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Characterizing Intraregular Semigroups by Intuitionistic Fuzzy Sets

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Abstract

In this paper, we give some theorems which characterizes the intraregular semigroups in terms of intuitionistic fuzzy left, right, and biideals.

Keywords: Intuitionistic fuzzy ideal; intuitionistic fuzzy biideal; Intraregular semiigroup.

1 Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The semigroup theory of fuzzy sets was deeply studied by many authors[1-6]. But in fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. In 1983, Antanassov [10] introduces the concept of intuitionistic fuzzy sets, which constitute a extension of fuzzy sets theory: intuitionistic fuzzy sets give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must smaller than or equal to 1. In this paper, we give some theorems which characterizes the intraregular semigroups in terms of intuitionistic fuzzy left, right, and biideals.

2 Preliminaries

Let S be a semigroup, a subsemigroup of S is a nonempty subset A of S such that $A^2 \subseteq A$ and a left (right) ideal of S is a nonempty subset A of S such that $SA \subseteq A \quad (AS \subseteq A)$, a two-ideal (or simply ideal) is a subset of S which is both a left and a right ideal of S.

Definition 1[12]: An intuitionistic fuzzy set A in S is an object

$$A = \{ \langle x, \mu_A(x), v_A(x) \rangle | \, x \in S \}$$

where, for all$x \in S$,$\mu_A(x) \in [0,1]$ and $v_A(x) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of x in S, and furthermore satisfy $\mu_A(x) + v_A(x) \leq 1$.

Definition 2[12]: Let $A,\ B$ be two intuitionistic fuzzy sets in S, then

$$A \subseteq B \quad \text{iff} \quad (\forall x \in S)\left(\mu_A(x) \leq \mu_B(x) \ \& \ v_A(x) \leq v_B(x)\right),$$

$$A \cap B \quad = \quad \{ \langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{ v_A(x), v_B(x)\} \rangle | \, x \in S \},$$

$$A \circ B \quad = \quad \{ \langle x, \mu_{A \circ B}(x), v_{A \circ B}(x) \rangle | \, x \in S \}$$

where

$$\mu_{A \circ B}(x) = \begin{cases} \sup\limits_{x=yz} \{\min\{\mu_A(y), \mu_B(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 0 \text{ otherwise} \end{cases},$$

$$v_{A \circ B}(x) = \begin{cases} \inf\limits_{x=yz} \{\max\{v_A(y), v_B(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 1 \text{ otherwise} \end{cases}.$$

Definition 3: If S be a semigroup, an intuitionistic fuzzy set A in S is called an intuitionistic fuzzy semigroup in S if

$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $v_A(xy) \leq \max\{v_A(x), v_A(y)\}$ for all $x, y \in S$.

And is called a intuitionistic fuzzy left (right) ideal of S if

$$\mu_A(xy) \geq \mu_A(y) \text{ and } v_A(x,y) \leq v_A(y)\,(\mu_A(xy) \geq \mu_A(x) \text{ and } v_A(x,y) \leq v_A(x))$$

for all $x, y \in S$. An intuitionistic fuzzy set A in S is called an intuitionistic fuzzy two-sided ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic right ideal of S.

Definition 4: An intuitionistic fuzzy semigroup A in S is called an intuitionistic fuzzy biideal of if

$$\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}, v_A(xyz) \leq \max\{v_A(x), v_A(z)\}$$

for all $x, y, z \in S$.

Let A be a subset of a semigroup S, then we denote

$$\tilde{A} = \left\{ \left\langle x, \mu_{\tilde{A}(x)}, v_{\tilde{A}(x)} \right\rangle \middle| x \in S \right\}$$

where

$$\mu_{\tilde{A}(x)} = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}, v_{\tilde{A}(x)} = \begin{cases} 0 & x \in A \\ 1 & \text{otherwise} \end{cases}.$$

Obviously $\tilde{A}$ is an intuitionistic fuzzy set in S, semigroup S also can be seen as an intuitionistic fuzzy set $\tilde{S} = \{\langle x, 1, 0\rangle | x \in S\}$. In the present paper, we will use S represent S and $\tilde{S}$.

Definition 5: A semigroup S is called intraregular if, for each element a of S, there exist element x and y of S such that $a = xa^2y$.

Lemma 1[13]**:** For a semigroup S the following conditions are equivalent:
(1) S is intraregular;
(2) $A \cap B \subset AB$ holds for every left ideal A and every right ideal B of S.

Lemma 2[15]**:** A nonempty subset A of a semigroup S is a biideal of S if and only if $\tilde{A}$ is an intuitionistic fuzzy biideal of S.

