On the Central Limit Theorem on IFS-events

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Abstract

A probability theory on IFS-events has been constructed in [3], and axiomatically characterized in [4]. Here using a general system of axioms it is shown that any probability on IFS-events can be decomposed onto two probabilities on a Łukasiewicz tribe, hence some known results from [5], [6] can be used also for IFS-sets. As an application of the approach a variant of Central limit theorem is presented.

Keywords. Probability theory, IFS-events

1 Introduction

An IFS-set $A$ on a space $\Omega$ as a couple $(\mu_A, \nu_A)$ is understood, $\mu_A : \Omega \rightarrow (0,1)$, $\nu_A : \Omega \rightarrow (0,1)$ such that $\mu_A(\omega) + \nu_A(\omega) \leq 1$ for any $\omega \in \Omega$ (see[1]). The function $\mu_A$ is called the membership function, the function $\nu_A$ is called the non membership function. An IFS-set $A = (\mu_A, \nu_A)$ is called IFS-event if $\mu_A, \nu_A$ are $\mathcal{S}$-measurable with respect to a given $\sigma$-algebra of subsets of $\Omega$.

In [3] P. Grzegorzewski and E. Mrowka considered a classical probability space $(\Omega, \mathcal{S}, P)$ and they suggested to define a probability measure on the set $\mathcal{G}$ of all IFS events as an interval valued function $P$ by the following way. Probability $P(A)$ of an event $A = (\mu_A, \nu_A)$ is the interval

$$P(A) = [\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP].$$

If $\nu_A = 1 - \mu_A$, then the interval is the singleton $\int_\Omega \mu_A dP$, hence the Grzegorzewski and Mrowka definition is an extension of the Zadeh definition. The probability $P$

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is the function $P : \mathcal{G} \to \mathcal{J}$, where $\mathcal{J}$ is the family of all compact subintervals of the unit interval $I = [0, 1]$. In [3] many properties of the mapping $P$ were discovered. Then in [4] it was proved that any function $P : \mathcal{G} \to \mathcal{J}$ satisfying some properties (as continuity, some kind of additivity etc.) has the form $(\ast)$.

Special attention should by devoted to the notion of additivity of $P$. Namely in fuzzy sets theory there are many possibilities how the define the intersection and the union of fuzzy sets. Recall that the representation theorem works with the Łukasiewicz connectives $\oplus$, $\odot$, hence the additivity has the form

$$A \odot B = (0, 1) \implies P(A \oplus B) = P(A) + P(B)$$

In the paper we shall use a more general situation. Instead of the set of all measurable functions with values in $(0, 1)$ we shall consider any Łukasiewicz tribe $T$. Instead of IFS-events we consider the family $\mathcal{F}$ of all couples $(f, g)$ of elements of $T$ such that $f + g \leq 1$. Then we define axiomatically the notion of a probability as a function from $\mathcal{F}$ to the family $\mathcal{J}$ of all compact subintervals of the unit interval. Moreover, we define the notion of an observable, that is an analogue of the notion of a random variable in the Kolmogorov theory. This notion is introduced here for the first time with regard to IFS-events. The main result of the paper are the representation theorems representing probabilities and observables in $\mathcal{F}$ by the corresponding notions in $T$. Consequently it is possible to transpose some known theorems from the probability theory on tribes to the more general case of IFS-events. As an illustration of the developed method the central limit theorem is presented.

In Section 2 we give the definitions of basic notions and some examples. Section 3 contains the representation theorems. In Section 4 and Section 5 a version of the central limit theorems is presented.

\section{Probabilities and observables}

Recall that a tribe is a non-empty family $T$ of functions $f : \Omega \to (0, 1)$ satisfying the following conditions:

(i) $f \in T \implies 1 - f \in T$;

(ii) $f, g \in T \implies f \oplus g = \min(f + g, 1) \in T$;

(iii) $f_n \in T \ (n = 1, 2, \ldots), f_n \not\to f \implies f \in T$.

Of course, a tribe is a special case of a $\sigma$-MV-algebra.

