

# A Model of Decision with Linguistic Knowledge \*

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## Abstract

The aim of this paper is to develop a new aggregating method for the decision problem in which the possible values of rewards are known in linguistic terms.

We show new operators for solving this problem, as well as the way in which OWA operators provide us with an adequate framework for representing the optimism degree of the decision maker in case we have no information about the real state.

**Keywords:** Decision-making, linguistic label, uncertainty, operators.

## 1 Introduction

A decision-making problem involves selecting the best alternative in view of the results and information that we have been able to obtain regarding the states of nature.

In many cases, the information available regarding the states of nature is obtained in an auxiliary manner. Normally, however, in the classical model this information is available in terms of a probability distribution, or it may so happen that we are faced with total ignorance.

On the other hand, the payments that the decision-maker perceives as a consequence of his/her choice which shall be numerical, are a function of the state of nature that is presented.

If we think that in everyday life the way of expressing an idea is more linguistic than numerical, this will mean that both the information available about the states of nature and the possible rewards are all known by us in linguistic terms.

That is why we shall first discuss the problem of decision-making under risk,

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where the information about the states of nature is known in terms of a probability distribution, whereas the possible rewards are values expressed in linguistic terms.

Finally, we shall deal with the case of decision-making under ignorance, where no information is available about the states of nature, and therefore in order to be able to act the decision-maker's attitude when faced with risk must be known. We shall solve the problem by introducing the OWA operators (Ordered Weighted Averaging), which offer us a more general framework than the one known till now.

## 2 The decision problem

Let us consider a decision-making problem with the following elements:

$$\{\Omega, D, \mathbf{r}, I, \leq\}$$

where:

1.  $\Omega$  is the set of states of nature which we suppose are finite.
2.  $D$  is the set of alternatives available to the decision-maker, among which the decision-maker must choose one.
3.  $\mathbf{r}: D \times \Omega \rightarrow R$  is a function where a real number corresponds to every decision  $d_i$  and to every state  $w_j$ .

$$(d_i, \omega_j) \longrightarrow \mathbf{r}(d_i, \omega_j) \equiv r_{ij}$$

With  $r_{ij}$  being the reward given to the decision-maker for having chosen the decision  $d_i$  when the state  $w_j$  has been presented  $\mathbf{r}$  can be represented by a matrix called the *Matrix of rewards*.

$D \setminus \Omega$	$w_1$	$w_2$	$\cdots$	$w_j$	$\cdots$	$w_m$
$A_1$	$r_{11}$	$r_{12}$	$\cdots$	$r_{1j}$	$\cdots$	$r_{1m}$
$A_2$	$r_{21}$	$r_{22}$	$\cdots$	$r_{2j}$	$\cdots$	$r_{2m}$
				$\vdots$		
$A_i$	$r_{i1}$	$r_{i2}$	$\cdots$	$r_{ij}$	$\cdots$	$r_{im}$
				$\vdots$		
$A_n$	$r_{n1}$	$r_{n2}$	$\cdots$	$r_{nj}$	$\cdots$	$r_{nm}$

In these conditions the rewards which can be obtained by the decision-maker for making a decision will be any value of the n-dimensional vector associated with the chosen alternative.

$$A_i \longrightarrow [r_{i1}, r_{i2}, \dots, r_{ij}, \dots, r_{im}] \equiv \vec{r}_i \in R^m$$

4. A relation of preferences  $\leq$  by the decision maker. We shall assume a coherent decision-maker, so we will try to maximize his profits or else minimize his losses.
5. Certain information about the real state of nature. Depending on the decision-maker's kind of knowledge about the real state, three environments appear in the literature.
  - (a) *Decision making under certainty.* The decision-maker knows which state is going to be presented. In particular, it is the case in which  $P(s_j) = 1$  or 0 for all  $j = 1, \dots, m$ . In this case, the decision-maker should choose the alternative which means the maximum payoff / minimum loss to this state of nature.
  - (b) *Decision-making under risk.* In the decision under risk a probability distribution about the states of nature is assumed to be known. In this case, the standard procedure is to use the expected value.
  - (c) *Decision making under ignorance.* The third environment involves the case of ignorance. Since we have no information available about the states of nature, the decision criteria take into account the decision-maker's attitude towards the risk.

