}(\alpha)]$ defined on the interval $[0, 1]$. It is easy to prove that the function $D$ has all the properties of a distance, provided that the functions $a^{(i)}, b^{(i)}$ belong to $L^2([0,1])$ and $\alpha > 0 \Rightarrow \varphi(\alpha) > 0$.

**Remark 5.** In some cases it is useful to choose $\varphi(\alpha) = 0$ on a suitable interval $[0, \pi]$. This happens when we establish that the $\alpha-$cuts corresponding to low degrees of membership ($\alpha \leq \pi$) do not affect the measure of the distance. In this case (2.8) does not represent a true distance because $D(\bar{A}, \bar{B}) = 0$ does not imply $\bar{A} = \bar{B}$. However, we may introduce a suitable equivalence relation (and its corresponding equivalence classes) by posing $\bar{A} \cong \bar{B} \iff D(\bar{A}, \bar{B}) = 0$; the function $D$ defines a true distance on the space of the equivalence classes, and
this distance may be usefully employed in many cases; in particular it may be used in the regression analysis, that is in the problem which motivates the research presented in this paper.

The practical choice of the function $\varphi$ depends on the structure of the fuzzy numbers we deal with. Thus if the membership function takes only a finite or countable set of values $\{\alpha_1 \ldots \alpha_R\}$, then it seems to be natural to select $\varphi$ in such a way that it is concentrated at these points.

$$\varphi(\alpha) = \sum_{i=1}^{R} f_i \delta(\alpha - \alpha_i)$$

On the contrary, if all the values are permitted, then we think that a measurable function has to be selected, that is $\varphi = \overline{\varphi}$. In particular the two following functions

$$\varphi(\alpha) = \overline{\varphi}(\alpha) = 1 \quad (2.9)$$
$$\varphi(\alpha) = \overline{\varphi}(\alpha) = \epsilon(\alpha - \frac{1}{2}) + 1 \quad \text{with} \quad 0 < \epsilon < 2 \quad (2.10)$$

seem to be quite useful. They are the simplest functions which allow us to distinguish between the case where all the $\alpha$-cuts have the same importance [(2.9)], and the case where the importance increases with the membership degree [(2.10)].

### 3 Topological equivalence

In the previous paragraph we introduced not one, but a class of distances, depending on the choice of the functions $\gamma$, $\varphi$. A very interesting problem, connected with this class is the following

Given two distances $D'$ and $D''$ of the type (2.8), are the topologies $T'$, $T''$ generated by them equivalent?

In order to answer this question, we observe that

a. the function $N(\tilde{X}) = \int_0^1 \varphi(\alpha) \nu(X_\alpha) \, d\alpha$, where $\nu$ is defined by (2.6), is a norm in the space of the fuzzy numbers,

b. the distance $D$ is generated by the norm $N$, 

c. the topologies $\mathcal{T}', \mathcal{T}''$ are equivalent iff the corresponding norms $N', N''$ are equivalent,

d. in order to prove that two norms $N', N''$ are equivalent, it suffices to prove that there exist two positive numbers $c', c''$ such that $c'[N']^2 \geq [N'']^2$ and $c''[N'']^2 \geq [N']^2$.

**Theorem 3.1** Let $\gamma', \gamma''$ and $\gamma', \gamma''$ be the functions which generate respectively the norms $N'$ and $N''$. The norms $N'$ and $N''$ are equivalent if the following conditions hold:

$$p' = \varphi'(0) > 0 \quad p'' = \varphi''(0) > 0 \quad (3.1)$$

$$q' = \varphi'(1) < +\infty \quad q'' = \varphi''(1) < +\infty \quad (3.2)$$

**Proof.** The squared norms $\nu', \nu''$ may be written in the form

$$[\nu'(a^{(1)}, a^{(2)})]^2 = \beta_{11}[a^{(1)}]_2^2 + \beta_{22}[a^{(2)}]_2^2 + 2\beta_{12}a^{(1)}a^{(2)}$$

$$[\nu''(a^{(1)}, a^{(2)})]^2 = \beta_{11}'[a^{(1)}]_2^2 + \beta_{22}'[a^{(2)}]_2^2 + 2\beta_{12}'a^{(1)}a^{(2)}$$

