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A Note on the Symmetric Difference in Lattices

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Abstract

The paper introduces a definition of symmetric difference in lattices with negation, presents its general properties and studies those that are typical of ortholattices, orthomodular lattices, de Morgan and boolean algebras.

Keywords. Symmetric difference, ortholattices, de Morgan algebras.

1 Introduction

Let $(L, \cdot, +; 0, 1)$ be a lattice with intersection $\cdot$, union $+$, zero 0, and unit 1. The order induced in L is "$a \leq b$ iff $a \cdot b = a$, or, equivalently, $a + b = b$", and it is $0 \leq a \leq 1$ for all a in L.

Provided a mapping $' : L \to L$ verifies: $0' = 1$; $1' = 0$; if $a \leq b$, then $b' \leq a'$, and $a'' = a$ for all a in L, the structure $(L, \cdot, +; 0, 1; ')$ is called a *lattice with negation* (LN). Obviously the mapping $'$ is one-to-one and onto.

The following properties are used to classify the big family of lattices with negation

- **Duality** (DUAL): $a + b = (a' \cdot b')'$, equivalent to $a \cdot b = (a' + b')'$.
- **Non Contradiction** (NC): $a \cdot a' = 0$
- **Excluded-Middle** (EM): $a + a' = 1$
- **Distributivity** (DIS): $a + b \cdot c = (a + b) \cdot (a + c)$, and $a \cdot (b + c) = a \cdot b + a \cdot c$,

for all a, b, c in L. It is immediate to prove that, under duality (DUAL), NC and EM are equivalent properties, and also that the two distributive laws are equivalent. With these properties:

- An **ortholattice** (OL), is a LN verifying DUAL and NC

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- A **de Morgan algebra** (DMA), is a LN verifying DUAL and DIS
- A **Kleene algebra** (KA), is a DMA such that $a \cdot a' \leq b + b'$ for all a, b in L.
- An **orthomodular lattice** (OML), is an OL with **relative complement**, that is, where $a \leq b$ implies $b = a + b \cdot a'$, or equivalently $a = b \cdot (b' + a)$. The relative complement $(b - a)$ of a in b is $b \cdot a'$.
- A **boolean algebra** (BA), is an OL verifying DIS, or a DMA verifying NC.

Obviously, OML $\subset$ OL, OL $\cap$ DMA=BA, BA $\subset$ OML, and BA $\subset$ KA $\subset$ DMA (see (3), (4)).

This paper was suggested by (1), where a general study of the symmetric difference of fuzzy sets is presented. When no confusion is possible, instead of $a \cdot b$ it will be written ab.

2 Symmetric difference in LN

Let it be L a LN. The mapping $\Delta : L \times L \to L$, defined by

$$\Delta(a, b) = (a + b)(a \cdot b)',$$

is called **symmetric difference**. This operation on L, verifies the following list of properties:

- $\Delta(a, a) = a \cdot a'$
- $\Delta(a, b) = \Delta(b, a)$
- $\Delta(a, 0) = a$, and $\Delta(a, 1) = a'$
- $\Delta(a, b) = \Delta(a + b, a \cdot b)$
- $a \cdot b \leq \Delta(a, b)'$, and $\Delta(a, b) \leq a + b$
- If $a \leq b$, then $\Delta(a, b) = b \cdot a'$.
- $\Delta(a, b) = 1$ iff $a + b = 1$ and $a \cdot b = 0$,

for all a, b in L. The proof of each property is straightforward.

Theorem 1. If a LN verifies DUAL, then:

- $\Delta(a', b') = \Delta(a, b) = (a + b) \cdot (a' + b')$
- $\Delta(a, a') = a + a'$
- $\Delta(a, b') = \Delta(a', b) = (a' + b) \cdot (a + b')$
- $\Delta(a, b') = a' \cdot b' + a \cdot b$

From the last formula it follows "$\Delta(a,b)=0$ *iff* $a'b'+ab=1$". Obviously,

Theorem 2. In an OL, $\Delta(a,a')=1$, and $\Delta(a,a)=0$.

