Extensions of Set Functions*

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Abstract

We establish a necessary and sufficient condition for a function defined on a subset of an algebra of sets to be extendable to a positive additive function on the algebra. It is also shown that this condition is necessary and sufficient for a regular function defined on a regular subset of the Borel algebra of subsets of a given compact Hausdorff space to be extendable to a measure. 

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1 Introduction

A standard method of constructing a measure in a given set $X$ is to define first an additive function on an algebra $\mathcal{A}$ of subsets of $X$ and then extend this function to a measure on the $\sigma$–algebra generated by $\mathcal{A}$. This ‘extension problem’ is an important part of the classical measure theory. Standard examples include Hahn’s extension theorem and the Borel measure in $[0, 1]$ (cf. [3, III.5]).

In the paper, we are concerned with the following problem: Let $X$ be a set and $\mathcal{A}$ be an algebra of subsets of $X$. Given a subset $S$

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of $\mathcal{A}$ and a real valued function $\alpha$ on $\mathcal{S}$, find necessary and sufficient conditions for $\alpha$ to be extendable to a positive additive function $\mu$ on $\mathcal{A}$.

The following condition is instrumental in our treatment of the extension problem:

$$\sum_{A \in \mathcal{F}} n(A) \chi_A(s) \geq 0, \forall s \in X \Rightarrow \sum_{A \in \mathcal{F}} n(A) \alpha(A) \geq 0, \quad [R]$$

for any finite family $\mathcal{F} \subseteq \mathcal{S}$, where coefficients $n(A)$'s are arbitrary integers and $\chi_A$ stands for the characteristic function of a set $A \subseteq X$.

We show that condition [R] is necessary (Section 2) and sufficient (Section 3) for $\alpha$ to be extendable to a positive additive set function. In the case when $X$ is a finite set, a stronger result is also established in Section 3. To obtain these results, we only assume that $X$ is a finite union of elements of $\mathcal{S}$ (this assumption is dropped in the case of a finite set $X$).

We make additional assumptions about the quadruple $(X, \mathcal{A}, \mathcal{S}, \alpha)$ when treating the extension problem for measures in sections 4 and 5. In both sections, $X$ is a compact Hausdorff space. In Section 4, $\mathcal{A}$ is the Borel algebra $\mathcal{B}$ of subsets of $X$, whereas in Section 5, $\mathcal{A}$ is the $\sigma$-algebra generated by $\mathcal{S}$. Assuming, in addition, that $\mathcal{S}$ and $\alpha$ satisfy some ‘regularity’ conditions, we show that [R] is a necessary and sufficient condition for $\alpha$ to be extendable to a positive regular measure on $\mathcal{A}$.

Our approach to the extension problem comes close to that of Bruno de Finetti in his “Probability Theory” [4] (Sections 9 and 10). In particular, his “convexity condition” (Section 15 in Appendix) is equivalent to condition [R], although de Finetti formulates it in rather different terms.

## 2 Condition [R]

The following lemma establishes a useful equivalent form of condition [R].

**Lemma 1.** [R] is equivalent to the following condition

$$\sum_{A \in \mathcal{F}} c(A) \chi_A(s) \geq 0, \forall s \in X \Rightarrow \sum_{A \in \mathcal{F}} c(A) \alpha(A) \geq 0, \quad (1)$$
for any finite family $\mathcal{F} \subseteq \mathcal{S}$, where coefficients $c(A)$'s are arbitrary real numbers.

Proof. It suffices to show that [R] implies (1). Suppose that for some real coefficients $c(A)$'s such that $\sum_{A \in \mathcal{F}} c(A) \chi_A \geq 0$ we have $\sum_{A \in \mathcal{F}} c(A) \alpha(A) < 0$. There are rational numbers $p(A)$'s such that $\sum_{A \in \mathcal{F}} p(A) \alpha(A) < 0$ and $p(A) \geq c(A)$ for all $A \in \mathcal{F}$. Clearly,

$$\sum_{A \in \mathcal{F}} p(A) \chi_A \geq \sum_{A \in \mathcal{F}} c(A) \chi_A \geq 0.$$  

Multiplying both inequalities $\sum_{A \in \mathcal{F}} p(A) \chi_A \geq 0$ and $\sum_{A \in \mathcal{F}} p(A) \alpha(A) < 0$ by a common multiple of the denominators of nonzero coefficients $p(A)$'s, we obtain a contradiction to [R].

\[ \square \]

Suppose that $\alpha$ is a restriction of a positive additive set function $\mu$ on $\mathcal{A}$. Note that the first sum in (1) is, by definition, a simple function on $X$. Then condition (1) states that the integral of a positive simple function is positive ([3, III.2.14]). Thus we have the following proposition.

