On the Identity of Fuzzy Material Conditionals

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Abstract

Given \(\mu, \eta : X \to [0,1]\) we study when the equality \(I^T_\mu = I^T_\eta\) holds, \(T\)
being a continuous \(t\)-norm, and \(I^T_\phi\) the elemental preorder:
\[
I^T_\phi(y/x) = \sup\{z : T(\theta(x) , z) \leq \theta(y)\}.
\]

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1 Introduction.

Given a subset \(A \subset X\), \(A \neq \emptyset\), the material conditional associated to \(A\) is the
relation in \(X \times X\) given by \(-_A = (A \times A) \cup (A' \times X)\), which is obviously a preorder.
The following theorem gives a characterization of a subset through its associated
material conditional \([?]\).

Theorem 1 If \(A\) and \(B\) are non-empty subsets of \(X\), then:
\[-_A = -_B \text{ if and only if } A = B\]

Proof. The sufficient condition is immediate.

Furthermore, since \(-_A = (A \times A) \cup (A' \times X) = (X \times X) - (A \times A')\), if \(-_A = -_B\),
that is, if \((X \times X) - (A \times A') = (X \times X) - (B \times B')\), then \(A \times A' = B \times B'\), which
holds if and only if \(A = B\). \(\Box\)
Given a fuzzy relation \( I : X \times X \rightarrow [0, 1] \), and a continuous t-norm \( T \), it is known [?] that \( I \) is a \( T \)-preorder if and only if
\[
I = \inf_{\mu \in \mathcal{F}} I^T_{\mu},
\]
\( \mathcal{F} \) being the set of the \( T \)-logical states of \( I \), that is
\[
\mathcal{F} = \{ \mu : X \rightarrow [0, 1] : T(\mu(x), I(y/x)) \leq \mu(y) \ \forall x, y \},
\]
and \( I^T_{\mu} \) the elemental preorder defined by
\[
I^T_{\mu}(y/x) = \sup \{ z \in [0, 1] : T(\mu(x), z) \leq \mu(y) \}.
\]
Choosing \( \mu = \varphi_A \), the classic characteristic function of \( A \), is
\[
I^T_{\varphi_A}(y/x) = \varphi_{A^c};
\]
and then it is natural to consider that the fuzzy elemental preorders are a generalization of the classic material conditionals\(^1\). We can therefore study whether it is possible to extend theorem 1 to the fuzzy preorders.

2 Equality of fuzzy material conditionals.

If we start to pay attention to the equivalence for the Min t-norm, we obtain that if the preorders \( I^\text{Min}_\mu, I^\text{Min}_\eta \) are equal, then \( \mu \) and \( \eta \) can only be different at the maximum values of \( \mu x \) and \( \eta y \).

**Theorem 2** \( I^\text{Min}_\mu = I^\text{Min}_\eta \) if and only if
\[
\mu x \neq \eta x \Rightarrow \mu x \geq \mu y \text{ and } \eta x \geq \eta y \text{ for all } y.
\]

Proof. Let \( I^\text{Min}_\mu(y/x) = I^\text{Min}_\eta(y/x) \), for each \( x, y \). We know that:
\[
I^\text{Min}_\mu(y/x) = \sup \{ z : \text{Min}(\theta x, z) \leq \theta y \} = \begin{cases} 1, & \text{if } \theta x \leq \theta y \\ \theta y, & \text{if } \theta x > \theta y \end{cases}.
\]
Let us choose an \( x \) such that \( \mu x \neq \eta x \), and let us suppose that there exists a \( y \) verifying \( \mu x < \mu y \). We will obtain \( I^\text{Min}_\mu(y/x) = 1 = I^\text{Min}_\eta(y/x) \), and then \( \eta x \leq \eta y \).