Theorem 3. Provided the order $\leq$ of a LN is total, $\Delta(a,b)=(a+b)\cdot(ab)'$ is the only operation on L verifying "$a\leq b$ implies $\Delta(a,b)=b\cdot a'$".

Proof. It is

$$\Delta(a,b)=\left\{\begin{array}{l} b\cdot a',\ \text{if } a\leq b \\ a\cdot b',\ \text{if } b\leq a \end{array}\right\}=(a+b)\cdot(a\cdot b').\square$$

3 The cases of symmetric difference in OL and DMA

Theorem 4. An OL is an OML *iff* $a+b=a\cdot b+\Delta(a,b)$.

Proof. 1) In a OML, since $a\cdot b\leq a+b$, is $a+b=a\cdot b+(a+b)\cdot(a\cdot b)'=a\cdot b+\Delta(a,b)$. 2) If in an OL is $a+b=a\cdot b+\Delta(a,b)$, from $a\leq b$ follows $b=a+b\cdot a'.\square$. Hence, in OML, is $\Delta(a,b)=(a+b)-(a\cdot b)$; $\Delta(a,b)$ is the relative complement of $a\cdot b$ in $a+b$. The theorem offers a re-definition of what s an OML.

Theorem 5. In a DMA, $\Delta(a,b)=a\cdot a'+b\cdot b'+a\cdot b'+a'\cdot b$.

Proof. $\Delta(a,b)=(a+b)\cdot(a'+b')=a\cdot a'+a\cdot b'+b\cdot a'+b\cdot b'.\square$

Corollary 6. In a BA, $\Delta(a,b)=a\cdot b'+a'\cdot b$.

Corollary 7. In a BA, $\Delta(a',b)=\Delta(a,b')=\Delta(a,b)'$.

Theorem 8. In an OML, $\Delta(a\cdot b,\Delta(a,b))=a+b$, and $\Delta(a+b,\Delta(a,b))=a\cdot b$.

Proof. 1) $\Delta(a\cdot b,\Delta(a,b))=(a\cdot b+\Delta(ab)).\ a\cdot b\cdot(a\cdot b)')'=a\cdot b+\Delta(a,b)=a+b$. 2) $\Delta(a+b,\Delta(a,b))=(a+b)\cdot\Delta(a,b)'=(a+b)\cdot((a+b)'+a\cdot b)=(a+b)\cdot(a\cdot b)=a\cdot b$, since $a\cdot b=(a+b)\cdot((a+b)'+a\cdot b)$ because of $a\cdot b\leq a+b.\square$

Theorem 9. In an OML, is $\Delta(a,b)=0$ *iff* $a=b$.

Proof. It is known that $\Delta(a,a)=0$. If $\Delta(a,b)=0$, is $a+b=a\cdot b+0=a\cdot b$ that means $a=b.\square$

Theorem 10. A LN verifying DUAL is an OL, if and only if $\Delta(a,a)=0$, or $\Delta(a,a')=1$.

Proof. Straigforward.$\square$

Theorem 11. An OML is a BA, if and only if $\Delta(a,b) = a \cdot b' + a' \cdot b$.

Proof. Necessity is corollary 6. If $\Delta(a,b) = a \cdot b' + a' \cdot b$, since $a \cdot b' \leq a + b'$ implies $(a+b')' \leq (a \cdot b')'$, it results $(a \cdot b')' = (a+b')' + (a \cdot b')' \cdot (a+b') = (a+b')' + \Delta(a \cdot b') = a' \cdot b + a'b' + ab \leq b + a' \cdot b'$, by using the hypothesis $\Delta(a,b) = a \cdot b' + a' \cdot b$. Since, $(a \cdot b')' = a' + b \geq a' \cdot b' + b$, it follows $(a \cdot b')' = b + a' \cdot b'$ that, among OML, characterizes BA (see (5)). $\square$

Last two theorems offer a re-definition of what is, respectively, an OL and an OML.

4 On the associativity of the symmetric difference

In proper ortholattices, Δ is not an associative operation. Even more, in general, it is not associative in orthomodular lattices. For example, in the OML with six elements $L = \{0, a, a', b', b', 1\}$ (chinesse lantern), where $0 < a < 1$, $0 < b < 1$, $0 < b' < 1$ and $0 < a' < 1$ but the four elements a, b, a', b' are not comparable, is $\Delta(a, \Delta(b, a')) = a'$ but $\Delta(\Delta(a,b), a') = a$.