**Proposition 1.** [R] is a necessary condition for a function $\alpha$ on $\mathcal{S}$ to be extendable to a positive additive set function on $\mathcal{A}$.

### 3 Extensions to positive additive set functions

We denote by $B_0$ the vector space of all simple functions (with respect to $\mathcal{A}$) on $X$ and denote by $B_0^\#$ – the algebraic dual space. The space $B_0^\#$ is isomorphic to the vector space of all additive set functions $\mu$ on $\mathcal{A}$. The isomorphism is given by

$$\mu \mapsto f_\mu \quad \text{where} \quad f_\mu(x) = \int x(s) \mu(ds). \quad (2)$$

The set $C$ of all positive simple functions on $X$ is a convex cone in $B_0$. Thus $B_0$ is an ordered vector space. A functional $f \in B_0^\#$ is
monotone if $x \geq y$ implies $f(x) \geq f(y)$. A functional $f$ is monotone if and only if it is positive, i.e., $x \geq 0$ implies $f(x) \geq 0$.

We shall use the following general fact about monotone linear extensions of linear functionals on ordered vector spaces ([1, Theorem 1, §6, ch. 2]).

**Theorem 1.** Let $L$ be a vector space with a cone $C$. Let $L_0$ be a subspace of $L$ such that for each $x$ in $L$, $x + L_0$ meets $C$ if and only if $-x + L_0$ meets $C$. Let $f_0$ in $L^\#$ be monotone. Then there exists an extension $f$ of $f_0$ which is monotone and in $L^\#$.

Now we prove the main theorem of this section.

**Theorem 2.** Let $S$ be a subset of $A$ such that $X$ is a finite union of sets in $S$ and let $\alpha$ be a function on $S$. Then $\alpha$ can be extended to a positive additive function $\mu$ on $A$ if and only if it satisfies condition [R].

**Proof.** Necessity was established in Proposition 1.

Sufficiency. Let $L_0$ be the subspace of $B_0$ generated by the characteristic functions of sets in $S$. For $x = \sum_{A \in \mathcal{F}} c(A)\chi_A \in L_0$ where $\mathcal{F}$ is a finite subset of $S$, we define

$$f_0(x) = \sum_{A \in \mathcal{F}} c(A)\alpha(A).$$

It follows immediately from (1) that $f_0$ is well-defined and is a positive linear functional on $L_0$.

Note that for any $x \in B_0$ the set $x + L_0$ meets the cone $C$ of positive functions in $B_0$. Indeed, let $X = \bigcup_{i=1}^n A_i$, $A_i \in S$ and define $x_0 = \sum_{i=1}^n \chi_{A_i} \in L_0$. Then, for $m = \sup_{s \in X} |x(s)|$, $x + mx_0 \in C$.

By Theorem 1, $f_0$ admits an extension to a positive linear functional $f$ on $B_0$. By defining $\mu(A) = f(\chi_A)$ for $A \in A$, we obtain an extension of $\alpha$ to a positive additive function on $A$.

Note that the assumption that $X$ is a finite union of sets in $S$ is essential in the theorem. Indeed, let $X$ be an infinite set, $A = 2^X$, and let $S$ be the family of all singletons in $A$. Let us define $\alpha(\{s\}) = 1$, $\forall s \in X$. Thus defined $\alpha$ satisfies condition [R] but cannot be extended to a monotone additive function on $A$. 


On the other hand, in the case of a finite set $X$ we have a stronger result.