In the preceding definition (instead of $\max(a, b)$) we have used the first Łukasiewicz operation $\oplus: (0, 1) \times (0, 1) \to (0, 1), a \oplus b = \min(a + b, 1)$. The second binary operation $\odot$ is defined by the equality $a \odot b = \max(a + b - 1, 0)$. It is easy to see that $\chi_A \oplus \chi_B = \chi_{A \cup B}, \chi_A \odot \chi_B = \chi_{A \cap B}$. Recall [5,6] that probability (= a state) on a Łukasiewicz tribe $T$ is any mapping $p : T \to (0, 1)$ satisfying the following conditions:

(i) $p(1_\Omega) = 1$;
(ii) if \( f \odot g = 0 \Omega \), then \( p(f \odot g) = p(f) + p(g) \);

(iii) if \( f_n \not/ f \), then \( p(f_n) \not/ p(f) \).

Example 1. Let \( \mathcal{S} \) be a \( \sigma \)-algebra of subsets of a set \( \Omega \), \( P : \mathcal{S} \to [0,1) \) be a probability measure. \( \chi_A \) be the characteristic function of a set \( A \in \mathcal{S} \). Put \( \mathcal{T} = \{ \chi_A; A \in \mathcal{S} \} \), \( p(\chi_A) = P(A) \). Then \( \mathcal{T} \) is a tribe and \( p \) is a probability on \( \mathcal{T} \).

Example 2. Again let \( (\Omega, \mathcal{S}, \mathcal{P}) \) be a probability space, \( \mathcal{T} \) be the set of all \( \mathcal{S} \)-measurable function \( f : \Omega \to [0,1) \), \( p(f) = \int_\Omega f \, dP \). Then \( \mathcal{T} \) is a tribe and \( p \) is a probability on \( \mathcal{T} \) defined by Zadeh [7].

During the whole text we fix the tribe \( \mathcal{T} \) and the generated family \( \mathcal{F} \).

**Definition 1.** By an IFS-event we understand any element of the family

\[
\mathcal{F} = \{ (\mu_A, \nu_A); \mu_A, \nu_A \in \mathcal{T}, \mu_A + \nu_A \leq 1 \}
\]

To define the notion of probability on IFS-events we need to introduce operations on \( \mathcal{F} \). Let \( A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \). Then we define

\[
A \oplus B = (\mu_A \oplus \mu_B, \nu_A \ominus \nu_B), \\
A \ominus B = (\mu_A \ominus \mu_B, \nu_A \ominus \nu_B).
\]

If \( A_n = (\mu_{A_n}, \nu_{A_n}) \), then we write

\[
A_n \not/ A \iff \mu_{A_n} \not/ \mu_A, \quad \nu_{A_n} \not/ \nu_A.
\]

If \( \nu_A = 1 - \mu_A, \nu_B = 1 - \mu_B \), then

\[
A \oplus B = (\mu_A \ominus \mu_B, (1 - \mu_A) \ominus (1 - \mu_B)) = (\mu_A \ominus \mu_B, 1 - \mu_A \ominus \mu_B),
\]

and similarly \( A \ominus B = (\mu_A \ominus \mu_B, 1 - \mu_A \ominus \mu_B) \).

A probability \( \mathcal{P} \) on \( \mathcal{F} \) is a mapping from \( \mathcal{F} \) to the family \( \mathcal{J} \) of all closed intervals \( \langle a, b \rangle \) such that \( 0 \leq a \leq b \leq 1 \). Here we define

\[
\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle, \\
\langle a_n, b_n \rangle \not/ \langle a, b \rangle \iff a_n \not/ a, \quad b_n \not/ b.
\]

**Definition 2.** By an IFS-probability on \( \mathcal{F} \) we understand any function \( \mathcal{P} : \mathcal{F} \to \mathcal{J} \) satisfying the following properties:

(i) \( \mathcal{P}(\langle 1\Omega, 0\Omega \rangle) = (1,1) = \{1\}; \mathcal{P}(\langle 0\Omega, 1\Omega \rangle) = (0,0) = \{0\}; \)

(ii) \( \mathcal{P}(A \oplus B) + \mathcal{P}(A \ominus B) = \mathcal{P}(A) + \mathcal{P}(B) \) for any \( A, B \in \mathcal{F} \);

(iii) if \( A_n \not/ A \), then \( \mathcal{P}(A_n) \not/ \mathcal{P}(A) \).