Among these attitudes the most representative are the following:

- *Pessimistic Attitude.* By means of this strategy, the decision-maker has the worst possible, point of view. He chooses the worst outcome for every alternative and then he selects the one which achieves the best/worst. This strategy is also known as Maximin.
- *Optimistic attitude.* Under this strategy the decision-maker assumes the best result possible, selecting the best outcome for each alternative and choosing the alternative which achieves the best/best. This strategy is also known as Maximax.
- *Hurwicz strategy.* With this approach the decision-maker selects a value  $\alpha \in [0, 1]$  which represents his degree of optimism. For each alternative we weight maximum and minimum values by  $\alpha$  and  $(1-\alpha)$  respectively.

$$H_i = \alpha \times \text{Max}_i r_{ij} + (1 - \alpha) \times \text{Min}_i r_{ij}$$

Finally, the alternative of achieving the maximum is chosen.

- *Normative approach.* In this approach, the decision-maker adds all values relating to every alternative in order to choose that alternative whose sum is the highest.

From this moment let us assume that the rewards  $r_{ij}$  can be expressed in linguistic terms. This will involve several problems such as the need to attach labels with numerical values.

Therefore, we shall have to introduce new operators similar to those which appear in the literature. In addition, these operators should be capable of solving the problem we indicate.

To start with, let us define the following:

### 3 Set of generalized labels

Let  $U = \{u_0 = 0, u_1, u_2, \dots, u_n\}$  be the ordinate set of linguistic values that represent the set of possible rewards, i.e.,  $U = \{u_i/i \in I \subset Z, I = 0, 1, \dots, n\}$  and so that  $u_i \leq u_j$  if  $i \leq j$ . Let  $P = \{0 = p_0, p_1, \dots, p_m\}$  with  $p_j \in [0, 1]$  and  $\sum p_j = 1$ . In this case,  $p_j$  represents the probability distribution about  $\Omega$  and accordingly the set of possible weightings of the different labels.

By means of these sets, which can be either linguistic or numerical, we define the set of generalized labels [4].

$$\mathcal{U} = U \times P$$

where the pair  $(u_i, p_j) \in \mathcal{U}$  represents the product of  $p_j \odot u_i$ .

- If  $p_j = 1 \rightarrow (u_i, 1) \equiv u_i$ .
- If  $p_j = 0 \rightarrow (u_i, 0) \equiv 0$ .

As a result of this, the label which represents the null value has to belong to the set of labels  $U$ .

#### 3.1 Addition

Let  $(u_{i_1}, p_{j_1})$  and  $(u_{i_2}, p_{j_2})$  be two generalized labels; we define their addition as:

$$\begin{aligned} \oplus : \mathcal{U}^2 &\longrightarrow \mathcal{U} \\ (u_{i_1}, p_{j_1}) \oplus (u_{i_2}, p_{j_2}) &= (u_s, p_{i_1} + p_{i_2}) \quad (1) \end{aligned}$$

where

$$s = \frac{(i_1 \times p_{j_1}) + (i_2 \times p_{j_2})}{p_{j_1} + p_{j_2}}$$

However, this value  $s$  is not necessarily an integer number, and  $u_s$  may not be a label of  $U$ . We can use a rounding method as the most straightforward way of achieving it.

As we have to add several generalized labels we should extend (1) in the following way:

$$(u_{i_1}, p_{j_1}) \oplus (u_{i_2}, p_{j_2}) \oplus \dots \oplus (u_{i_n}, p_{j_n}) = (u_s, \sum_{k=1}^n p_{j_k}) = (u_s, 1)$$

with

$$s = \text{round} \sum_{k=1}^n (i_k \times p_{j_k}) \quad (2)$$

**Properties:**

1. Commutativity: The order of the addition of two generalized labels does not affect the result of the addition.
2. Associativity: Due to the way in which the operation has been defined in (2).
3. Neutral Element: Any label such as:

$$(u_i, 0) \quad \forall i = 0, 1, \dots, n$$

## 4 Decision under risk

When we have a probability distribution about the states of nature which informs us about the uncertainty associated to the experience, the mathematical expectation is the classic operator which for each decision weights the different payoffs.