**Theorem 3.** Let $X$ be a finite set, $S \subseteq A$, and $\alpha$ be a function on $S$. Then $\alpha$ can be extended to a positive additive function $\mu$ on $A$ if and only if it satisfies condition [R] with coefficients from a finite set of integers.

*Proof.* Again, we need to prove sufficiency only. Let $X' = \bigcup S$ and $A'$ be the algebra of subsets of $X'$ consisting of sets in $A$ that are subsets of $X'$. By Theorem 2, $\alpha$ can be extended to a positive additive set function $\mu'$ on $A'$. For an $A \in A$, we define $\mu(A) = \mu'(A \cap X')$. Clearly, $\mu$ is a positive additive set function on $A$.

Let us consider characteristic functions of sets in $S$ as integral vectors in $\mathbb{R}^{\{X\}}$ and let $C$ be the intersection of the subspace generated by these vectors with the positive cone in $\mathbb{R}^{\{X\}}$. The cone $C$ is a rational polyhedral cone and therefore has an integral Hilbert basis (Theorem 16.4 in [5]). Thus we can use only vectors from this basis in the right side of the implication in [R]. It follows that in the case of finite set $X$ coefficients in [R] can be taken from a finite set of integers.

\[ \square \]

**Remark.** It was noted by Jean-Paul Doignon (personal communication) that sufficiency of condition [R] in the finite case is a direct consequence of Farkas' lemma [5, Corollary 7.1d].

## 4 Extensions to measures I

The following example shows that, in general, condition [R] is not sufficient for a function $\alpha$ to be extendable to a positive measure ($\sigma$–additive set–function) on a $\sigma$–algebra $A$.

**Example 1.** Let $X = [0, 1]$ and $S = \{[0, t) : t \in (0, 1]\} \cup \{[0, t] : t \in [0, 1]\}$. Note that $X \in S$. We define $\alpha(\{0\}) = 0$ and $\alpha(A) = 1$ if $A$ is $[0, t)$ or $[0, t]$ for $0 < t < 1$. It is easy to verify that thus defined $\alpha$ satisfies condition [R].
Let $\mu$ be a $\sigma$–additive extension of $\alpha$ to the $\sigma$–algebra $\mathcal{B}$ of Borel subsets of $[0, 1]$. We have

$$
\begin{align*}
\mu((t, 1]) &= 1 - \alpha([0, t]) = 0, \quad \text{for } t > 0, \\
\mu((0, 1]) &= 1 - \alpha(\{0\}) = 1, \\
\mu((s, t]) &= 1 - \alpha([0, s]) - \mu((t, 1]) = 0, \quad \text{for } 0 < s < t.
\end{align*}
$$

By $\sigma$–additivity of $\mu$,

$$
1 = \mu((0, 1]) = \mu\left(\bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k}\right]\right) = \sum_{k=1}^{\infty} \mu\left(\left(\frac{1}{k+1}, \frac{1}{k}\right]\right) = 0,
$$

a contradiction. On the other hand, by condition [R], there is an additive extension of $\alpha$ to $\mathcal{B}$.

This example suggests that in order to keep [R] as a necessary and sufficient condition for extendibility of a set function to a measure, some constrains should be imposed on the quadruple $(X, \mathcal{A}, \mathcal{S}, \alpha)$. Namely, we assume that $X$ is a compact Hausdorff space and introduce the following ‘regularity’ conditions on $\mathcal{S}$ and $\alpha$.

**Definition 1.** (i) A family $\mathcal{S}$ of subsets of $X$ is said to be regular if

(a) For each $E \in \mathcal{S}$ and a closed set $F \subseteq E$ there is $E' \in \mathcal{S}$ such that

$$
F \subseteq E' \subseteq \text{cl}E' \subseteq E.
$$

(b) For each $E \in \mathcal{S}$ and an open set $G \supseteq E$ there is $E'' \in \mathcal{S}$ such that

$$
E \subseteq \text{int}E'' \subseteq E'' \subseteq G.
$$

(ii) A function $\alpha$ on a family $\mathcal{S}$ is said to be regular if for each $E \in \mathcal{S}$ and $\varepsilon > 0$ there is a set $F$ in $\mathcal{S}$ whose closure is contained in $E$ and a set $G$ whose interior contains $E$ such that $|\alpha(G) - \alpha(F)| < \varepsilon$.