Theorem 12. In a DMA, Δ is associative if and only if the algebra is a KA.

Proof. Let Δ be associative. Then, from $\Delta(a, \Delta(b,b)) = \Delta(a, b \cdot b') = aa' + ab + ab' + bb'$, and $\Delta(\Delta(a,b), b) = \Delta(a'b + ab', b) = ab + ab' + bb'$, it follows:

$$aa' + ab + ab' + bb' = ab + ab' + bb',$$

and

$$aa' \leq ab + ab' + bb' \leq b + b',$$

for all a, b. Hence, the DMA is a KA. Conversely, if the algebra is a KA, it is $a \cdot a' \cdot (b + b') = a \cdot a'$, and $b \cdot b' \cdot (a + a') = b \cdot b'$, for all a, b. Therefore, after some computions one obtains:

$$\Delta(a, \Delta(b,c)) = \Delta(\Delta(a,b), c) = a \cdot a' + b \cdot b' + c \cdot c' + a \cdot b' \cdot c' + a' \cdot b \cdot c' + a' \cdot b' \cdot c + a \cdot b \cdot c. \square$$

Thus this theorem offers a re-definition of what is a KA.

Corollary 13. In the DMA ([0,1], min, max, 1-id), Δ is associative.

Proof. This is a known result (see (1)) that now follows from min $(x, 1-x) \leq \max(y, 1-y)$ for all x, y in $[0,1]$. $\square$.

Obviously, if $J \subset [0,1]$ is closet under min, max and 1-id, it is (J, min, max, 1-id) a DMA with Δ associative.

Corollary 14. In a BA, Δ is associative.

Proof. This is a well known result (see, for example, (2)), that now follows from $a \cdot a' = 0 < 1 = b + b'$, and gives $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c) = a \cdot b' \cdot c' + a' \cdot b \cdot c' + a' \cdot b' \cdot c + a \cdot b \cdot c.$ $\Box$

Theorem 15. An OML is a BA if and only if Δ is an associative operation.

Proof. By last corollary, in a BA, Δ is associative. If L is an OML, the following formulas are well known to hold (see ref (4)):

$$((a' + b)(a + b') + b')((a' + b)' + b) = (a + b')(b + ab') = (a + b')b + ab' \ [*]$$

If Δ is associative in L, from $\Delta(a', \Delta(b, b')) = \Delta(\Delta(a', b), b')$ it, respectively, follows:

i) $\Delta(a', \Delta(b, b')) = \Delta(a', (b + b')(bb')') = \Delta(a', 1) = a$

ii) $\Delta(\Delta(a, b), b') = \Delta(((a'+b)(a'b)', b') = ((a'+b)(a+b')+b')((a'+b)(a'b)'b')' = ((a' + b)(a + b') + b')((a' + b)' + b) = (a + b')(b + ab')$, because of [1].

Thus, $(a + b')(b + ab') = a$. Hence, it can be written $ab + ab' = (a + b')(b + ab')b + (a + b')(b + ab')b'$ [$**$]. Since $(b + ab')b = b$ and, applying the orthomodular law to $ab' \leq b'$, it follows $(b + ab')b' = ab'$, finally, from [*] and [**] it results :

$$ab + ab' = (a + b')b + (a + b')ab' = (a + b')b + ab' = a \text{ for all } a, b \text{ in } L. \Box$$

This theorem also offers a re-definition of what is a BA.

5 The ring $(L, \Delta, \cdot)$

It is immediate that if L is a KA, (L, Δ) is a *commutative semigroup* with neutral element 0, but, in general, this semigroup is not a group. For example, it is easy to see that in the KA ([0,1], min, max, 1-id), for no $a \in (0, 1)$ it exists $a^{-1} \in [0, 1]$ such that $\Delta(a, a^{-1}) = 0$. In all DMA L, the subset of *boolean elements*, bool $(L) = \{a \in L; a \cdot a' = 0\}$, is not empty since $\{0, 1\} \subset$ bool (L). Obviously, L is a BA if and only if $L \subset$ bool (L).