\( \mathcal{P} \) is called separating, if \( \mathcal{P}(\langle f, g \rangle) = \langle p(f), 1 - q(g) \rangle \) for some \( p, q : \mathcal{T} \to [0,1) \).
Example 3. ([3]). Let \((\Omega, \mathcal{S}, P)\) be a probability space \(\mathcal{T} = \{f; f : \Omega \to (0,1), f \text{ is } \mathcal{S} \text{ measurable}\}\), and for \(A \in \mathcal{F}, A = (\mu_A, \nu_A)\), put

\[
\mathcal{P}(A) = \left\langle \int_\Omega \mu_A \, dP, 1 - \int_\Omega \nu_A \, dP \right\rangle.
\]

Then \(\mathcal{P}\) is probability with respect to Definition 2. Indeed,

\[
\mathcal{P}(1_\Omega, 0_\Omega) = \left\langle \int_\Omega 1_\Omega \, dP, 1 - \int_\Omega 0_\Omega \, dP \right\rangle = (1, 1),
\]

\[
\mathcal{P}(0_\Omega, 1_\Omega) = \left\langle \int_\Omega 0_\Omega \, dP, 1 - \int_\Omega 1_\Omega \, dP \right\rangle = (0, 0).
\]

The property (iii) has been proved in [3], we shall prove (ii). We have

\[
\mathcal{P}(A \oplus B) + \mathcal{P}(A \odot B) =
\]

\[
\mathcal{P}((\mu_A \odot \mu_B, \nu_A \odot \nu_B)) + \mathcal{P}(\mu_A \odot \mu_B, \nu_A \odot \nu_B)
\]

\[
= \left\langle \int (\mu_A \odot \mu_B) \, dP, 1 - \int (\nu_A \odot \nu_B) \, dP \right\rangle
\]

\[
+ \left\langle \int (\mu_A \odot \mu_B) \, dP, 1 - \int (\nu_A \odot \nu_B) \, dP \right\rangle
\]

\[
= \left\langle \int (\mu_A \odot \mu_B + \mu_A \odot \mu_B) \, dP, 2 - \int (\nu_A \odot \nu_B + \nu_A \odot \nu_B) \, dP \right\rangle
\]

\[
= \left\langle \int \mu_A \, dP + \int \mu_B \, dP, 1 - \int \nu_A \, dP + 1 - \int \nu_B \, dP \right\rangle
\]

\[
= \mathcal{P}(A) + \mathcal{P}(B).
\]

Moreover, in [4] it has been proved that under two additional conditions any IFS-probability \(\mathcal{P}\) on the family \(\mathcal{F}\) generated by a \(\sigma\)-algebra \(\mathcal{S}\), has the above form.

More generally, if \(p, q : \mathcal{T} \to (0,1), p \leq q\) are probabilities, then \(\mathcal{P} : \mathcal{F} \to \mathcal{J}\) defined by \(\mathcal{P}((f,g)) = (p(f), 1 - q(g))\), is a probability. In the special case \(\mathcal{P}((f, 1 - f)) = (p(f), q(f))\).

The second important notion in the probability theory is the notion of a random variable. According to the terminology used in quantum structures we shall speak about observables instead of random variables. Recall that an observable with values in \(\mathcal{T}\) is a mapping \(x : \mathcal{B}(R) \to \mathcal{T}\) (\(\mathcal{B}(R)\) being the \(\sigma\)-algebra of Borel subsets of \(R\)) satisfying the following properties:

(i) \(x(R) = 1_\Omega\);

(ii) if \(A \cap B = \emptyset\), then \(x(A) \odot x(B) = 0_\Omega\), and \(x(A \cup B) = x(A) + x(B)\);

(iii) if \(A_n \nrightarrow A\) then \(x(A_n) \nrightarrow x(A)\).