Furthermore, \( I^\text{Min}_\mu(x/y) = \mu x = I^\text{Min}_\eta(x/y) < 1 \), and so \( I^\text{Min}_\mu(x/y) = \eta x = \mu x \), which gives a contradiction; so, it must be \( \mu x \geq \mu y \) for all

\(^1\)The elemental preorders are not the unique generalizations of the material conditional; see for example [?].
y. In a similar way, \( \eta x \geq \eta y \) is obtained for all \( y \).

Reciprocally, let us suppose that if \( \mu x \neq \eta x \) then \( \mu x \geq \mu y \) and \( \eta x \geq \eta y \), for all \( y \). We will prove that the equality \( I_{\mu}^{Min} = I_{\eta}^{Min} \) holds.

- If \( \mu x = \eta x \) and \( \mu y = \eta y \), clearly \( I_{\mu}^{Min}(y/x) = I_{\eta}^{Min}(y/x) \).

- If \( \mu x = \eta x \) and \( \mu y \neq \eta y \), then \( \mu y \geq \mu x \), \( \eta y \geq \eta x \), and \( I_{\mu}^{Min}(y/x) = I_{\eta}^{Min}(y/x) = 1 \).

- If \( \mu x \neq \eta x \) and \( \mu y = \eta y \), we obtain \( \mu x \geq \mu y \), \( \eta x \geq \eta y \), and \( I_{\mu}^{Min}(y/x) = \mu y = \eta y = I_{\eta}^{Min}(y/x) \).

Finally, if \( \mu x \neq \eta x \) and \( \mu y \neq \eta y \), since \( \mu x \geq \mu y \), \( \eta x \geq \eta y \), and \( \mu y \geq \mu x \), \( \eta y \geq \eta x \), obviously \( \mu x = \mu y \), \( \eta x = \eta y \), and \( I_{\mu}^{Min}(y/x) = I_{\eta}^{Min}(y/x) \). □

**Note.** If \( T \) is Archimedean with additive generator \( h \), then from:
\[
I_{T}(y/x) = \sup \{ z / T(\mu x, z) \leq \mu y \} = \sup \{ z / h^{-1}(h(\mu x) + h(z)) \leq \mu y \},
\]
it follows:

- If \( \mu x \leq \mu y \), then \( h(\mu x) \geq h(\mu y) \), and \( h(\mu x) + h(z) \geq h(\mu y) \) for all \( z \). So, \( h^{-1}(h(\mu x) + h(z)) \leq h^{-1}(h(\mu y)) = \mu y \) for all \( z \), and \( I_{\mu}^{T}(y/x) = 1 \).

- If \( \mu x > \mu y \), \( h(\mu x) < h(\mu y) \) and \( h(\mu y) - h(\mu x) \in [0, h(0)] \). Let us see that \( I_{\mu}^{T}(y/x) = h^{-1}(h(\mu y) - h(\mu x)) \). In fact, \( h^{-1}(h(\mu x) + h(\mu y) - h(\mu x)) = h^{-1}(h(\mu x) + h(\mu y) - h(\mu x)) = h^{-1}(h(\mu y)) = \mu y \) and for all \( z > h^{-1}(h(\mu y) - h(\mu x)) \), it is \( h(z) < h(\mu y) - h(\mu x) \), \( h(\mu x) + h(z) < h(\mu y) \leq h(0) \) and \( h^{-1}(h(\mu x) + h(z)) > h^{-1}(h(\mu y)) = \mu y \).

Therefore, we obtain
\[
I_{\mu}^{T}(y/x) = \begin{cases} 
1, & \text{if } \mu x \leq \mu y \\
h^{-1}(h(\mu y) - h(\mu x)), & \text{if } \mu x > \mu y.
\end{cases}
\]

The following theorem asserts that in the case of strict Archimedean \( t \)-norms, the preorders associated to \( \mu \) and \( \eta \) are equal if and only if \( \mu \) and \( \eta \) have "similar forms".

**Theorem 3** If \( T \) is a strict Archimedean \( t \)-norm with additive generator \( h \), \( I_{\mu}^{T} = I_{\eta}^{T} \) if and only if there exists \( k \in R \) such that for all \( x \), \( h(\mu x) = k + h(\eta x) \).