where

$$\beta_{11}' = \int_0^1 \gamma'(t)t^2 dt, \quad \beta_{22}' = \int_0^1 \gamma'(t)(1-t)^2 dt,$$

$$\beta_{12}' = \int_0^1 \gamma'(t)t(1-t)dt,$$

$$\beta_{11}'' = \int_0^1 \gamma''(t)t^2 dt, \quad \beta_{22}'' = \int_0^1 \gamma''(t)(1-t)^2 dt,$$

$$\beta_{12}'' = \int_0^1 \gamma''(t)t(1-t)dt.$$

Both $\nu'$ and $\nu''$ are positive definite quadratic forms. Thus, by spectral theorem, there exists a regular (non-degenerate) coordinate transformation

$$(a^{(1)}, a^{(2)}) \longrightarrow (\overline{a}^{(1)}, \overline{a}^{(2)})$$

which reduces both $\nu'$ and $\nu''$ to the normal form, and more precisely to the forms

$$[\nu'(a^{(1)}, a^{(2)})]^2 = [\overline{a}^{(1)}]_2^2 + [\overline{a}^{(2)}]_2^2 \quad (3.3)$$

$$[\nu''(a^{(1)}, a^{(2)})]^2 = \beta_{11}[\overline{a}^{(1)}]_2^2 + \beta_{22}[\overline{a}^{(2)}]_2^2 \quad (3.4)$$
where both $\beta_1$ and $\beta_2$ are strictly positive real values. By letting $c = \max(\beta_1, \beta_2)$, $c' = [\min(\beta_1, \beta_2)]^{-1}$, we obtain immediately, from (3.3)(3.4)

$$c'[\nu'(a^{(1)}, a^{(2)})]^2 \geq [\nu'(a^{(1)}, a^{(2)})]^2$$

$$c''[\nu''(a^{(1)}, a^{(2)})]^2 \geq [\nu'(a^{(1)}, a^{(2)})]^2$$

Now let us suppose that the following inequalities hold:

$$q' = \sup \{ \varphi'(\alpha) \mid \alpha \in [0, 1] \} < +\infty \quad (3.5)$$

$$q'' = \sup \{ \varphi''(\alpha) \mid \alpha \in [0, 1] \} < +\infty \quad (3.6)$$

$$p' = \inf \{ \varphi'(\alpha) \mid \alpha \in [0, 1] \} > 0 \quad (3.7)$$

$$p'' = \inf \{ \varphi''(\alpha) \mid \alpha \in [0, 1] \} > 0 \quad (3.8)$$

Then it follows immediately that

$$\frac{1}{q''} \int_0^1 \varphi''(\alpha) [\nu''(a^{(1)}, a^{(2)})]^2 \, d\alpha \leq \int_0^1 [\nu''(a^{(1)}, a^{(2)})]^2 \, d\alpha$$

$$\leq c' \int_0^1 [\nu'(a^{(1)}, a^{(2)})]^2 \, d\alpha$$

$$\leq c' \frac{1}{p'} \int_0^1 \varphi'(\alpha) [\nu'(a^{(1)}, a^{(2)})]^2 \, d\alpha$$

If the conditions (3.6) (3.7) are fulfilled, we can let $C' = \frac{c''}{p'}$; then from the above relation we obtain

$$C'[\mathcal{N}'(\bar{A})]^2 \geq [\mathcal{N}''(\bar{A})]^2$$

In the same way we can prove that if conditions (3.5)(3.8) hold, by letting $C'' = \frac{c''}{p''}$, we have

$$C''[\mathcal{N}''(\bar{A})]^2 \geq [\mathcal{N}'(\bar{A})]^2$$

Since $\varphi'$ and $\varphi''$ are not decreasing, the conditions (3.5)—(3.8) are equivalent to (3.1)(3.2); therefore theorem 3.1 is completely proved.

We will note that the conditions (3.1)(3.2) are fulfilled by the function (2.9) ($\varphi \equiv 1$), and by all the functions $\varphi$ of type (2.10) with $\epsilon \neq 2$, which we think may have a wide utilization.
References