Definition 3. A mapping \(x : \mathcal{B}(R) \to \mathcal{F}\) is called an IFS-observable, if it satisfies the following conditions:
(i) \(x(R) = (1_\Omega, 0_\Omega)\);
(ii) if \(A \cap B = \emptyset\), then \(x(A) \circ x(B) = (0_\Omega, 1_\Omega)\), and \(x(A \cup B) = x(A) \oplus x(B)\);
(iii) if \(A_n \not\sim A\), then \(x(A_n) \not\sim x(A)\).

3 Representation theorems

**Theorem 1.** \(\mathcal{P} : \mathcal{F} \to \mathcal{J}\) is a separating IFS-probability if and only if \(p, q : \mathcal{T} \to \langle 0, 1 \rangle\) are probabilities.

**Proof.** Let \(A = (f_1, f_2), B = (g_1, g_2) \in \mathcal{F}\). Then
\[
A \circ B = (f_1 \oplus g_1, f_2 \oplus g_2),
A \circ B = (f_1 \circ g_1, f_2 \circ g_2),
\mathcal{P}(A \circ B) = \langle p(f_1 \circ g_1) + (1 - q(f_2 \circ g_2)),
\mathcal{P}(A \circ B) = \langle p(f_1 \circ g_1) + (1 - q(f_2 \circ g_2)),
\mathcal{P}(A \oplus B) + \mathcal{P}(A \circ B) = \langle p(f_1 \oplus g_1) + p(f_1 \circ g_1), 2 - q(f_2 \circ g_2) - q(f_2 \circ g_2)),
\mathcal{P}(A) + \mathcal{P}(B) = \langle p(f_1, 1 - q(g_1)) + (p(g_1), 1 - q(g_2))
\mathcal{P}(A) + \mathcal{P}(B) = \langle p(f_1, 1 - q(g_1)) + (p(g_1), 1 - q(g_2))\rangle = \langle p(f_1) + p(g_1), 2 - q(g_1) - q(g_2)\rangle,
\]

hence
\[
p(f_1 \oplus g_1) + p(f_1 \circ g_1) = p(f_1), p(g_1),
q(f_2 \circ g_2) + q(f_2 \circ g_2) = q(f_2), q(g_2),
\]

for all \(f_1, f_2, g_1, g_2 \in \mathcal{T}\).

By these two equalities the additivity of \(p\) and \(q\) follows. If \(I = (1_\Omega, 0_\Omega)\), then
\[
\langle p(1_\Omega), 1 - q(0_\Omega)\rangle = \mathcal{P}(I) = \{1\},
\]

hence \(p(1_\Omega) = 1\).

On the other hand, if \(O = (0_\Omega, 1_\Omega)\), then
\[
\langle p(0_\Omega), 1 - q(1_\Omega)\rangle = \mathcal{P}(O) = \{0\},
\]

hence \(1 - q(1_\Omega) = 0, q(1_\Omega) = 1\).

Now we prove the continuity of \(p\) and \(q\). First let \(f_n \in \mathcal{T}\), \(n = 1, 2, \ldots\), \(f_n \not\sim f\). Put \(F_n = (f_n, 1 - f_n)\). Then \(F_n \in \mathcal{F}, F_n \not\sim F = (f, 1 - f)\). Therefore
\[
\langle p(f_n), 1 - q(f_n)\rangle = \mathcal{P}(F_n) \not\sim \mathcal{P}(F) = \langle p(f), 1 - q(f)\rangle,
\]

hence \(p(f_n) \not\sim p(f), 1 - q(f_n) \not\sim 1 - q(f), q(f_n) \not\sim q(f)\). \(\Box\)