**Proof.** As \( T \) is strict, \( h^{-1} = h^{-1} \). Let \( I_{\mu}^{T} = I_{\eta}^{T} \), and let us choose some \( x, y \).
Firstly, let us point out that if \( \mu x < \mu y \), then \( I^T_n(x/y) = I^T_\mu(x/y) = h^{-1}(h(\mu x) - h(\mu y)) < 1 \); so \( \eta x < \eta y \). Similarly, if \( \eta x < \eta y \), it is \( \mu x < \mu y \).

So, if \( \mu x < \mu y \), \( I^T_\mu(x/y) = h^{-1}(h(\mu x) - h(\mu y)) = I^T_n(x/y) = h^{-1}(h(\eta x) - h(\eta y)) \), if and only if \( h(\mu x) - h(\mu y) = h(\eta x) - h(\eta y) \), if and only if \( h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y) \).

Also, if \( \mu y < \mu x \), then \( h(\mu y) - h(\eta y) = h(\mu x) - h(\eta x) \).

Finally, if \( \mu x = \mu y \) then \( \eta x = \eta y \), and newly \( h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y) \).

Therefore \( h(\mu x) - h(\eta x) \) is a constant \( k \) for all \( x \), and \( h(\mu x) = k + h(\eta x) \) for each \( x \).

Reciprocally, if there exists \( k \in R \) such that for all \( x \) the equality \( h(\mu x) = k + h(\eta x) \) holds, since \( h(\mu x) - h(\eta x) = k = h(\mu y) - h(\eta y) \), we obtain \( h(\mu x) + h(\eta y) = h(\mu y) + h(\eta x) \); then \( \mu x \leq \mu y \) if and only if \( h(\mu x) \geq h(\mu y) \), if and only if \( h(\eta x) \geq h(\eta y) \), if and only if \( \eta x \leq \eta y \).

So, if \( \mu x \leq \mu y \), then \( \eta x \leq \eta y \) and \( I^T_n(x/y) = 1 = I^T_\mu(x/y) \); and if \( \mu x > \mu y \) then \( \eta x > \eta y \), and \( I^T_\mu(x/y) = h^{-1}(h(\mu y) - h(\mu x)) = h^{-1}(h(\mu y) - h(\eta y)) = h^{-1}(h(\eta y) - h(\eta x)) = I^T_n(x/y) \). \( \square \)

In the particular case in which \( \mu \) and \( \eta \) "have points", that is, they are normalized, we can get:

**Corollary 1** If \( T \) is Archimedean and strict, and \( \mu \) and \( \eta \) are such that there exist \( x, y \) with \( \mu x = 1, \eta y = 1 \), then

\[ I^T_\mu = I^T_\eta \text{ if and only if } \mu = \eta. \]

**Proof.** The sufficient condition is clear.

On the other hand, if \( I^T_\mu = I^T_\eta \), there exists \( k \in R \) such that for all \( z \) \( h(\mu z) = k + h(\eta z) \).

In particular, \( h(\mu x) = 0 = k + h(\eta x) \leq h(\eta x) \), and necessarily \( k \leq 0 \).

Analogously, \( h(\mu y) = k + h(\eta y) = k + 0 \), and then \( k \geq 0 \).

So \( k = 0 \), and then \( h(\mu z) = h(\eta z) \), and \( \mu z = \eta z \) for all \( z \). \( \square \)

Now, for the case in which \( \mu \) and \( \eta \) "have not some points", we have
Corollary 2 If $T$ is Archimedean and strict, and $\mu$ and $\eta$ are such that there exist $x, y$ with $\mu x = 0$, $\eta y = 0$, then

$$I^T_\mu = I^T_\eta \text{ if and only if } \mu = \eta$$

Proof. If $I^T_\mu = I^T_\eta$, there exists $k \in R$ with $h(\mu z) = k + h(\eta z)$ for all $z$.