In this section, $\mathcal{A}$ is the Borel algebra $\mathcal{B}$ of subsets of $X$.

**Example 2.** Since $X$ is a normal space, the families of all open sets and of all closed sets in $X$ are examples of regular families of Borel sets (cf. [2, VII.3.2(2)]).
Example 3. Let \( X = [0, 1] \) and \( S \) be the family of all intervals in the form \([a, b)\). Clearly, \( S \) is a regular family of Borel sets.

Example 4. Let \( S = \mathcal{B} \) and let \( \alpha = \mu - \) a regular positive additive set function on \( \mathcal{B} \) in the usual sense (cf. [3, III.5.11]). Then \( \alpha \) is a regular function in the sense of Definition 1.

Lemma 2. Let \( \mu \) be a regular positive measure on the Borel algebra \( \mathcal{B} \) and let \( S \) be a regular family of Borel sets. Then the restriction of \( \mu \) to \( S \) is a regular function on \( S \).

Proof. Let \( E \in S \) and \( \varepsilon > 0 \). Since \( \mu \) is regular and positive, there is a closed set \( F \subseteq E \) and an open set \( G \supseteq E \) such that \( \mu(G) - \mu(F) < \varepsilon \). Since \( S \) is regular, there are \( E', E'' \in S \) such that \( F \subseteq E' \subseteq \text{cl}E' \subseteq E \subseteq \text{int}E'' \subseteq E'' \subseteq G \). Since \( \mu \) is positive, \( \mu(E'') - \mu(E') < \varepsilon \). Therefore the restriction of \( \mu \) to \( S \) is a regular set function on \( S \).

Lemma 3. Let \( S \) be a regular family of Borel sets such that \( X \) is a finite union of sets in \( S \) and let \( \alpha \) be a regular function on \( S \) satisfying condition \([R]\). Then \( \alpha \) is extendable to a regular positive measure on \( \mathcal{B} \).

Proof. By Theorem 2, \( \alpha \) admits an extension to a positive additive set function \( \mu \) on \( \mathcal{B} \). Since \( \mu \) is bounded, it defines a bounded positive linear functional \( f \) on the Banach space \( \mathcal{B} \) of all uniform limits of functions in \( B_0 \) endowed with the norm \( \| \cdot \|_\infty \). This functional is given by [3, IV.5.1]

\[
f(x) = \int x(s) \mu(ds), \quad x \in B_0.
\]

By the Riesz representation theorem [3, IV.6.3] the restriction of this functional (which we denote by the same symbol \( f \)) to the space \( C(X) \) of continuous functions on \( S \) is given by

\[
f(x) = \int x(s) \mu^*(ds), \quad x \in C(X),
\]

where \( \mu^* \) is a regular positive measure on \( \mathcal{B} \).
Now it suffices to show that $\mu^*(E) = \mu(E)$ on $S$. Let $E \subseteq S$ and $\varepsilon > 0$. Since $\mu^*$ is positive and regular there is a closed set $F$ and an open set $G$ such that

$$F \subseteq E \subseteq G, \quad \mu^*(F) \cdot \mu^*(E) \cdot \mu^*(G), \quad \text{and} \quad \mu^*(G) - \mu^*(F) < \varepsilon.$$  

Since $S$ is regular, there are $E', E'' \in S$ such that

$$F \subseteq E' \subseteq \text{cl}E' \subseteq E \subseteq \text{int}E'' \subseteq E'' \subseteq G \quad \text{and} \quad \mu(E'') - \mu(E') < \varepsilon.$$  

We denote $F' = \text{cl}E'$ and $G' = \text{int}E''$. Since $\mu$ and $\mu^*$ are positive,

$$\mu(G') - \mu(F') < \varepsilon \quad \text{and} \quad \mu^*(G') - \mu^*(F') < \varepsilon. \quad (3)$$