**Theorem 2.** Let \(x : \mathcal{B}(R) \to \mathcal{F}\). For any \(A \in \mathcal{B}(R)\) denote \(x(A) = (x^2(A), 1 - x^2(A))\). Then \(x\) is IFS-observable if and only if \(x^2 : \mathcal{B}(R) \to \mathcal{T}, x^4 : \mathcal{B}(R) \to \mathcal{T}\) are observables.
Proof. Since 

\[(1_\Omega, 0_\Omega) = x(R) = (x^\flat(R), 1 - x^\sharp(R)),\]

we obtain

\[x^\flat(R) = 1_\Omega, x^\sharp(R) = 1_\Omega.\]

Let \(A \cap B = \emptyset.\) Then

\[(0_\Omega, 1_\Omega) = x(A) \odot x(B)
= (x^\flat(A), 1 - x^\sharp(A)) \odot (x^\flat(B), 1 - x^\sharp(B))
= (x^\flat(A) \odot x^\flat(B), (1 - x^\sharp(A)) \oplus (1 - x^\sharp(B))),\]

hence \(0_\Omega = x^\flat(A) \odot x^\flat(B).\) Further

\[1_\Omega = (1 - x^\sharp(A)) \oplus (1 - x^\sharp(B)) = (1 - x^\sharp(A) + 1 - x^\sharp(B)) \wedge 1,\]

hence

\[1 - x^\sharp(A) + 1 - x^\sharp(B) \geq 1,\]
\[1 \geq x^\sharp(A) + x^\sharp(B),\]
\[x^\flat(A) \odot x^\flat(B) = (x^\sharp(A) + x^\sharp(B) - 1) \lor 0 = 0.\]

Moreover,

\[(x^\flat(A \cup B), 1 - x^\sharp(A \cup B)) =
= x(A \cup B) = x(A) \oplus x(B)
= (x^\flat(A), 1 - x^\sharp(A)) \oplus (x^\flat(B), 1 - x^\sharp(B))
= (x^\flat(A) \odot x^\flat(B), (1 - x^\sharp(A)) \oplus (1 - x^\sharp(B)))
= (x^\flat(A) + x^\flat(B), (1 - x^\sharp(A) + 1 - x^\sharp(B) - 1) \lor 0)
= (x^\flat(A) + x^\flat(B), 1 - (x^\sharp(A) + x^\sharp(B))).\]

Therefore

\[x^\flat(A \cup B) = x^\flat(A) + x^\flat(B),\]
\[x^\sharp(A \cup B) = x^\sharp(A) + x^\sharp(B).\]

Finally, let \(A_n \not\subset A.\) Then

\[(x^\flat(A_n), 1 - x^\sharp(A_n)) = x(A_n) \not\subset x(A) = (x^\flat(A), 1 - x^\sharp(A)),\]

hence

\[x^\flat(A_n) \not\subset x^\flat(A), \quad 1 - x^\sharp(A_n) \not\subset 1 - x^\sharp(A), \quad \text{i. e.} \quad x^\sharp(A_n) \not\subset x^\sharp(A). \quad \square\]
It is easy to see that the mappings
\[ p_{x^3} = p \circ x^3 : B(R) \to (0, 1), \quad q_{x^2} = q \circ x^2 : B(R) \to (0, 1) \]
are probability measures. Therefore we define
\[ E(x^3) = \int_R t \, dp_{x^3}(t), \quad E(x^2) = \int_R t \, dq_{x^2}(t) \]
if these integrals exist. In this case we say that \( x \) is integrable. Further we define
\[ \sigma^2(x^3) = \int_R (t - E(x^3))^2 \, dp_{x^3}(t), \quad \sigma^2(x^2) = \int_R (t - E(x^2))^2 \, dq_{x^2}(t) \]
if these integral exists. In this case we say that \( x \) belongs to \( L^2 \).