As $\mu x = 0$, it is $h(0) = k + h(\eta x) \leq k + h(0)$, and then $0 \leq k$. And since $\eta y = 0$, $h(\mu y) = k + h(0) \leq h(0)$, and $k \leq 0$.

We get that $k = 0$ and $\mu = \eta$. □

Theorem 4 If $T$ is a non-strict Archimedean $t$-norm, with additive generator $h$, then:

$$I^T_\mu = I^T_\eta \text{ if and only if there exists } k \in R \text{ such that for all } x \text{ it is } h(\mu x) = k + h(\eta x).$$

Furthermore, if that holds, for all $x$ it is $k \leq h(0) - h(\eta x)$.

Proof. Let $I^T_\mu = I^T_\eta$ be, and let us choose any $x, y$.

- If $\mu x < \mu y$, $h(\mu x) > h(\mu y)$ and $I^T_\mu(x/y) = h^{(-1)}(h(\mu x) - h(\mu y)) = I^T_\eta(x/y) < 1$, and then $\eta x < \eta y$, $h(\eta x) > h(\eta y)$ and $I^T_\eta(x/y) = h^{(-1)}(h(\eta x) - h(\eta y)) = h^{(-1)}(h(\mu x) - h(\mu y))$, which implies (because of $h(\eta x) - h(\eta y) \in [0, h(0)]$ and $h(\mu x) - h(\mu y) \in [0, h(0)]$) that $h(\eta x) - h(\eta y) = h(\mu x) - h(\mu y)$ and $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$.

- In a similar way if $\mu y < \mu x$, then $\eta y < \eta x$ and $h(\mu y) - h(\mu x) = h(\mu y) - h(\eta x)$.

- In the case in which $\mu x = \mu y$, $\eta x = \eta y$ and $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$.

Then for all $x, y$ holds $h(\mu x) - h(\eta x) = h(\mu y) - h(\eta y)$, and there exists $k \in R$ such that for all $x$ $h(\mu x) - h(\eta x) = k$ and $h(\mu x) = k + h(\eta x)$.

In this case, if there exists $x$ with $k > h(0) - h(\eta x)$, we obtain the contradiction $h(\mu x) = k + h(\eta x) > h(0) - h(\eta x) + h(\eta x) = h(0)$. So $k \leq h(0) - h(\eta x)$ for all $x$.

Reciprocally, let us suppose that there exists $k \in R$ such that for all $x$ $k \leq h(0) - h(\eta x)$ and $h(\mu x) = k + h(\eta x)$.

- If $\mu x < \mu y$, $k + h(\eta x) \geq k + h(\eta y)$, $h(\eta x) \geq h(\eta y)$, and $\eta x \leq \eta y$. Therefore, $I^T_\eta(y/x) = I^T_\eta(y/x) = 1$.

- If $\mu x > \mu y$, $k + h(\eta x) < k + h(\eta y)$, $h(\eta x) < h(\eta y)$, and $\eta x > \eta y$.

Then, $I^T_\mu(y/x) = h^{(-1)}(h(\mu y) - h(\mu x)) = h^{(-1)}(k + h(\eta y) - k + h(\eta x)) = h^{(-1)}(h(\eta y) - h(\eta x)) = I^T_\eta(y/x)$. □

Let us point out that newly, if the preceders associated to $\mu$ and to $\eta$ concur, $\mu$ and $\eta$ must have "similar forms", but now, furthermore, their distance is bounded.
by $h(0) - \sup_x \{h(\eta x)\}$.

Analogously, in the case in which $\mu$ and $\eta$ "have points", and in the case in which its complements "have points", it holds:

**Corollary 3** If $T$ is Archimedean non-strict and $\mu$ and $\eta$ are such that there exist $x, y$ with $\mu x = 1$, $\eta y = 1$, or there exist $x, y$ with $\mu x = 0$, $\eta y = 0$, then

$$T^T_\mu = T^T_\eta \text{ if and only if } \mu = \eta.$$  

**Proof.** Similar to the case of the Archimedean strict t-norms.

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**References**