For the problem we address, in which the rewards are linguistic, we propose the following operator.

### 4.1 The set of probabilities is numerical and the payoffs linguistic

$$E : (U \times P)^m \longrightarrow U$$

$$\Sigma_j (u_j \times p_j) \longrightarrow u_{s_i}$$

Considering  $U = \{u_i / i = 0, 1, \dots, n\}$  to be the ordinate finite set of the possible linguistic labels, and since each utility  $u_{ij}$  is a label  $u_j \in U$ , we define, as in the above, its order as  $O(u_{ij}) = O(u_j) = j$ .

Then

$$E_2[A_i] = u_{s_i}$$

where

$$s_i = \text{round}(U_i)$$

and

$$U_i = \sum_j (O(u_j) \times p_j)$$

As the set  $U$  is finite and normally its granularity is low, this means that several decisions achieve the same expectation. In this case, in order to take a decision, the action for which the  $Max_i U_i$  is obtained should be chosen.

$$U \longrightarrow R$$

$$u_{si} \longrightarrow U_i$$

### Algorithm

1. For each alternative we calculate its expectation

$$E_2[A_i]$$

2. We choose as optimum that decision for which the maximum is reached.

$$E_2[A^*] = Max_i u_{si}$$

3. If the decision is not unique, choose the  $A^*$  optimum so that

$$E_2[A^*] = Max_i U_i$$

## 4.2 Example

Let us suppose that the information about the five states of nature is probabilistic and given by  $p(w_i) = i/15$  for  $i=1,2,3,4,5$ . Let us also we consider the following matrix of rewards expressed by means of linguistic terms:

	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$
$a_1$	$u_7$	$u_4$	$u_4$	$u_6$	$u_5$
$a_2$	$u_2$	$u_5$	$u_7$	$u_4$	$u_7$
$a_3$	$u_5$	$u_5$	$u_7$	$u_4$	$u_3$
$a_4$	$u_6$	$u_4$	$u_3$	$u_5$	$u_2$

where

$U = \{0, u_1 = \text{very low}, u_2 = \text{low}, u_3 = \text{something less than fair}, u_4 = \text{fair}, u_5 = \text{more than fair}, u_6 = \text{high}, u_7 = \text{very high}\}$

The solution is very simple

$$E[a_1] = 7 \times 1/15 + 4 \times 2/15 + 4 \times 3/15 + 6 \times 4/15 + 5 \times 5/15 = 5.60$$

$$s = \text{round}(5.60) = 6 \implies E[a_1] = u_6$$

in the same way

$$E[a_2] = u_6$$

$$E[a_3] = u_5$$

$$E[a_4] = u_4$$

the best alternative being  $a_2$ .

## 5 Decision under ignorance

Not having any information about the states of nature, and supposing that the criteria of dominance does not provide any result, it is the decision-maker who must decide, taking into account a subjective rule.

We suppose that the utilities are linguistic. In this case, the classical strategies of decision maximax and minimax are not the most suitable, since with the set  $U$  being finite and of little granularity, it is likely that two alternatives reach the same maximum and/or minimum. Due to this, we also remove the Hurwicz approach and we consider it more suitable to insert the OWA operators.

### 5.1 Operators

**Definition** (Yager) An ordered weighted averaging operator (OWA) of dimension  $n$  is a function

$$F : R^n \rightarrow R$$

that has associated with it a weighting vector  $W$

$$W = [w_1, w_2, \dots, w_n]$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

such that

1.  $w_i \in [0, 1]$
2.  $\sum_i w_i = 1$

where for any set of values  $a_1, \dots, a_n$

$$F(a_1, \dots, a_n) = \sum_i w_i \star b_i$$

where  $b_i$  is the  $i$ -largest of the  $a_1, \dots, a_n$ .