Lemma 3[15]**:** For a semigroup S, the following conditions are equivalent:
(1) S is regular;
(2) $A \circ B = A \cap B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of S.

Lemma 4[15]**:** For an intuitionistic fuzzy set A of a semigroup S, the following conditions are equivalent:
(1) A is an intuitionistic fuzzy left ideal of S.
(2) $S \circ A \subseteq A$.

Lemma 5[15]**:** For an intuitionistic fuzzy set A of a semigroup S, the following conditions are equivalent:
(1) A is an intuitionistic fuzzy right ideal of S.
(2) $A \circ S \subseteq A$.

Lemma 6[14,15]**:** Let A be a nonempty subset of a semigroup S, then

(1) A is a subsemigroup of S if and only if $\tilde{A}$ is an intuitionistic fuzzy semigroup of S.

(2) A is a left (right, two-sided) ideal of S if and only if $\tilde{A}$ is an intuitionistic fuzzy left (right, two-sided) of S.

3 Characterizing intraregular semigroups

As is well known, a semigroup S is intraregular if and only if it is a semilattice of simple semigroups. Now we shall give a characterization of an intraregular semigroup by intuitionistic fuzzy right ideals and intuitionistic fuzzy left ideals.

Theorem 1: For a semigroup S the following conditions are equivalent:

(1) S is intraregular;

(2) $A \cap B \subset B \circ A$ holds for every intuitionistic fuzzy right ideal A and every intuitionistic left ideal B of S.

Proof: First assume that (1) holds. Let A be any intuitionistic fuzzy biideal of S, and a any element of S. Since S is intraregular, there exists element x, y in S such that $a = xa^2y$. Then we have

$$\begin{aligned} \mu_{B\circ A}(a) &= \sup_{a=yz}\{\min\{\mu_B(y), \mu_A(z)\}\} \\ &\geq \min\{\mu_B(ax), \mu_A(ay)\} \\ &\geq \min\{\mu_B(a), \mu_A(a)\} \\ &= \mu_{A\cap B}(a) \end{aligned}$$

$$\begin{aligned} v_{B\circ A}(a) &= \inf_{a=yz}\{\max\{v_B(y), v_A(z)\}\} \\ &\leq \max\{v_B(ax), v_A(ay)\} \\ &\leq \max\{v_B(a), v_A(a)\} \\ &= v_{A\cap B}(a) \end{aligned}$$

so we have $A \cap B \subseteq B \circ A$. Thus we obtain (1) implies (2).

Conversely, assume (2) holds, let R and L be a right ideal and a left ideal of S, respectively. By Lemma6, $\tilde{R}$ and $\tilde{L}$ be the intuitionistic fuzzy right ideal and intuitionistic fuzzy left ideal of S. Let a be any element of $L \cap R$, hen we have

$$\begin{aligned} \mu_{\tilde{L}\circ\tilde{R}}(a) &= \sup_{a=yz}\left\{\min\left\{\mu_{\tilde{L}(y)}, \mu_{\tilde{R}(z)}\right\}\right\} \\ &\geq \mu_{\tilde{L}\circ\tilde{R}}(a), \\ &\geq \min\left\{\mu_{\tilde{L}(a)}, \mu_{\tilde{R}(a)}\right\} \\ &= \min\{1, 1\} \\ &= 1 \end{aligned}$$

$$\begin{aligned} v_{\tilde{L}\circ\tilde{R}}(a) &= \inf_{a=yz}\left\{\max\left\{v_{\tilde{L}(y)}, v_{\tilde{R}(z)}\right\}\right\} \\ &\le v_{\tilde{L}\circ\tilde{R}}(a) \\ &\le \max\left\{v_{\tilde{L}(a)}, v_{\tilde{R}(a)}\right\} \\ &= \max\{0,0\} \\ &= 0. \end{aligned}$$

This implies that there exist $b, c \in S, a = bc$ such that

$$\mu_{\tilde{L}(b)} = 1, \mu_{\tilde{R}(c)} = 1, v_{\tilde{L}(b)} = 0, v_{\tilde{R}(c)} = 0.$$

Then we have

$$a = bc \in LR,$$

And so we have

$$L \circ R \subset LR.$$

This is follows from Lemma1 that S is intraregular, and (2) implies (1). This completes the proof.