Theorem 16. Let L be a KA. (L, Δ) is a group *if and only if* L=bool (L).

Proof. If $L =$ bool (L), $\Delta(a, a) = aa' = 0$, and this means $a^{-1} = a$. Provided (L, Δ) is a group and $a \in L$, there is a unique $a^{-1} \in L$ such that $\Delta(a, a^{-1}) = 0$. Then, by theorem 5, it follows

$$a \cdot a' + a^{-1} \cdot (a^{-1})' + a' \cdot a^{-1} + a \cdot (a^{-1})' = 0,$$

that implies $a \cdot a' = a^{-1} \cdot (a^{-1})' = a' \cdot a^{-1} = a \cdot (a^{-1})' = 0$, and, in particular $a \in$ bool (L). Hence, $L \subset$ bool (L). $\Box$

Corollary 17. In the group (L, Δ), it is $a^{-1} = a$, for all $a \in L$.

Proof. Since L needs to be a boolean algebra. $a = a \cdot a^{-1} + a \cdot (a^{-1})'$, and $a^{-1} = a^{-1} \cdot a + a^{-1} \cdot a'$. Since $a' \cdot a^{-1} = a \cdot (a^{-1})' = 0$, it follows $a = a \cdot a^{-1}$ and $a^{-1} = a^{-1} \cdot a$, that is $a \leq a^{-1}$ and $a^{-1} \leq a$.□

Corollary 18. If L is a DMA, (L, Δ) is a group *if and only if* L is a BA. The elements of such group are self-inverse.

Proof. Follows immediately from theorems 5 and 15, and corollary 16.□

In general, the intersection $\cdot$ does not distribute over Δ. In proper KA it is not $a \cdot \Delta(b, c) = \Delta(a \cdot b, a \cdot c)$. For example, in ([0,1],min, max, 1-id) is $0.7 \cdot \Delta(0.8, 0.9) = 0.2$, but $\Delta(0.7 \cdot 0.8, 0.7 \cdot 0.9) = \Delta(0.7, 0.7) = 0.3$. Also in the orthomodular case of the *chinesse lantern*, is $\Delta(a \cdot b, b \cdot b') = \Delta(0, 0) = 0$, but $a \cdot \Delta(b, b') = a$: neither in proper ortholattices is $a \cdot \Delta(b \cdot c) = \Delta(a \cdot b, a \cdot c)$ verified.

Theorem 19. If L is a BA, $a \cdot \Delta(b, c) = \Delta(a \cdot b, a \cdot c)$, for all a, b, c in L.

Proof. This result, proven in (5), follows from $a\Delta(b, c) = abc' + ab'c$, and $\Delta(a \cdot b, a \cdot c) = (ab + ac)(abc)' = (ab + ac) \cdot (a' + b' + c') = abc' + ab'c$.□.

Thus, only in the case of a BA the structures $(L, \Delta, \cdot)$ is a ring.

6 Conclusions

The definition of symmetric difference, $\Delta(a, b) = (a + b)(ab)'$, adopted in section 2 for all lattices with negation, proves its adequacy when particularized to ortholattices, orthomodular lattices, de Morgan and boolean algebras. It should be pointed out that $\Delta(a, b)$ results to be, in orthomodular lattices, the relative complement of $a \cdot b$ in $a + b$; that only in boolean algebras is $\Delta(a, b) = a' \cdot b + a \cdot b'$; that only in orthocomplement lattices is $\Delta(a, b) = 0$ if and only if $a = b$, and that the associativity of Δ seems to be restricted to those de Morgan algebras that are Kleene algebras, like, for example, boolean algebras and, the mentioned chains (J, min, max, 1-id).

Note that the usual way of defining Δ in boolean algebras of sets, $\Delta(a \cdot b) = (b - a) + (a - b)$, has no strict theoretical sense because of $b - a$ requires $a \leq b$ and $a - b$ requires $b \leq a$. Such formula can conduce to some conceptual mistakes as those shown in (1).

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