**Definition.** Assume that  $X$  is a set of elements. A bag (or multi-set) drawn from  $X$  is any collection of elements, each of which is contained in  $X$ . A bag is different from a subset in that it allows multiple copies of the same element. A bag is similar to a set in that the ordering of elements does not matter. If  $A$  is a bag consisting of  $a, b, c, d$ , we denote this  $A = \langle a, b, c, d \rangle$ .

### Properties

1.  $F$  satisfies a generalized commutativity (symmetry)

$$F(a_1, a_2, \dots, a_m) = F(b_1, b_2, \dots, b_m)$$

if the bags  $\langle a_1, a_2, \dots, a_m \rangle$  and  $\langle b_1, b_2, \dots, b_m \rangle$  are equal.

2.  $F$  is monotonic

$$F(a_1, a_2, \dots, a_m) \leq F(b_1, b_2, \dots, b_m)$$

if  $a_i \leq b_i$  for all  $i=1,2,\dots,m$

3.  $F$  is idempotent

$$F(a, a, \dots, a) = a$$

for every  $a \in R$ .

If we take into account the previous definitions, and since we have no any information about the states of nature and we are not capable of considering subjective probabilities, we shall suggest the following:

**Criterion of preference.** Given two decisions,  $A_1$  and  $A_2$ , we shall say that the first is preferred to the second if it is verified that

$$F(u_{11}, u_{12}, \dots, u_{1m}) \leq F(u_{21}, u_{22}, \dots, u_{2m})$$

If  $\langle u_{11}, u_{12}, \dots, u_{1m} \rangle$  and  $\langle u_{21}, u_{22}, \dots, u_{2m} \rangle$  are equal then we say they are indifferent.

On the contrary, if the decision-maker wants to weight some results more than others, according to their degree of optimism about the result, we shall take into account that for any degree of optimism, the operator  $F$  which represents it, is always between the extreme operators maximum and minimum.

$$\text{Min}(a_1, a_2, \dots, a_m) \leq F(a_1, a_2, \dots, a_m) \leq \text{Max}(a_1, a_2, \dots, a_m)$$

Taking into account the decision-maker's attitude, we reproduce the classical criteria of decision under ignorance when the rewards are linguistic, and we operate with the concept of the generalized label previously inserted and the OWA operators.

- Pessimistic Attitude: If we select  $w$  where

$$w_* = [0, 0, \dots, 1]^T$$

then  $F(a_1, a_2, \dots, a_m) = \text{Min}_j[a_j] = u_{*j} \in U$ , which is the aggregation rule used in the pessimistic strategy.



- Optimistic Attitude: If we select  $w^*$  where

$$w^* = [1, 0, \dots, 0]^T$$

then  $F(a_1, a_2, \dots, a_m) = \text{Max}_j [a_j] = l_j^* \in U$ , which is what is used in the optimistic strategy.

- Hurwicz Strategy: If we select

$$w_H = [\alpha, 0, \dots, 0, (1 - \alpha)]^T$$

then  $F(a_1, a_2, \dots, a_m) = \alpha \times \text{Max}[a_j] + (1 - \alpha) \times \text{Min}[a_j] = \alpha \times l_j^* + (1 - \alpha) \times u_{*j} = u_j \in U$ . This operation is the same as in Subsection 4.2.

This is the formulation used in the Hurwicz strategy.

- Hurwicz Strategy Generalized: If we select

$$w_{HG} = [\alpha_1, \alpha_2, \dots, \alpha_m]^T$$

with  $\sum \alpha_i = 1$  and whose values  $\alpha_i$  are the result of the mathematical programming problem proposed by O'Hagan.

*Maximize*

$$\sum_j w_j \ln(w_j) \quad (\text{entropy})$$

*Subject to:*

$$\sum_j (hn(j) \times w_j) = \beta$$

$$\sum_j w_j = 1$$

$$w_j \geq 0 \quad \text{for all } j = 1, \dots, m$$

where  $\beta$  is the degree of optimism and  $hn(j) = \frac{(n-j)}{(n-1)}$ .

Then  $F(a_1, a_2, \dots, a_m) = \alpha_1 \times \text{Max}[a_j] + \alpha_m \times \text{Min}[a_j] = u_j \in U$ .