Since $X$ is a normal space, by Urysohn’s lemma, there is a continuous function $x$ such that

$$0 \cdot x(s) \cdot 1, \quad \text{for all } s \in X,$$

$$x(s) = 1, \quad \text{for all } s \in F',$$

$$x(s) = 0, \quad \text{for all } s \notin G'.$$  

For a natural number $n$, we define a family of $n+1$ intervals in $[0, 1]$ by

$$I_k = \begin{cases} \left[ \frac{k-1}{n}, \frac{k}{n} \right), & \text{for } 1 \cdot k \cdot n; \\ \{1\}, & \text{for } k = n + 1. \end{cases}$$

The family of Borel sets $E_k = x^{-1}(I_k)$, $1 \cdot k \cdot n+1$, forms a partition of $X$. Clearly, $\bigcup_{k=2}^{n} E_k \subseteq G' \setminus F'$. Therefore, by the first inequality in $(3)$,

$$\sum_{k=2}^{n} \mu(E_k) \cdot \mu(G') - \mu(F') < \varepsilon. \quad (4)$$

Let $x_n$ be a function defined by $x_n(s) = \frac{k-1}{n}$ for $s \in E_k$, $1 \cdot k \cdot n+1$. Thus

$$|f(x) - f(x_n)| \cdot \|f\| \cdot \|x - x_n\| < \frac{1}{n} \|f\| \quad (5)$$
Further,

\[ x_n = \sum_{k=1}^{n+1} \frac{k-1}{n} \chi_{E_k} = \sum_{k=2}^{n} \frac{k-1}{n} \chi_{E_k} + \chi_{E_{n+1}}. \]

Thus

\[ f(x_n) = \sum_{k=2}^{n} \frac{k-1}{n} \mu(E_k) + \mu(E_{n+1}), \]

which implies, by (4),

\[ f(x_n) - \mu(E_{n+1}) = \sum_{k=2}^{n} \frac{k-1}{n} \mu(E_k) < \varepsilon. \]

This inequality together with one in (5) imply

\[ |f(x) - \mu(E_{n+1})| < \varepsilon + \frac{1}{n} \|f\|. \quad (6) \]

Clearly, \( F' \subseteq E_{n+1} \subseteq G' \), and \( F' \subseteq E \subseteq G' \). Thus, by (3),

\[ |\mu(E_{n+1}) - \mu(E)| < \varepsilon. \quad (7) \]

Since \( f(x) = \int x(s) \mu^*(ds) \), we have \( \mu^*(F') \cdot f(x) \cdot \mu^*(G') \). On the other hand, \( \mu^*(F') \cdot \mu^*(E) \cdot \mu^*(G') \). By the second inequality in (3),

\[ |\mu^*(E) - f(x)| < \varepsilon. \quad (8) \]

Combining inequalities (6), (7), and (8), we have

\[ |\mu^*(E) - \mu(E)| < 3\varepsilon + \frac{1}{n} \|f\|. \]

Hence, \( \mu^*(E) = \mu(E) = \alpha(E) \).

Combining the results of Lemma 2 and Lemma 3, we have the following theorem.

**Theorem 4.** Let \( S \) be a regular family of Borel sets such that \( X \) is a finite union of sets in \( S \). A function \( \alpha \) on \( S \) is extendible to a regular positive measure on \( B \) if and only if it is regular and satisfies condition \([R]\).
5 Extensions to measures II

In this section we make a different assumption about components of the quadruple \((X, \mathcal{A}, \mathcal{S}, \alpha)\). Namely, let \(X\) again be a compact Hausdorff space, \(\mathcal{S}\) be a family of subsets of \(X\), and let \(\mathcal{A}\) be the \(\sigma\)-algebra generated by \(\mathcal{S}\).

**Lemma 4.** Let \(\mathcal{S}\) be a regular family of subsets of \(X\). The restriction of a regular positive measure \(\mu\) on \(\mathcal{A}\) to \(\mathcal{S}\) is a regular function on \(\mathcal{S}\).