**Theorem 3.** Let \( \mathcal{P} : \mathcal{F} \to \mathcal{J} \) be a separating IFS-probability given by \( \mathcal{P}(f, g) = (p(f), 1 - q(g)), \) \( x : B(R) \to \mathcal{F} \) be an IFS-observable given by \( x(A) = (x^3(A), 1 - x^2(A)) \). Then \( \mathcal{P} \circ x : B(R) \to \mathcal{J} \) is given by
\[ \mathcal{P} \circ x(A) = (p(x^3(A)), q(x^2(A))) \]

**Proof.** Evidently
\[ \mathcal{P} \circ x(A) = \mathcal{P}(x(A)) = \mathcal{P}((x^3(A), 1 - x^2(A))) = (p(x^3(A)), 1 - q(1 - x^2(A))) = (p(x^3(A)), q(x^2(A))) \]

\[ \square \]

4 Independence

**Definition 4.** An \( n \)-dimensional IFS-observable is a mapping \( h : B(R^n) \to \mathcal{F} \) satisfying the following conditions:

(i) \( h(R^n) = (1, 0_n) \);

(ii) if \( A \cap B = \emptyset \), then \( h(A) \cap h(B) = (0_2, 1_1) \), and \( h(A \cup B) = h(A) + h(B) \);

(iii) if \( A_n \not\subset A \), then \( h(A_n) \not\subset h(A) \).

Recall that observables \( (x_1, \ldots, x_n) : B(R) \to \mathcal{T} \) are called independent if there exists \( n \)-dimensional observable \( h : B(R^n) \to \mathcal{T} \) such that
\[ p(h(A_1 \times A_2 \times \cdots \times A_n)) = p(x_1(A_1)) \cdot p(x_2(A_2)) \cdots \cdot p(x_n(A_n)) \]
for any \( (A_1, \ldots, A_n) \in B(R) \).

**Definition 5.** IFS-observables \( x_1, \ldots, x_n : B(R) \to \mathcal{F} \) are called independent with respect to an IFS-probability \( \mathcal{P} \), if there exists \( n \)-dimensional observable \( h : B(R) \to \mathcal{F} \) such that
\[ \mathcal{P}(h(A_1 \times A_2 \times \cdots \times A_n)) = \mathcal{P}(x_1(A_1)) \otimes \mathcal{P}(x_2(A_2)) \otimes \cdots \otimes \mathcal{P}(x_n(A_n)) \]
for any \((A_1, A_2, \ldots, A_n) \in B(R)\). Here
\[
\langle a_1, b_1 \rangle \otimes \langle a_2, b_2 \rangle \otimes \cdots \otimes \langle a_n, b_n \rangle = \langle a_1a_2 \ldots a_n, b_1b_2 \ldots b_n \rangle
\]
for any \((a_i, b_i) \in \mathcal{J}(i = 1, 2, \ldots, n)\).

**Theorem 4.** Let \(\mathcal{P} : \mathcal{F} \to \mathcal{J}\) be a separating probability. Then IFS-observables
\(x_1, x_2, \ldots, x_n \in B(R) \to \mathcal{F}\) are independent if and only if the corresponding observables
\(x_1^{'}, x_2^{'}, \ldots, x_n^{'} : B(R) \to T\) are independent as well as \(x_1^{''}, x_2^{''}, \ldots, x_n^{''} : B(R) \to T\).

**Proof.** Let \(A_1, A_2, \ldots, A_n \in B(R)\). Then by Theorem 3
\[
\langle p(h^3(A_1 \times \cdots \times A_n)), q(h^2(A_1 \times \cdots \times A_n)) \rangle = \\
\mathcal{P}(h(A_1 \times \cdots \times A_n)) = \mathcal{P}(x_1(A_1)) \otimes \mathcal{P}(x_2(A_2)) \otimes \cdots \otimes \mathcal{P}(x_n(A_n))
\]

\[
= \langle p(x_1^3(A_1)), q(x_1^2(A_1)) \rangle \otimes \cdots \otimes \langle p(x_n^3(A_n)), q(x_n^2(A_n)) \rangle
\]

\[
= \langle p(x_1^3(A_1)) \cdot p(x_2^3(A_2)) \cdots \cdot p(x_n^3(A_n)), q(x_1^2(A_1)) \cdot q(x_2^2(A_2)) \cdots \cdot q(x_n^2(A_n)) \rangle
\]

hence
\[
p(h^3(A_1 \times \cdots \times A_n)) = p(x_1^3(A_1)) \cdot p(x_2^3(A_2)) \cdots \cdot p(x_n^3(A_n)),
\]
\[
q(h^2(A_1 \times \cdots \times A_n)) = q(x_1^2(A_1)) \cdot q(x_2^2(A_2)) \cdots \cdot q(x_n^2(A_n)).
\]