With the similar operation as in the Hurwicz Strategy.

- Normative Approach: If we select

$$w_N = [1/n, 1/n, \dots, 1/n]^T$$

then  $F(a_1, a_2, \dots, a_m) = 1/n \sum_j a_j = u_j \in U$ . This function is essentially the normative strategy.

## 5.2 Example

Let us suppose the same example for the case of risk, in which we have no information about  $\Omega$ .

	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$
$a_1$	$u_7$	$u_4$	$u_4$	$u_6$	$u_5$
$a_2$	$u_2$	$u_5$	$u_7$	$u_4$	$u_7$
$a_3$	$u_5$	$u_5$	$u_7$	$u_4$	$u_3$
$a_4$	$u_6$	$u_4$	$u_3$	$u_5$	$u_2$

For a degree of optimism  $\beta = .75$  and  $m = 5$  the solution of the mathematical programming problem is

$$[.46, .26, .15, .08, .05]$$

The associated bags are the following:

$$\langle u_7, u_6, u_5, u_4, u_4 \rangle$$

$$\langle u_7, u_7, u_5, u_4, u_2 \rangle$$

$$\langle u_7, u_5, u_5, u_4, u_3 \rangle$$

$$\langle u_6, u_5, u_4, u_3, u_2 \rangle$$

Let us see how the two latter bags are dominated by the first. The solutions for  $a_1$  and  $a_2$  are the following.

$$s = 6.05 \text{ with } E[a_1] = u_6$$

$$s = 6.21 \text{ with } E[a_2] = u_6$$

the best option being  $a_2$ .

## 6 Conclusion

In this paper, we have offered an alternative method to those already known for solving the problems of probabilistic decision-making when the values of the possible rewards are not numerical but rather they are known by us in linguistic terms.

For this purpose we developed an aggregation method using the proposed operators that have most of the desirable properties.

## References

- [1] Bellman, R. and Zadeh, L.A., "Decision making in a fuzzy environment". *Management Science*, 17, 1970, pp. 141-164.

- [2] Bolaños, M.J., Lamata, M.T. and Moral, S., "Decision making problems in a general environment". *Fuzzy Sets and Systems*, 18, 1988, pp. 135-144.
- [3] Bonissone, P.P., "A fuzzy sets based linguistic approach". Approximate Reasoning in *Decision Analysis*, M.M.Gupta, E.Sanchez (eds), North Holland Publishing Company, 1982, pp. 329-339.
- [4] Delgado, M., Verdegay J.L. and Vila, A., "On aggregation operations of linguistic label". *International Journal of Intelligent Systems*, 8, 1993, pp. 351-370.
- [5] Fishburn, P.C., "*Utility theory for decision making*". John Wiley 1970.
- [6] Lamata, M.T., "Problemas de decisión con información general". Tesis doctoral. Universidad de Granada, 1986.
- [7] Lamata, M.T. and Moral, S., "Decision; Utilidades lingüísticas". III Congreso Español sobre tecnologías y lógica fuzzy, 1993, pp. 321-328.
- [8] Lamata, M.T. and Moral, S., "Calculus with linguistic probabilities and belief". *Advances in the Dempster-Shafer theory of Evidence*. (M.Fedrizzì, J.Kacprzyk, R.R.Yager, eds). John Wiley (1994), pp. 133-152.
- [9] O'Hagan, M., "Aggregating template rule antecedents in real-time expert systems with fuzzy set logic". *Int. J. of Man-Machine Studies* (to appear).
- [10] Smith, J.Q., *Decision Analysis. A Bayesian Approach*, 1988.
- [11] Yager, R.R., "Decision making under Dempster-Shafer uncertainties". *Int.J.of Int. Sys*, 20, 1992, pp. 233-245.
- [12] Yager, R.R., "Families of OWA operators". *Fuzzy Sets and Systems*, 59, 1993, pp. 125-148.
- [13] Zadeh, L.A., "Fuzzy sets and information granularity". *Advances in fuzzy sets theory and applications*, M.M.Gupta, E. Sanchez (eds), North Holland Publishing Company, New York, 1979, pp. 3-19.