**Proof.** Let \(E \in \mathcal{S}\) and \(\varepsilon > 0\). Since \(\mu\) is regular and positive, there is \(F \in \mathcal{A}\) such that \(\text{cl}F \subseteq E\) and \(G \in \mathcal{A}\) such that \(\text{int}G \supseteq E\) such that \(\mu(G) - \mu(F) < \varepsilon\). Since \(\mathcal{S}\) is regular, there are \(E', E'' \in \mathcal{S}\) such that

\[
F \subseteq \text{cl}F \subseteq E' \subseteq E \subseteq \text{int}E'' \subseteq E'' \subseteq \text{int}G \subseteq G.
\]

Since \(\mu\) is positive, \(\mu(E'') - \mu(E') < \varepsilon\). Therefore the restriction of \(\mu\) to \(\mathcal{S}\) is a regular set function on \(\mathcal{S}\).

\(\square\)

**Lemma 5.** Let \(\mathcal{S}\) be a regular family of subsets of \(X\) such that \(X\) is a finite union of sets in \(\mathcal{S}\) and let \(\alpha\) be a regular function on \(\mathcal{S}\) satisfying condition [R]. Then \(\alpha\) is extendable to a regular positive measure \(\mu\) on the \(\sigma\)-algebra \(\mathcal{A}\) generated by \(\mathcal{S}\).

**Proof.** Let \(\mathcal{A}_0\) be the algebra generated by \(\mathcal{S}\). By Theorem 2, \(\alpha\) admits an extension to a positive additive set function \(\mu\) on \(\mathcal{A}_0\). It suffices to show that \(\mu\) is a regular function on \(\mathcal{A}_0\). Indeed, by Theorem 14 in [3, III.5], a regular function on \(\mathcal{A}_0\) admits an extension to a positive measure on \(\mathcal{A}\).

Let \(\varepsilon > 0\) and \(A\) and \(B\) be two sets in \(\mathcal{S}\). Since \(\mathcal{S}\) is a regular family and \(\alpha\) is a regular function, there are \(A_1, A_2 \in \mathcal{S}\) and \(B_1, B_2 \in \mathcal{S}\) such that

\[
A_1 \subseteq \text{cl}A_1 \subseteq A \subseteq \text{int}A_2 \subseteq A_2, \quad \alpha(A_2) - \alpha(A_1) < \varepsilon/2,
\]

and

\[
B_1 \subseteq \text{cl}B_1 \subseteq B \subseteq \text{int}B_2 \subseteq B_2, \quad \alpha(B_2) - \alpha(B_1) < \varepsilon/2,
\]
We have
\[ \mu(A_1 \cup B_1) + \mu(A_1 \cap B_1) = \mu(A_1) + \mu(B_1) = \alpha(A_1) + \alpha(B_1) \]
and
\[ \mu(A_2 \cup B_2) + \mu(A_2 \cap B_2) = \mu(A_2) + \mu(B_2) = \alpha(A_2) + \alpha(B_2). \]

Hence,
\[ \mu(A_2 \cup B_2) - \mu(A_1 \cup B_1) \]
\[ = [\alpha(A_2) - \alpha(A_1)] + [\alpha(B_2) - \alpha(B_1)] < \varepsilon, \]
implying
\[ \mu(A_2 \cup B_2) - \mu(A_1 \cup B_1) < \varepsilon \quad \text{and} \quad \mu(A_2 \cap B_2) - \mu(A_1 \cap B_1) < \varepsilon. \]

Clearly,
\[ \text{cl}(A_1 \cup B_1) \subseteq A \cup B \subseteq \text{int}(A_1 \cup B_1) \quad \text{and} \]
\[ \text{cl}(A_1 \cap B_1) \subseteq A \cap B \subseteq \text{int}(A_1 \cap B_1). \]

Thus the regularity condition for \( \mu \) is satisfied for unions and intersections of sets in \( S \). Hence, \( \mu \) is a regular function on \( A_0 \).

Combining the results of Lemma 4 and Lemma 5, we have the following theorem.

**Theorem 5.** Let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by a regular family \( \mathcal{S} \) of subsets of \( X \) such that \( X \) is a finite union of sets in \( \mathcal{S} \). A function \( \alpha \) on \( \mathcal{S} \) is extendable to a regular positive measure on \( \mathcal{A} \) if and only if it is regular and satisfies condition \([R]\).

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References