\(\Box\)

5 **Central limit theorem**

A sequence \((x_n)_{n=1}^{\infty}\) of IFS-observables is called independent, if \((x_1, x_2, \ldots, x_n)\) are
independent for any \(n\). They are equally distributed if
\[
x_n((\infty, t)) = x_1((\infty, t))
\]
for any \(n \in N\) and \(t \in R\).

If \(k : R^n \to R\) is any Borel function and \(x_1^{'}, \ldots, x_n^{'} : B(R) \to T\) are observable,
and \(h^3\) their joint observable we define \(k(x_1^{'}, \ldots, x_n{'})\) by the formula
\[
k(x_1^{'}, \ldots, x_n{'}) = h^3(k^{-1}(A))
\]
for any \(A \in B(R)\). E.g.
\[
\frac{\sqrt{n}}{\sigma_1} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^{'} - a_1 \right) : B(R) \to T
\]
is defined by the formula
\[
\frac{\sqrt{n}}{\sigma_1} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^{'} - a_1 \right) (A) = h^3(k^{-1}(A)),
\]
where
\[
k(u_1, u_2, \ldots, u_n) = \frac{\sqrt{n}}{\sigma_1} \left( \frac{1}{n} \sum_{i=1}^{n} u_i - a_1 \right).
\]
Theorem 5. Let $\mathcal{P} : \mathcal{F} \rightarrow \mathcal{J}$ be a separating probability. Let $(x_n)_{1}^{\infty}$ be a sequence of independent equally distributed IFS-observables from $L^2$, $E(x_1^\flat) = a_1$, $E(x_1^\sharp) = a_2$, $\sigma^2(x_1^\flat) = \sigma_1^2$, $\sigma^2(x_1^\sharp) = \sigma_2^2$, $y_n^\flat = \frac{\sqrt{n}}{\sigma_1} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^\flat - a_1 \right)$, $y_n^\sharp = \frac{\sqrt{n}}{\sigma_2} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^\sharp - a_2 \right)$, $y_n = (y_n^\flat, 1 - y_n^\sharp)$. Then

$$\lim_{n \to \infty} \mathcal{P}(y_n((-\infty, t))) = \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left( -\frac{u^2}{2} \right) du \right\}$$

for every $t \in \mathbb{R}$.

Proof. By Theorem 4 $(x_n^\flat)_{1}^{\infty}$, $(x_n^\sharp)_{1}^{\infty}$ are independent and evidently equally distributed. Let $h_n^\flat$, be the joint observable of $x_1^\flat, \ldots, x_n^\flat$,

$$k_n(h_1, \ldots, h_n) = \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_i - a_1 \right), \quad y_n^\flat = h_n^\flat \circ k_n^{-1}.$$

Then by [5], Theorem 3.12

$$\lim_{n \to \infty} p(y_n^\flat((-\infty, t))) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left( -\frac{u^2}{2} \right) du.$$

Similarly

$$\lim_{n \to \infty} q(y_n^\sharp((-\infty, t))) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left( -\frac{u^2}{2} \right) du. \quad \square$$

6 Conclusions

The paper is concerned in the probability theory on IFS-events. The main result of the paper is an original method of achieving new results of the probability theory on IFS-events by the corresponding results holding for fuzzy events. The method can be developed in two directions. First instead of a tribe of fuzzy sets one could try to consider any MV-algebra. Secondly, instead of independency of observables a kind of compatibility could be introduced and then conditional probabilities could be considered.

References