Theorem 2: For a semigroup S the following conditions are equivalent:
(1)S is both regular and intraregular;
(2)$A \circ A = A$ for every intuitionistic fuzzy biideals A of S;
(3)$A \cap B \subset (A \circ B) \cap (B \circ A)$ for all intuitionistic fuzzy biideals A and B of S;
(4) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for every intuitionistic fuzzy biideal A and every left ideal B of S;
(5) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for every intuitionistic fuzzy biideal A and every intuitionistic fuzzy right ideal B of S;
(6) $A \cap B \subset (A \circ B) \cap (B \circ A)$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of S.

Proof: It is clear that (3) implies (4), (4) implies (6), (3) implies (5), (5) implies (6) , and (3) implies (2). So we will prove that (1) implies (3), (6) implies (1), and (2) implies (1). In the following, we will prove that (1) implies (3), (6) implies (1), and (2) implies (1).

First assume that (1) holds. In order to prove that (3) holds, let A and B be any intuitionistic fuzzy biideals of S, and a any element of S. Then, since S is regular, there exists an element x in S such that

$$a = axa (= axaxa).$$

And, since S is intraregular, there exist elements y and z in S such that

$$a = ya^2z.$$

Thus we have

$$a = axa = axaxa = ax(ya^2z)xa = (axya)(azxa).$$

Since A and B are both intuitionistic fuzzy biideals of S, we have

$$\mu_A(axya) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a),$$
$$\mu_B(axya) \geq \min\{\mu_B(a), \mu_B(a)\} = \mu_B(a),$$

And

$$v_A(axya) \leq \max\{v_A(a), v_A(a)\} = v_A(a),$$
$$v_B(axya) \leq \max\{v_B(a), v_B(a)\} = v_B(a).$$

Thus we have

$$\begin{aligned}
\mu_{A\circ B}(a) &= \sup_{a=pq}[\min\{\mu_A(p), \mu_B(q)\}] \\
&\geq \min\{\mu_A(axya), \mu_B(azxa)\} \\
&\geq \min\{\mu_A(a), \mu_B(a)\} \\
&= \mu_{A\cap B}(a), \\
v_{A\circ B}(a) &= \inf_{a=pq}[\max\{v_A(p), v_B(q)\}] \\
&\leq \max\{v_A(axya), v_B(azxa)\} \\
&\leq \max\{v_A(a), v_B(a)\} \\
&= v_{A\cap B}(a).
\end{aligned}$$

and so we have

$$A \cap B \subset A \circ B.$$

It can be seen in a similar way that

$$A \cap B \subset B \circ A.$$

Thus we obtain that

$$A \cap B \subset (A \circ B) \cap (B \circ A),$$

and that (1) implies (3).

Assume that (6) holds. Let A and B be any intuitionistic fuzzy right ideal and any intuitionistic fuzzy left ideal of S, respectively. Then we have

$$A \cap B \subset (A \circ B) \cap (B \circ A) \subset (B \circ A),$$

Then it follows from Theorem1 that S is intraregular. On the other hand,

$$A \cap B \subset (A \circ B) \cap (B \circ A) \subset (A \circ B).$$

By Lemma5 and Lemma4 we obtain

$$A \circ B \subseteq A \circ S \subseteq A,$$

$$A \circ B \subseteq S \circ B \subseteq B,$$

and we have

$$A \circ B \subseteq A \cap B.$$

Thus we obtain that

$$A \circ B = A \cap B.$$

Thus it follows from Lemma3 that S is regular. Thus we obtain that (6) implies (1).

Finally, assume that (2) holds. In order to prove that (1) holds, let C be any biideal of S, and a any element of C. Since it follows from lemma that $\tilde{C}$ is an intuitionistic fuzzy biideal of S, we have

$$\sup_{a=pq}\left[\min\left\{\mu_{\tilde{C}(p)}, \mu_{\tilde{C}(q)}\right\}\right] = \mu_{\tilde{C}\circ\tilde{C}}(a) = \mu_{\tilde{C}(a)} = 1,$$
$$\inf_{a=pq}\left[\max\left\{v_{\tilde{C}(p)}, v_{\tilde{C}(q)}\right\}\right] = v_{\tilde{C}\circ\tilde{C}}(a) = v_{\tilde{C}(a)} = 0.$$

This implies that there exist elements b and c of S with $a = bc$ such that

$$\mu_{\tilde{C}(b)} = \mu_{\tilde{C}(c)} = 1, v_{\tilde{C}(b)} = v_{\tilde{C}(c)} = 0.$$

Then we have

$$a = bc \in CC,$$

And so we have

$$C \subset CC.$$

Since C is a biideal of S, the converse inclusion always holds. Thus we obtain that

$$C^2 = C.$$

Then it follows from Lemma2 that S is both regular and intraregular. Therefore we obtain that (2) implies (1). This completes the proof.

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